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An analysis of the Stokes system with pressure dependent viscosity

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Abstract. *In this paper we study the existence and uniqueness of the solution of the Stokes system, describing the flow of a viscous fluid, in case of pressure dependent viscosity.*

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1. Introduction

In his classical paper from 1848 [22] Stokes predicted that the viscosity of the fluid can depend on the pressure. Those effects for various liquids have been measured in many engineering papers, starting from the beginning of the 20-th century (see e.g. [1, 2, 10, 13, 15] etc.). That effect is usually neglected as it becomes important only in case of high pressure. Several models have been used to describe that relation since. The most popular is probably the exponential law

$$\mu = \mu_0 \exp(\alpha\, p) \tag{1}$$

usually called the Barus formula [1]. Here μ_0 and α are the constants depending on the fluid. The formula seems to be reasonable for mineral oil, unless the pressure is very high (larger then 0.5 MPa). The coefficient α typically ranges between 1 and 10^{-8} Pa^{-1}. The lower end of the range corresponding to *paraffinic* and the upper end to the *nephtenic* oils (see Jones et al [14]). That formula is frequently used by engineers, sometimes combined with temperature dependence. In case of two above mentioned laws explicit solutions of the equations of motion, for some particular situations like unidirectional and plane-parallel flows, were found in [12]. Discussion on other possibilities for the viscosity-pressure formula and some historical remarks on the subject can be found in the same paper. Several engineering papers can be found discussing other possible laws and their consistency. We mention for instance [19] and [21].

From mathematical point of view supposing that the viscosity depends on the pressure makes the Navier-Stokes system much more complicated. Not only that it brings in additional nonlinearity to the momentum equation, but it changes the nature of the pressure as it cannot be eliminated from the system using Helmholtz decomposition and it cannot be seen as just a Lagrange multiplier. First important contribution was made by Renardy [20], where the viscosity function $p \mapsto \mu(p)$ is assumed to be sublinear at infinity and its derivative is assumed to be bounded on $\mathbf{R}$. In three interesting papers Gazzola and Gazzola and Secchi have proved the existence theorems for stationary and evolutional case under the assumption that the flow is governed by almost conservative force, has small initial velocity (in non-stationary case) and that μ is a smooth function globally bounded from below by some positive constant (hypothesis that rules out the Barus formula). The approach is based on the local inverse function theorem and relies on the fact that given data are small enough. Since our next goal is to do the asymptotic analysis of the system in thin domain, the small data assumption is not acceptable as such condition, in general, depends on the domain, which is not practical when the domain shrinks.

Interesting results were found about evolution Navier- Stokes system with pressure and shear dependent viscosity in two papers [11] and [16], but under certain technical assumptions on the viscosity that are not fulfilled by the Barus formula. More precisely they assume that $\mu = \mu(p, |D\,\mathbf{u}|^2)$, satisfies

$$C_1 \left(1 + |\mathbf{D}|^2\right)^{\frac{r-2}{2}} |\mathbf{B}|^2 \le \frac{\partial}{\partial \mathbf{D}} \mu\left(p, |\mathbf{D}|^2\right) (\mathbf{B} \otimes \mathbf{B}) \le C_2 \left(1 + |\mathbf{D}|^2\right)^{\frac{r-2}{2}} |\mathbf{B}|^2$$
$$\frac{\partial}{\partial p} \mu\left(p, |\mathbf{D}|^2\right) |\mathbf{D}|^2 \le \gamma_0 \left(1 + |\mathbf{D}|^2\right)^{\frac{r-2}{4}} \le \gamma_0$$
$$\forall \ \mathbf{D}, \mathbf{B} \in \mathbf{R}^{n\times n} \text{ symmetric }, \ p \in \mathbf{R}, \ 1 < r < 2 \ .$$

Those assumptions allow them to derive the uniform a priori estimates for the Galerkin approximation. Also, for simplicity, they take periodic boundary conditions which is not useful for applications that we have in mind.

Reynolds lubrication equation with pressure dependent viscosity was studied in [17]. It's well-posedness was proved as well as an apropriate maximum principle. Finally the nonlocal effects obtained by homogenization of the same equation in case of large Reynolds number were studied. An asymptotic model for flow of the fluid with pressure dependent viscosity through thin domain was derived in [18].

We use an approach here that does not need any small data assumption and very general assumption on μ satisfied by Barus formula and other empiric laws that can be found in the literature. Indeed, we assume that the function $p \mapsto \mu(p)$ is convex in vicinity of $-\infty$, that $\lim_{s\to-\infty} \mu'(s) = 0$ and $\int_{-\infty}^0 \frac{ds}{\mu(s)} = +\infty$. Those assumption are irrelevant from the physical point of view since the dependence of the viscosity on the pressure becomes significant only for large positive pressure, while all our assumptions concern the behavior of the viscosity for large negative pressures. Typically, the viscosity is taken to be constant for negative pressures, and thus all our assumptions are trivially fulfilled. We also work with physically relevant Dirichlet boundary condition. However the approach works only for the stationary Stokes system. i.e. we do not handle neither the inertial term nor the time derivative.

2. Position of the problem

The mathematical model can be written in the following form: Let $\Omega \subset \mathbf{R}^d$, $d = 2, 3$ be a bounded smooth domain. We assume that the boundary is at least of class C^2. The unknowns in the model are $\mathbf{u}$ - the velocity , p - the pressure . We recall that the stationary motion of the incompressible viscous laminar flow is governed by the Navier-Stokes equations. Thus we write the following system

$$\begin{cases} -2\mathrm{div}\,[\mu(p)\,\mathbf{D}\,\mathbf{u}] + (\mathbf{u}\,\cdot\,\nabla\,)\,\mathbf{u} + \nabla p = \mathbf{0}\,, \quad \mathrm{div}\,\mathbf{u} = 0 \ \text{ in } \ \Omega\,, \\ \mathbf{u} = \mathbf{g} \ \text{ on } \ \partial\Omega\,. \end{cases} \tag{2}$$

If the Reynolds number is not too large it is reasonable to neglect the inertial term and replace the Navier-Stokes by the Stokes system

$$\begin{cases} -2\mathrm{div}\,[\mu(p)\,\mathbf{D}\,\mathbf{u}] + \nabla p = \mathbf{0}\,, \quad \mathrm{div}\,\mathbf{u} = 0 \ \text{ in } \ \Omega\,, \\ \mathbf{u} = \mathbf{g} \ \text{ on } \ \partial\Omega\,. \end{cases} \tag{3}$$

We assume that the function $\mathbf{g}$ satisfies the following regularity and compatibility conditions

$$\mathbf{g} \in W^{2-1/\beta,\beta}(\Omega)^d\,, \qquad \text{for some } \ \beta > d \tag{4}$$

$$\int_{\partial\Omega} \mathbf{g}\,\cdot\,\mathbf{n} = 0\,. \tag{5}$$

For the dependence of the viscosity on the pressure we assume that $p \mapsto \mu(p)$ is a C^2 function and that it satisfies the following conditions:

$$\int_{-\infty}^{0} \frac{ds}{\mu(s)} = +\infty \tag{6}$$

$$\lim_{s\to-\infty} \mu'(s) = 0\,, \quad \text{and } \mu \text{ is convex in vicinity of } \ -\infty\,. \tag{7}$$

3. The idea

For the sake of simplicity, in this chapter, we assume that μ satisfies the Barus formula (1). Then

$$\begin{aligned} \nabla p \quad &= 2\mathrm{div}\ (\mu(p)\ \mathbf{Du}) = \mu(p)\,\Delta\mathbf{u} + 2\mu'(p)\,\nabla p\,\mathbf{Du} \\ &= \mu_0\, e^{\alpha p}\ \ (\Delta\mathbf{u} + 2\alpha\,\nabla p\ \mathbf{Du})\ . \end{aligned}$$

Dividing by $\mu_0\, e^{\alpha p}$ we get

$$-\Delta\mathbf{u} + \frac{1}{\mu_0}\, e^{-\alpha p}\,\nabla p = 2\alpha\ \nabla p\ \mathbf{Du}\,.$$

We now look for the function q such that

$$\frac{1}{\mu_0}\, e^{-\alpha p}\,\nabla p = \nabla q\,.$$

Obviously there is a continuum of such functions given by

$$q = \frac{1}{\alpha\mu_0}\,\left(e^{-\alpha\sigma} - e^{-\alpha p}\right)\,, \tag{8}$$

where $\sigma \in \mathbf{R}$ is arbitrary . We can use that liberty in choice of σ to get what we want. Now

$$\nabla p = \mu_0\, e^{\alpha p}\ \nabla q = \frac{\mu_0}{e^{-\alpha\sigma} - \alpha\mu_0 q}\ \nabla q\,.$$

Thus our system becomes

$$-\Delta\mathbf{u} + \nabla q = 2\frac{\alpha\,\mu_0}{e^{-\alpha\sigma} - \alpha\mu_0 q}\ \nabla q\ \mathbf{Du}\,. \tag{9}$$

Obviously

$$\lim_{\sigma\to-\infty} \frac{\alpha\,\mu_0}{e^{-\alpha\sigma} - \alpha\mu_0 q} = 0$$

meaning that the right-hand side can be made as small as we need by picking large $|\sigma|$, $\sigma < 0$.

Equation (9) is to be complemented by

$$\operatorname{div}\mathbf{u} = 0\,, \qquad \mathbf{u} = \mathbf{g}\ \text{ on } \partial\Omega$$

and it makes a nonlinear Stokes-like system, but with nonlinearity that can be made as small as we want. Under the circumstances, it is not too complicated to prove that it has a solution. Does it mean that our original problem (3) has a solution? Well, it does if we can invert the transformation (8) and reconstruct p from q. It is easy to see that

$$p(x) = \frac{1}{\alpha}\ln\left(\frac{1}{e^{-\alpha\sigma} - \alpha\,\mu_0\,q(x)}\right)\,.$$

For that formula to make sense, we need to make sure that

$$q(x) < \frac{1}{\alpha\,\mu_0\,e^{\alpha\sigma}}\,,\ \forall\ x \in\ \Omega\,.$$

Once again that condition can be met by choosing σ small enough, i.e. $\sigma < 0$ and $|\sigma|$ large enough.

The uniqueness of such solution can be proved since the transformed system is the Stokes system with small nonlinear perturbation, and if the transformed system has a unique solution, so does the original system. At least as long as we look only for regular solutions.

4. The main results

The statement of the main result is that our problem has a solution, which is unique, under some technical conditions. In standard Stokes (or Navier-Stokes) system the pressure is obviously determined only up to a constant, so it is not unique unless we impose some additional condition, like prescribing the value of its integral over Ω or prescribing the value of the pressure in some point of the domain Ω. That is less obvious here, since the pressure appears in the viscosity formula. However, it turns out that similar condition is needed here to fix the pressure.

4.1. Existence theorem

Theorem 1. *Let* $\mathbf{g}$ *satisfy (4) and (5). Assume, in addition that (6) and (7) hold. Then the problem (3) has a solution* $(\mathbf{u}, p) \in \mathbf{X} = W^{2,\beta}(\Omega)^d \times W^{1,\beta}(\Omega)$.

4.2. Existence proof

4.2.1. Transformation of the system

We slightly generalize the idea presented in chapter 3, for general viscosity-pressure relation, and give all the technical details.

So, for some $\sigma \in \mathbf{R}$ we define two mappings

$$B(p, \sigma) = \int_\sigma^p \frac{ds}{\mu(s)} \tag{10}$$

As $p \mapsto \frac{1}{\mu(p)}$ is continuous and positive, B is of class C^1. Since

$$\frac{\partial B}{\partial p}(p, \sigma) = \frac{1}{\mu(p)} > 0$$

the function $B(\,\cdot\,, \sigma) : \mathbf{R} \mapsto \mathbf{R}$ is strictly increasing (and thus injective), for any parameter σ. Furthermore

$$Im\, B(\,\cdot\,, \sigma) = [M_\sigma^-, M_\sigma^+]\,,$$

where

$$M_\sigma^+ = \lim_{p \to +\infty} B(p, \sigma) = \int_\sigma^{+\infty} \frac{ds}{\mu(s)}\,, \quad M_\sigma^- = \lim_{p \to +\infty} B(p, \sigma) = -\int_{-\infty}^\sigma \frac{ds}{\mu(s)}\,. \tag{11}$$

If the above integrals are divergent we take their value to be $+\infty$. We can now define another function

$$H(\,\cdot\,, \sigma) = B^{-1}(\,\cdot\,, \sigma)\,. \tag{12}$$

Thus $q \mapsto H(q,\sigma)$ is an inverse of the function $p \mapsto B(q,\sigma)$, while σ is treated only as a parameter. Obviously $H(\,\cdot\,,\sigma) : [M_\sigma^-, M_\sigma^+] \to \mathbf{R}$ is well defined, strictly increasing and smooth. Furthermore

$$\frac{\partial H}{\partial q}(q,\sigma) = \mu(p) = \mu(H(q,\sigma))\,.$$

Next we define the function

$$b(q,\sigma) = 2\,\mu'(p) = 2\,\mu'(H(q,\sigma)) \tag{13}$$

that we need in the sequel. Function μ' is defined on $\mathbf{R}$, but $H(\,\cdot\,,\sigma)$ is defined only on $[M_\sigma^-, M_\sigma^+]$, thus the domain of $b(\,\cdot\,,\sigma)$ is $[M_\sigma^-, M_\sigma^+]$. As μ and H are smooth b is continuous. Using the assumptions on μ we prove the following important technical result:

LEMMA 1. *Let $\mu : \mathbf{R} \to \mathbf{R}$ satisfy the following hypothesis*

1.) $\displaystyle\int_{-\infty}^{0} \frac{ds}{\mu(s)} = +\infty$

2.) μ is convex in vicinity of $-\infty$ and $\displaystyle\lim_{s\to-\infty} \mu'(s) = 0\,.$

Then

$$M_\sigma^- = -\infty\,, \qquad \lim_{\sigma\to-\infty} M_\sigma^+ = +\infty \tag{14}$$

$$\lim_{\sigma\to-\infty} b(q,\sigma) = 0\,, \qquad \textit{for any } q \in \mathbf{R}\,. \tag{15}$$

Proof. First of all, 1.) means that $M_\sigma^- = -\infty$ and that

$$\lim_{\sigma\to-\infty} M_\sigma^+ = \int_{-\infty}^{+\infty} \frac{ds}{\mu(s)} = \infty\,.$$

Next, since $b(q,\sigma) = \mu'\left(H(q,\sigma)\right)$ we only need to see that assumption 1.) implies that

$$\lim_{\sigma\to-\infty} H(q,\sigma) = -\infty\,. \tag{16}$$

But that is clear, because the mapping $\sigma \mapsto H(q,\sigma)$ is strictly increasing and unbounded from below. Indeed

$$\frac{\partial}{\partial\sigma} H(q,\sigma) = \frac{\mu\left(H(q,\sigma)\right)}{\mu(\sigma)} > 0\,.$$

On the other hand the mapping $\sigma \mapsto H(q,\sigma)$ cannot be bounded from below. Supposing that there exists some $\gamma(q) = \inf_{\sigma\in\mathbf{R}} H(q,\sigma) \in \mathbf{R}$ implies that

$$q = \int_{\sigma}^{H(q,\sigma)} \frac{ds}{\mu(s)} \geq \int_{\sigma}^{\gamma(q)} \frac{ds}{\mu(s)} .$$

Now the assumption 1.) leads to

$$q = \lim_{\sigma\to-\infty} \inf \int_{\sigma}^{H(q,\sigma)} \frac{ds}{\mu(s)} \geq \lim_{\sigma\to-\infty} \int_{\sigma}^{\gamma(q)} \frac{ds}{\mu(s)} = \int_{-\infty}^{\gamma(q)} \frac{ds}{\mu(s)} = +\infty .$$

Thus $\sigma \mapsto H(q,\sigma)$ is strictly increasing and unbounded from below and consequently

$$\lim_{\sigma\to-\infty} H(q,\sigma) = -\infty .$$

Then, since μ' tends to zero at $-\infty$ we have

$$b(q,\sigma) = 2\,\mu'(H(q,\sigma)) \to 0 , \quad \text{as } \sigma \to -\infty ,$$

We now define the new unknown

$$q = B(p,\sigma) \tag{17}$$

We can now rewrite our system in terms of new unknown and it reads

$$-\Delta\mathbf{u} + \nabla q = b(q,\sigma)\,\mathbf{D}\,\mathbf{u}\;\nabla q , \quad \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega , \tag{18}$$

$$\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega . \tag{19}$$

Recall that, due to (15)

$$\lim_{\sigma\to-\infty} b(q,\sigma) = 0 , \qquad \text{for any } q . \tag{20}$$

□

EXAMPLE 1. *In case of Barus law $\mu(p) = \mu_0\, e^{\alpha\, p}$ we have*

$$B(p,\sigma) = \frac{1}{\alpha\,\mu_0}\left(e^{-\alpha\sigma} - e^{-\alpha p}\right) , \quad H(q,\sigma) = \frac{1}{\alpha}\, ln\, \frac{1}{e^{-\alpha\sigma} - \alpha\,\mu_0 q} .$$

Furthermore

$$b(q,\sigma) \equiv 2\,\mu'(H(q,\sigma)) = \frac{2\,\alpha\,\mu_0}{e^{-\alpha\sigma} - \alpha\,\mu_0 q} , \quad M_{\sigma}^{+} = \frac{e^{-\alpha\,\sigma}}{\alpha\,\mu_0} , \quad M_{\sigma}^{-} = -\infty .$$

4.2.2. The proof of Theorem 1

The idea is to define the mapping $T : X = W^{2,\beta}(\Omega)^d \times W^{1,\beta}(\Omega) \to X$ by taking $T(\mathbf{v}, q) = (\mathbf{w}, \pi)$ where $(\mathbf{w}, \pi)$ is the solution to the problem

$$-\Delta \mathbf{w} + \nabla \pi = b(q, \sigma)\, \mathbf{D}\, \mathbf{v}\ \nabla q \quad \text{in } \Omega\,, \tag{21}$$

$$\operatorname{div} \mathbf{w} = 0 \ \text{ in } \Omega\,, \quad \mathbf{w} = \mathbf{g} \ \text{ on } \partial\Omega\,, \quad \int_\Omega \pi = 0\,. \tag{22}$$

The classical result by Cattabriga [3](see Appendix A) implies, for any $\beta > d$ and any $(\mathbf{v}, q) \in W^{2,\beta}(\Omega)^d \times W^{1,\beta}(\Omega)$ the existence of such $(\mathbf{w}, \pi) \in W^{2,\beta}(\Omega) \times W^{1,\beta}(\Omega)$. Since we have imposed

$$\int_\Omega \pi = 0\,, \tag{23}$$

the solution is unique. Furthermore, assuming that

$$|(\mathbf{v}, q)|_X \equiv |\mathbf{v}|_{W^{2,\beta}(\Omega)} + |q|_{W^{1,\beta}(\Omega)} \le M\,, \quad \int_\Omega q = 0\,. \tag{24}$$

and using (45), we obtain

$$|\mathbf{w}|_{W^{2,\beta}(\Omega)} + |\pi|_{W^{1,\beta}(\Omega)} \le C_\beta \Big(|b(q, \sigma)|_{L^\infty(\Omega)} |\mathbf{D}\, \mathbf{v}|_{L^\infty(\Omega)} |\nabla q|_{L^\beta(\Omega)} + \tag{25}$$

$$+ |\mathbf{g}|_{W^{2-1/\beta,\beta}(\partial\Omega)} \Big) \le C_\beta \left(M^2 C(\beta, \infty)^2 |b(q, \sigma)|_{L^\infty(\Omega)} + |\mathbf{g}|_{W^{2-1/\beta,\beta}(\partial\Omega)} \right) .$$

For any $x \in \overline{\Omega}$

$$|q(x)| \le C(\beta, \infty)\ M \equiv \overline{M}\,. \tag{26}$$

As μ is convex in vicinity of $-\infty$, there exists some $s_0 < 0$ such that $s \mapsto \mu(s)$ is convex for all $s < s_0$.

For any $\eta < 0$ there exists $\sigma_0 < 0$ such that for any $\sigma < \sigma_0$

$$H(q(x), \sigma) < H(\overline{M}, \sigma) < \eta\,.$$

We choose $\eta < s_0$. Due to the convexity of μ we know that μ' is increasing for $s < -s_0$ so that for all $\sigma < \sigma_0$

$$b(q(x), \sigma) = \mu'(H(q(x), \sigma)) < \mu'(H(\overline{M}, \sigma)) = b(\overline{M}, \sigma)\,. \tag{27}$$

Now, due to the Lemma 1

$$\lim_{\sigma \to -\infty} |b(\overline{M}, \sigma)| = 0\,. \tag{28}$$

Thus, for any $x \in \overline{\Omega}$ and for any $\varepsilon > 0$, there exists σ_0 such that for any $\sigma < \sigma_0$

$$|b(q(x), \sigma)|_{L^\infty(\Omega)} \le b(\overline{M}, \sigma) \le \frac{\varepsilon}{C_\beta\, M^2 C(\beta, \infty)^2} \,.$$

For

$$G = |\mathbf{g}|_{W^{2-1/\beta,\beta}(\partial\Omega)} \tag{29}$$

we choose

$$M = 2\, C_\beta\; G \tag{30}$$

and for $\varepsilon < G$ (24) and (25) imply that

$$|(\mathbf{w}, \pi)|_X \equiv |\mathbf{w}|_{W^{2,\beta}(\Omega)} + |\pi|_{W^{1,\beta}(\Omega)} < M \,. \tag{31}$$

It proves that T maps the ball $B_M \subset X$ of radius M in itself. To apply the Tychonoff fixed point theorem it remains to prove that T is weakly continuous. As the ball B_M is metrizable in weak topology, weak sequential continuity will be enough. To do so, we assume that the sequence $(\mathbf{v}_n, q_n)$ converges weakly in X to some $(\mathbf{v}, q)$. Due to the compact embedding $W^{1,\beta}(\Omega) \subset C(\overline{\Omega})$ we know that $\mathbf{D}\mathbf{v}_n$ is bounded and strongly convergent in $C(\overline{\Omega})$, while ∇q_n is bounded in $L^\beta(\Omega)$.

Let $(\mathbf{w}_n, \pi_n) = T(\mathbf{v}_n, q_n)$. Since T maps B_M in itself, the sequence (w_n, π_n) is bounded in X. Furthermore, we can extract a subsequence, denoted by the same symbol, such that

$$\mathbf{w}_n \rightharpoonup \mathbf{w} \quad \text{weakly in } \; W^{2,\beta}(\Omega)^d \tag{32}$$

$$q_n \rightharpoonup q \quad \text{weakly in } \; W^{1,\beta}(\Omega)\,. \tag{33}$$

Compact embedding $W^{1,\beta}(\Omega) \subset C(\overline{\Omega})$ and (31) implies the strong convergence

$$\mathbf{w}_n \to \mathbf{w} \quad \text{in } \; C^1(\overline{\Omega})^d \tag{34}$$

$$q_n \to q \quad \text{in } \; L^\sigma(\Omega) \;\; \text{and in } \;\; C(\overline{\Omega})\,. \tag{35}$$

Due to the continuity of b, we consequently get

$$b(q_n, \sigma) \to b(q, \sigma) \quad \text{in } \; C(\overline{\Omega})\,. \tag{36}$$

That is enough to pass to the limit in (21), (22). Proving that $T(v_n, q_n) \rightharpoonup T(v, q)$ weakly in X means that $T : B_M \to B_M$ is weakly continuous. Now the Tychonoff fixed point theorem implies the existence of solution $(\mathbf{u}, q) \in B_M$ of the transformed system (18), (19). To prove that $(\mathbf{u}, p)$, $p = H(q, \sigma)$, with H defined by (12), is the solution to the original system (3) we need to verify that $q(x)$ is in the range of function $H(\,\cdot\,, \sigma)$, $Im\, H(\,\cdot\,, \sigma) = [M_\sigma^-, M_\sigma^+]$, for some $\sigma < 0$. We recall that $M_\sigma^\pm$ were defined by (11) . Under the assumption (7) Lemma 1 states that $M_\sigma^- = -\infty$ so that we only need to verify that $q(x) \le M_\sigma^+$, $\forall x \in \Omega$, which is fulfilled (again Lemma 1) for sufficiently large negative σ. Indeed, we know that $q(x) \le \overline{M}$. It is therefore sufficient to chose σ such that $M_\sigma^+ \ge \overline{M}$.

4.3. Uniqueness theorem

The uniqueness of the solution can, in general, be proved only if the given boundary data is not too large and the pressure is not too high. The construction that we have used to prove the existence of the solution leads to the unique solution of that form. We recall that for such solution the velocity $\mathbf{u}$ and the transformed pressure $q = B(p, \sigma)$ remain inside the ball B_M in X for an appropriate choice of M. However, for general boundary data, we cannot rule out the possibility that there are some other solutions for which $(\mathbf{u}, q) \notin B_M$. For our solution (27) and (28) hold, so that the right hand side in the transformed equation (19) can be made as small as we want. Thus supposing that the problem (19) has two solutions $(\mathbf{u}, q), (\mathbf{w}, \eta)$ in $\mathbf{X}$. We denote by

$$\mathbf{E} = \mathbf{u} - \mathbf{w}\,, \qquad e = q - \eta\,.$$

Then, obviously

$$\mathbf{E} = 0 \;\text{ on }\; \partial\Omega\,. \tag{37}$$

On the other hand, due to (13), (15), for given $\varepsilon > 0$ we can choose σ in definition of b such that

$$|b(q,\sigma)\,\nabla q\, D\,\mathbf{u}|_{L^\beta(\Omega)} \le \frac{\varepsilon}{2\,C_\beta}\,, \quad |b(\eta,\sigma)\,\nabla \eta\, D\,\mathbf{w}|_{L^\beta(\Omega)} \le \frac{\varepsilon}{2\,C_\beta}\,, \tag{38}$$

where $C_\beta > 0$ is the constant from (45) Since the difference $(\mathbf{E}, e)$ satisfies the Stokes system

$$-\Delta \mathbf{E} + \nabla e = b(q,\sigma)\,\nabla q\, D\,\mathbf{u} - b(\eta,\sigma)\,\nabla \eta\, D\,\mathbf{w}$$

and the boundary condition (37), the standard a priori estimate (45) implies

$$|\mathbf{E}|_{W^{2,\beta}(\Omega)} + |e|_{W^{1,\beta}(\Omega)/\mathbf{R}} \le \varepsilon\,.$$

As ε was arbitrary, we conclude that $\mathbf{E} = 0$ and $e = const$. That proves the uniqueness of the velocity since it is independent on choice of σ. On the other hand q and η do depend on σ. However their difference does not since

$$e(x) = q(x) - \eta(x) = \int_{\pi(x)}^{p(x)} \frac{ds}{\mu(s)}\,,$$

with $p = H(q,\sigma)$, $\pi = H(\eta,\sigma)$. So, if we prescribe the mean value of the transformed pressure , i.e. if we put

$$\int_\Omega q = \int_\Omega \eta = 0\,, \tag{39}$$

we have $q = \eta$ and then

$$\int_{\pi(x)}^{p(x)} \frac{ds}{\mu(s)} = 0 \quad \text{for any} \quad x \in \Omega \,.$$

Functions $1/\mu, p, \pi$ are continuous and $1/\mu$ is positive, so that $p(x) = \pi(x)$. Thus, there can be only one solution constructed by transforming the pressure $q = B(p, \sigma)$ and solving (19).

As for the general case, we were only able to prove weaker result.
First of all, we need to assume that μ is increasing, which is physically reasonable. We also require that μ is convex, not only in vicinity of $-\infty$ but also in vicinity of $+\infty$. More precisely, we assume that there exists some $\xi_0 \geq 0$ such that $\xi \mapsto \mu(\xi)$ is convex for $\xi > \xi_0$. Next, suppose that μ is smooth and define

$$\overline{\mu'} = \sup_{\xi \leq \xi_0} |\mu'(\xi)| \,.$$

It is well known (see e.g. Galdi [5]) that, for any $\psi \in L^2(\Omega)$ such that

$$\int_\Omega \psi = 0$$

there exists $\mathbf{z} \in H^1(\Omega)$ such that

$$\begin{cases} \operatorname{div} \mathbf{z} = \psi \\ \mathbf{z} = 0 \ \text{ on } \ \partial\Omega \\ |\mathbf{z}|_{H^1(\Omega)} \leq C_{div} |\psi|_{L^2(\Omega)} \ , \end{cases} \tag{40}$$

with C_{div} depending only on the domain Ω.

We can now formulate the uniqueness result:

THEOREM 2. *Let $\overline{M}$ be defined by (26), (30) and (29). Assume that*

$$\overline{\mu'} \ \overline{M}\, C_{div} \left(1 + \sqrt{\mu(\xi_0)}\right) < 1 \,. \tag{41}$$

Then the problem (3) has only one solution $(\mathbf{u}, p) \in \mathbf{X} = W^{2,\beta}(\Omega)^d \times W^{1,\beta}(\Omega)$ such that

$$p(x) \leq \xi_0 \,, \quad x \in \Omega \,.$$

Proof. First of all, it is sufficient to prove the uniqueness of the pressure, since it implies the uniqueness of the velocity. Indeed, for given pressure the viscosity is given and the system is linear with respect to velocity.

If $p(x) \leq \xi_0$ then

$$\mu(p(x)) \leq \mu(\xi_0) \,, \quad x \in \Omega \,.$$

Let $(w,\pi)\in X$ be the solution constructed in Theorem 1 and let (u,p) be any other solution. Then, subtracting the equations (3) for $(\mathbf{u},p)$, $(\mathbf{w},\pi)$ and testing with $\mathbf{u}-\mathbf{w}$ gives

$$\int_\Omega \mu(p)\,|\,\mathbf{D}\,(\mathbf{u}-\mathbf{w})|^2 = \int_\Omega [\mu(p)-\mu(\pi)]\;\mathbf{D}\,(\mathbf{u}-\mathbf{w})\,\mathbf{D}\,\mathbf{w} \le$$
$$\le \overline{\mu'}\,|\mathbf{D}\,\mathbf{w}|_{L^\infty(\Omega)}\,|\mathbf{D}(\mathbf{u}-\mathbf{w})|_{L^2(\Omega)}\,.$$

Since

$$|\mathbf{D}\,\mathbf{w}|_{L^\infty(\Omega)} \le C(\infty,\beta)|\mathbf{w}|_{W^{2,\beta}(\Omega)} \le \overline{M}$$

we get

$$|\sqrt{\mu(p)}\;\mathbf{D}\,(\mathbf{u}-\mathbf{w})|_{L^2(\Omega)} \le \overline{\mu'}\;\;\overline{M}\,. \tag{42}$$

Now, taking $\mathbf{z}$ such that

$$\operatorname{div}\mathbf{z} = p-\pi \;\text{ in }\;\Omega\,,\quad \mathbf{z}=0 \text{ on } \partial\Omega$$
$$|\mathbf{z}|_{H^1(\Omega)} \le C_{div}\,|p-\pi|_{L^2(\Omega)}\,,$$

and testing (3) with it, we obtain, using (42)

$$\int_\Omega (p-\pi)^2 = \int_\Omega (p-\pi)\operatorname{div}\mathbf{z} = \int_\Omega [\mu(p)-\mu(\pi)]\;\mathbf{D}\,\mathbf{w}\;\mathbf{D}\,\mathbf{z}\,+$$
$$+\int_\Omega \mu(p)\,\mathbf{D}\,(\mathbf{u}-\mathbf{w})\,\mathbf{D}\,\mathbf{z} \le \overline{\mu'}\;\;\overline{M}\,C_{div}\,\left(1+\sqrt{\mu(\xi_0)}\right)\,|p-\pi|^2_{L^2(\Omega)}\,.$$

The result now follows from the assumption (41). □

A. Technical results

It is well known (see e.g. Cattabriga [3]) that the Stokes system

$$-\Delta\mathbf{v}+\nabla\pi=\mathbf{f}\;\;,\;\;\operatorname{div}\mathbf{v}=0\;\;\text{in}\;\;\Omega \tag{43}$$
$$\mathbf{v}=\mathbf{h}\;\;\text{on}\;\;\partial\Omega\,. \tag{44}$$

with $\mathbf{f}\in L^\beta(\Omega)^d$ and $\mathbf{h}\in W^{2-1/\beta,\beta}(\partial\Omega)^d$, satisfying

$$\int_{\partial\Omega}\mathbf{h}\,\cdot\,\mathbf{n}=0\,,$$

admits a solution $(\mathbf{v},\pi)\in\mathbf{X}=W^{2,\beta}(\Omega)^d\,\times\,W^{1,\beta}(\Omega)$ which is unique (π up to an additive constant) . Furthermore it satisfies the following estimate

$$|\mathbf{v}|_{W^{2,\beta}(\Omega)}+|\pi|_{W^{1,\beta}(\partial\Omega)/\mathbf{R}} \le C_\beta\left(|\mathbf{f}|_{L^\beta(\Omega)}+|\mathbf{h}|_{W^{2-1/\beta,\beta}(\partial\Omega)}\right)\,. \tag{45}$$

We recall that for $\beta>n$ the embedding $W^{1,\beta}(\Omega)\subset L^\infty(\Omega)$ holds true. We denote by $C(\beta,\infty)$ the constant such that

$$|\phi|_{L^\infty(\Omega)} \le C(\beta,\infty)\,|\phi|_{W^{1,\beta}(\Omega)}\;\;,\;\forall\,\phi\in W^{1,\beta}(\Omega)\,. \tag{46}$$

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References

[1] C. Barus, *Isotherms, isopiestics and isometrics relative to viscosity*, Amer. J. Sc. **45** (1893), 87–96.

[2] P.W. Bridgman, *The viscosity of liquids under pressure*, Proc. Nat. Acad. Sci. **11** (1925), 603–606

[3] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Semin. Mat. Univ. Padova **31** (1961), 308–340.

[4] G. Delmaz and M. Godet, *Traction, load and film thickness in lightly-loaded lubricated point contacts* , J. Mech. Eng. Sci. **15** (1973) 400–409.

[5] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Steady-state problems, Springer, 1994.

[6] F. Gazzola, *On stationary Navier-Stokes equations with a pressure dependent viscosity*, Rend. Inst. Lomb. Sci. Lett. Sez. A **128** (1994), 107–119.

[7] F. Gazzola, *A note on the evolution Navier-Stokes equations with a pressure dependent viscosity*, Z. Angew. Math. Phys **48** (1997), 760–773.

[8] F.Gazzola and P. Secchi, *Some results about stationary Navier-Stokes equations with a pressure-dependent viscosity*, Proceedings of the International conference on Navier-Stokes equations, Varenna, 1997, Pitman research notes, Math. Ser. 388, 1998, 174–183.

[9] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer, 2001.

[10] E.M. Griest, W. Webb and R.W. Schiessler, *Effect of pressure on viscosity of higher hydrocarbons and their mixtures*, J. Chem. Phys. **29(4)** (1958), 711–720.

[11] J. Hron, J. Málek, J. Nečas and K.R. Rajagopal, *Numerical simulations and global existence of solutions for two-dimensional flows of fluids with shear and pressure dependent viscosities*, Math. Comput. Simulation **61** (2003), 297–315.

[12] J. Hron, J. Málek and K.R. Rajagopal, *Simple flows of fluids with pressure dependent viscosities*, Proc. R. Soc. Lond. A **457** (2001), 1603–1622.

[13] S.C. Jain, R. Sinhasan amd D.V. Singh, *Consideration of viscosity variation in determining the performance characteristics of circular bearings in the laminar and turbulent regimes*, Wear **86** (1983), 233–245.

[14] W.R. Jones, *Pressure viscosity measurement for several lubricants*, ASLE Trans. **18** (1975), 249–262.

[15] A. Kalogirou, S. Polyiadji and G.C. Georgiou, *Incompressible Poiseuille flow of Newtonian liquids with pressure-dependent viscosity*, J. Non-Newtonian Fluid Mech. **166** (2011), 413–419.

[16] J. Málek , J. Nečas and K.R. Rajagopal, *Global existence of solutions for flows fluids with pressure and shear dependent viscosities*, Appl. Math. Lett. **15** (2002), 961–967.

[17] E. Marušić-Paloka and S. Marušić, *Analysis of the Reynolds Equation for Lubrication in Case of Pressure-Dependent Viscosity*, Math. Probl. Eng. bf 2013 (2013) , 1–10.

[18] E. Marušić-Paloka and I. Pažanin, *A note on the pipe flow with a pressure-dependent viscosity*, J. Non-Newtonian Fluid Mech. **197** (2013) , 5–10.

[19] K.R. Rajagopal and A.Z. Szeri, *On an inconsistency in the derivation of the equations of hydrodynamic lubrication*, Proc. R. Soc. Lond. A **459** (2003), 2771–2786.

[20] M. Renardy, *Some remarks on the Navier-Stokes equations with pressure-dependent viscosity*, Comm. Part. Diff. Eq. **11(7)** (1986), 779–793.

[21] C.J.A. Roelands, *Correlation aspects of the viscosity-pressure relationship of lubricating oils*, PhD thesis, Delft University of technology, Netherlands, 1966.

[22] G.G. Stokes, *Notes on hydrodynamics. On the dynamical equations.* Cambridge and Dublin Mathematical Journal, III, (1848) 121–127.

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