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Compact groups with a dense free abelian subgroup[1]

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Abstract. *The compact groups having a dense infinite cyclic subgroup (known as monothetic compact groups) have been studied by many authors for their relevance and nice applications. In this paper we describe in full details the compact groups K with a dense free abelian subgroup F and we describe the minimum rank $r_t(K)$ of such a subgroup F of K. Surprisingly, it is either finite or coincides with the density character $d(K)$ of K.*

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1. Introduction

Dense subgroups (with some additional properties) of compact groups have been largely studied for instance in [2, 3, 4, 7, 25]. Moreover, large independent families of dense pseudocompact subgroups of compact connected groups are built in [23], while potential density is studied in [13, 14, 15].

This note is dedicated mainly to the study of the class $\mathcal{F}$ of those Hausdorff topological groups that have a dense free abelian subgroup. These groups are necessarily abelian, so in this paper we are concerned exclusively with Hausdorff topological abelian groups, and we always use the additive notation. Moreover, we mainly consider the groups K in $\mathcal{F}$ that are also compact. The choice of compactness is motivated by the fact that many non-discrete topological abelian groups possess no proper dense subgroups at all (see [25] for a locally compact abelian group with this property), whereas every infinite compact group K admits proper dense subgroups. Indeed, it is known that $d(K) < |K|$, where $d(K)$ denotes the density character of K (i.e., the minimum cardinality of a dense subgroup of K). We recall also that

$$d(K) = \log w(K);$$

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here as usual $w(G)$ denotes the weight of a topological abelian group G, and for an infinite cardinal κ we let $\log \kappa = \min\{\lambda : 2^\lambda \geq \kappa\}$ the logarithm of κ.

Let us start the discussion on the class $\mathcal{F}$ from a different point of view that requires also some historical background.

The *topological generators* of a topological (abelian) group G have been largely studied by many authors in [1, 8, 10, 12, 16, 17, 18, 19]; these are the elements of subsets X of G generating a dense subgroup of G. In this case we say that G is *topologically generated* by X. In particular, a topological group having a finite topologically generating set X is called *topologically finitely generated* (*topologically s-generated*, whenever X has at most s element).

Usually, various other restraints apart finiteness have been imposed on the set X of topological generators. These restraints are mainly of topological nature and we collect some of them in the next example.

EXAMPLE 1.1. *In the sequel G is a topological group with neutral element e_G and X is a topologically generating set of G.*

(a) *When X is compact, G is called* compactly generated. *This provides a special well studied subclass of the class of σ-compact groups.*

(b) *The set X is called a* suitable set *for G if $X \setminus \{e_G\}$ is discrete and closed in $G \setminus \{e_G\}$. This notion was introduced in 1990 by Hofmann and Morris. They proved that every locally compact group has a suitable set in [21] and dedicated the entire last chapter of the monograph [22] to the study of the minimum size $s(G)$ of a suitable set of a compact group G. Properties and existence of suitable sets are studied also in the papers [8, 10, 12, 16, 17, 18, 19].*

   Clearly, a finite topologically generating set X is always suitable.

(c) *The set X is called* totally suitable *if X is suitable and generates a totally dense subgroup of G [19]. The locally compact groups that admit a totally suitable set are studied in [1, 19].*

(d) *The set X is called a* supersequence *if $X \cup \{e_G\}$ is compact, so coincides with the one-point compactification of the discrete set $X \setminus \{e_G\}$. Any infinite suitable set X in a countably compact group G is a supersequence converging to e_G [17]. This case is studied in detail in [12, 16, 26].*

Now, with the condition $G \in \mathcal{F}$ we impose a purely algebraic condition on the topologically generating set X of the topological abelian group G. Indeed, clearly a topological abelian group G belongs to $\mathcal{F}$ precisely when G has a topologically generating set X that is *independent*, i.e., X generates a dense free abelian subgroup of G. In case G is discrete, the *free rank* $r(G)$ of G is the maximum cardinality of an independent subset of G; in this paper we call it simply rank. Imitating the discrete case, one may introduce the following invariant measuring the minimum cardinality of an independent generating set X of G.

Definition 1.2. *For a topological abelian group $G \in \mathcal{F}$, the* topological free rank *of G is*

$$r_t(G) = \min\{r(F) : F \text{ dense free abelian subgroup of } G\}.$$

Let

$$\mathcal{F}_{\text{fin}} = \{G \in \mathcal{F} : r_t(G) < \infty\}.$$

Obviously, $d(G) \leq r_t(G)$ whenever $G \in \mathcal{F} \setminus \mathcal{F}_{\text{fin}}$, whereas always

$$d(G) \leq r_t(G) \cdot \omega \tag{1}$$

holds true.

We describe the compact abelian groups K in $\mathcal{F}$ in two steps, depending on whether the topological free rank $r_t(K)$ of K is finite.

We start with the subclass $\mathcal{F}_{\text{fin}}$, i.e., with the compact abelian groups K having a dense free abelian subgroup of finite rank. The complete characterization of this case is given in the next Theorem A, proved in Section 3. We recall that the case of rank one is that of monothetic compact groups, and the characterization of monothetic compact groups is well known (see [20]). Moreover, every totally disconnected monothetic compact group is a quotient of the universal totally disconnected monothetic compact group $M = \prod_p \mathbb{J}_p$, where $\mathbb{J}_p$ denotes the p-adic integers. Furthermore, an arbitrary monothetic compact group is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M$ (see Proposition 3.4 below and [20, Section 25]).

For a prime p and an abelian group G, the p-socle of G is $\{x \in G : px = 0\}$, which is a vector space over the field $\mathbb{F}_p$ with p many elements; the *p-rank* $r_p(G)$ of G is the dimension of the p-socle over $\mathbb{F}_p$. Moreover, we give the following

Definition 1.3. *Let G be an abelian group. For p a prime, set*

$$\rho_p(G) = r_p(G/pG).$$

Moreover, let $\rho(G) = \sup_p \rho_p(G)$.

Several properties of the invariant ρ_p for compact abelian groups are given in Section 3.

In the next theorem we denote by $c(K)$ the connected component of the compact abelian group K.

Theorem A. *Let K be an infinite compact abelian group and $n \in \mathbb{N}_+$. Then the following conditions are equivalent:*

(a) $K \in \mathcal{F}$ *and* $r_t(K) \leq n$*;*

(b) $w(K) \leq \mathfrak{c}$ *and* $K/c(K)$ *is a quotient of* M^n*;*

(c) $w(K) \leq \mathfrak{c}$ *and* $\rho(K) \leq n$ *(i.e.,* $\rho_p(K) \leq n$ *for every prime* p*);*

(d) K *is a quotient of* $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M^n$*.*

The case of infinite rank is settled by the next theorem characterizing the compact abelian groups that admit a dense free abelian subgroup, by making use of dense embeddings in some power of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (endowed with the quotient topology inherited from $\mathbb{R}$). Here $\widehat{K}$ denotes the Pontryagin dual of the compact abelian group K; in this case $\widehat{K}$ is a discrete abelian group, and

$$w(K) = |\widehat{K}|.$$

Note that, if K is an infinite compact abelian group and $K \in \mathcal{F}$, then there exists a dense free abelian subgroup of K of infinite rank, as $r(K) \geq \mathfrak{c}$.

Theorem B. *Let* K *be an infinite compact abelian group and* κ *an infinite cardinal. Then the following conditions are equivalent:*

(a) $K \in \mathcal{F}$ *and* $r_t(K) \leq \kappa$*;*

(b) $\widehat{K}$ *admits a dense embedding in* $\mathbb{T}^\lambda$ *for some* $\lambda \leq \kappa$*;*

(c) $d(K) \leq r(K)$ *and* $d(K) \leq \kappa$*.*

Theorem B has as an easy consequence the next characterization. To prove the second part of the corollary take $\kappa = d(K)$ in item (c) of Theorem B.

Corollary 1. *Let* K *be an infinite compact abelian group. Then* $K \in \mathcal{F}$ *if and only if* $d(K) \leq r(K)$*. In this case* K *has a dense free abelian subgroup of rank* $d(K)$*.*

According to Corollary 1, a compact abelian group K admits a dense free abelian subgroup of rank exactly $d(K)$. Hence, we see now in Corollary 2 that the inequality in (1) becomes an equality in case K is compact. Furthermore, as a consequence of Theorem A and Theorem B respectively, we can see that $r_t(K)$ is equal either to $\rho(K)$ or to $d(K)$ depending on its finiteness or infiniteness respectively.

Corollary 2. *Let* K *be an infinite compact abelian group. If* $K \in \mathcal{F}$*, then*

$$d(K) = r_t(K) \cdot \omega.$$

Moreover,

$$r_t(K) = \begin{cases} \rho(K) & \text{if } K \in \mathcal{F}_{\text{fin}}, \\ d(K) & \text{if } K \in \mathcal{F} \setminus \mathcal{F}_{\text{fin}}. \end{cases}$$

Roughly speaking, if K is an infinite compact abelian group in $\mathcal{F}$, Theorem B asserts that $d(K) \leq r(K)$. Moreover, if F is a dense free abelian subgroup of K of infinite rank, then $r(F)$ can range between $d(K)$ and $r(K)$. We underline that the maximum $r(K)$ can be reached by $r(F)$ since $r(K) \geq \mathfrak{c}$, and that also the minimum is a possible value of $r(F)$ by the equality in Corollary 2.

The proof of Theorem B is given in Section 4. It makes use of the following concepts introduced and studied in [9]; as usual, for an abelian group G we denote $mG = \{mx : x \in G\}$ and $G[m] = \{x \in G : mx = 0\}$ for $m \in \mathbb{N}_+$, where $\mathbb{N}_+$ denotes the set of positive natural numbers.

Definition 1.4. [9] *Let G be a topological abelian group.*

(i) The group G is w-divisible *if $w(mG) = w(G) \geq \omega$ for every $m \in \mathbb{N}_+$.*

(ii) The divisible weight *of G is $w_d(G) = \inf\{w(mG) : m \in \mathbb{N}_+\}$.*

This definition is different from the original definition from [9], where instead of $w(mG) = w(G) \geq \omega$ one imposes the stronger condition $w(mG) = w(G) > \omega$, which rules out all second countable groups. Since this is somewhat restrictive from the point of view of the current paper, we adopt this slight modification here.

So an infinite topological abelian group G is w-divisible if and only if $w(G) = w_d(G)$. In particular, an infinite discrete abelian group G is w-divisible if and only if $|mG| = |G|$ for every $m \in \mathbb{N}_+$. Moreover, it is worth to note here that every infinite monothetic group is w-divisible.

Another consequence of Theorem B is that the class $\mathcal{F}$ contains all w-divisible compact abelian groups, so in particular all connected and all torsion-free compact abelian groups:

Corollary 3. *If K is a w-divisible compact abelian group, then $K \in \mathcal{F}$.*

2. Some general properties of the class $\mathcal{F}$

The next lemma ensuring density of subgroups is frequently used in the sequel.

Lemma 2.1. [4] *Let G be a topological group and let N be a quotient of G with $h : G \twoheadrightarrow N$ the canonical projection. If H is a subgroup of G such that $h(H)$ is dense in N and H contains a dense subgroup of* $\ker h$*, then H is dense in G.*

In the next result we give two stability properties of the class $\mathcal{F}$.

Proposition 2.2. *The class $\mathcal{F}$ is stable under taking:*

(a) arbitrary direct products;

(b) extensions.

Proof. (a) Assume $G_i \in \mathcal{F}$ for all $i \in I$. Let F_i be a dense free abelian subgroup of G_i for every $i \in I$. Then the direct sum $F = \bigoplus_{i\in I} F_i$ is a dense free abelian subgroup of $\prod_{i\in I} G_i$.

(b) Assume that G is a topological abelian group with a closed subgroup $H \in \mathcal{F}$ such that also $G/H \in \mathcal{F}$. Let F and F_1 be dense free abelian subgroups of H and G/H, respectively. Let $q : G \to G/H$ be the canonical projection. Since q is surjective, we can find a subset X of G such that $q(X)$ is an independent subset of G/H generating F_1 as a free set of generators. Then X is independent, so generates a free abelian subgroup F_2 of G and the restriction $q\restriction_{F_2}: F_2 \to F_1$ is an isomorphism. In particular, $F_2 \cap \ker q = 0$, so $F_2 \cap F = 0$ as well. Therefore, $F_3 = F + F_2$ is a free abelian subgroup of G. Moreover, F_3 contains the dense subgroup $F_1 \subseteq H = \ker q$ and $q(F_3) = q(F_2) = F_1$ is a dense subgroup of G/H. Therefore, F_3 is a dense subgroup of G by Lemma 2.1. □

The next claim is used essentially in the proof of Proposition 2.4, which solves one of the implications of Theorem A.

Lemma 2.3. *Let G be an abelian group, K a subgroup of G of infinite rank $r(K)$ and let F be a finitely generated subgroup of G. If $s \in \mathbb{N}_+$ and H is an s-generated subgroup of G, then there exists a free abelian subgroup F_1 of rank s of G, such that $F_1 \cap F = 0$ and $H \subseteq K + F_1$.*

Proof. Since $F + H$ is a finitely generated subgroup of G, the intersection $N = K \cap (F + H)$ is a finitely generated subgroup of K. Therefore, the rank $r(K/N)$ is still infinite. In particular, there exists a free abelian subgroup F_2 of rank s of K, such that $F_2 \cap N = 0$. Then also

$$F_2 \cap (F + H) = 0. \tag{2}$$

Let $x_1, \dots, x_s$ be the generators of H and let $t_1, \dots, t_s \in K$ be the free generators of F_2. Let $z_i = x_i + t_i$ for $i = 1, \dots, s$ and $F_1 = \langle z_1, \dots, z_s\rangle$. Then, obviously $H \subseteq K + F_1$ as $x_i = t_i - z_i \in K + F_1$.

The subgroup F_1 is free since any linear combination $k_1z_1 + \ldots + k_sz_s = 0$ produces an equality $k_1x_1 + \ldots + k_sx_s = -(k_1t_1 + \ldots + k_st_s) \in F_2$. Since $k_1x_1 + \ldots + k_sx_s \in H$, from (2) one can deduce that $k_1t_1 + \ldots + k_st_s = 0$. Since $t_1, \dots, t_s$ are independent, this gives $k_1 = \ldots = k_s = 0$. This concludes the proof that F_1 is free.

It remains to prove that $F_1 \cap F = 0$. So let $x \in F_1 \cap F$ and let us verify that necessarily $x = 0$. Since $x \in F_1$, there exist some integers $a_1, \dots, a_s$ such that $x = a_1z_1 + \ldots + a_sz_s = (a_1x_1 + \ldots + a_sx_s) + (a_1t_1 + \ldots + a_st_s)$. Then $a_1t_1 + \ldots + a_st_s \in F_2 \cap (F + H)$ and this intersection is trivial by (2). By the independence of $t_1, \ldots, t_s$ we have $a_1 = \ldots = a_s = 0$ and hence $x = 0$ as desired. $\square$

Proposition 2.4. *Let G be a topological abelian group and K a closed subgroup of G with $r(K) \geq \omega$. If K contains a dense free abelian subgroup F of rank $m \in \mathbb{N}_+$ and G/K is topologically s-generated for some $s \in \mathbb{N}_+$, then G admits a dense free abelian subgroup E of rank $m+s$ (i.e., $G \in \mathcal{F}$ and $r_t(G) \leq m+s$).*

Proof. Let $q : G \to G/K$ be the canonical projection and consider G/K endowed with the quotient topology of the topology of G. Let Y be a subset of size s of G/K generating a dense subgroup of G/K. Pick a subset X of size s of G such that $q(X) = Y$ and let $H = \langle X \rangle$. By Lemma 2.3 there exists a free abelian subgroup F_1 of rank s of G such that $F_1 \cap F = 0$ and $H \subseteq K + F_1$. Hence, $q(F_1)$ is a dense subgroup of G/K as it contains $q(H) = \langle Y \rangle$. Let $E = F + F_1$. Then E is a free abelian subgroup of G of rank $m + s$. Moreover, E contains a dense subgroup of $K = \ker q$ and $q(E) = q(F_1)$ is dense in G/K. Therefore, E is dense in G by virtue of Lemma 2.1. $\square$

Corollary 2.5. *If G is a topologically finitely generated abelian group with infinite rank $r(G)$, then $G \in \mathcal{F}_{\mathrm{fin}}$.*

Proof. Let H be the finitely generated dense subgroup of G. Then $H = L \times F$, where L is a free abelian subgroup of finite rank and F is a finite subgroup. Then the closure K of L in G is a closed finite index subgroup of G, so K is open too. Since K has finite index in G, its rank $r(K)$ is infinite. By Proposition 2.4, G contains a dense free abelian subgroup F of finite rank. $\square$

In particular, topologically finitely generated compact abelian groups belong to $\mathcal{F}$. The same holds relaxing compactness to pseudocompactness since non torsion pseudocompact abelian groups have rank at least $\mathfrak{c}$ as proved in [6].

We recall the following result stated in [11] giving, for an infinite compact abelian group K, several equivalent conditions characterizing the density character $d(K)$ of K.

Proposition 2.6. [11, Exercise 3.8.25] *Let K be an infinite compact abelian group and κ an infinite cardinal. Then the following conditions are equivalent:*

(a) $d(K) \leq \kappa$;

(b) there exists a homomorphism $f : \bigoplus_\kappa \mathbb{Z} \to K$ *with dense image;*

(c) there exists an injective homomorphism $\widehat{K} \to \mathbb{T}^\kappa$;

(d) $|\widehat{K}| \leq 2^\kappa$;

(e) $w(K) \leq 2^\kappa$*;*

(f) there exists a continuous surjective homomorphism $(M \times \widehat{\mathbb{Q}})^{2^\kappa} \to K$*.*

Proof. (a)⇒(b) Since $d(K) \leq \kappa$ and it is infinite, then there exists a dense subgroup D of G with $|D| \leq \kappa$, so in particular there exists a homomorphism $f : \bigoplus_\kappa \mathbb{Z} \to D$, which has dense image in K.

(b)⇒(a) is obvious and (b)⇔(c) is an easy application of Pontryagin duality.

(c)⇔(d) follows from the fact that $\mathbb{T}^\kappa$ is divisible with $r(\mathbb{T}^\kappa) = r_p(\mathbb{T}^\kappa) = 2^\kappa$ for every prime p.

(d)⇔(e) follows from the fact that $w(K) = |\widehat{K}|$.

(c)⇔(f) follows from the fact that as a discrete abelian group $\mathbb{T}^\kappa$ is isomorphic to $\bigoplus_{2^\kappa} \mathbb{Q} \oplus \bigoplus_{2^\kappa} \mathbb{Z}(p^\infty) = (\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})^{2^\kappa}$. □

Remark 2.7. *As a by-product of this proposition we show an easy argument for the well known equality* $d(K) = \log w(K)$ *for a compact group* K *in case* K *is* abelian *(the argument in the non-abelian case makes use of the highly non-trivial fact of the dyadicity of the compact groups). The inequality* $\log w(K) \leq d(K)$ *follows from the well known fact that* $w(K) \leq 2^{d(K)}$*. Since* $w(K) \leq 2^{\log w(K)}$ *obviously holds by the definition of* $\log$*, the equivalence of (a) and (e) from above proposition, applied to* $\kappa = \log w(K)$ *gives the desired inequality* $d(K) \leq \log w(K)$*.*

The equivalent conditions of Proposition 2.6 appear to be weaker than those of Theorem B (see also Lemma 4.2). On the other hand, these conditions become equivalent to those of Theorem B assuming the infinite compact abelian group K to be in $\mathcal{F}$; indeed, the point is that $K \in \mathcal{F}$ is equivalent to $d(K) \leq r(K)$ by Corollary 1 in the Introduction.

3. Compact abelian groups with dense free subgroups of finite rank

In the next lemma we give a computation of the value of the invariant ρ_p of a compact abelian group K in terms of the p-rank of the discrete dual group $\widehat{K}$.

Lemma 3.1. *For a prime* p *and a compact abelian group* K*, we have that*

$$\rho_p(K) = \begin{cases} r_p(\widehat{K}) & \text{if } \rho_p(K) \text{ is finite,} \\ 2^{r_p(\widehat{K})} & \text{if } \rho_p(K) \text{ is infinite.} \end{cases}$$

Proof. Let $G = \widehat{K}$. Then $K/pK \cong \widehat{G[p]}$. If K/pK is finite then $K/pK \cong G[p]$ and so $r_p(K/pK) = r_p(G[p])$. Assume now that K/pK is infinite; therefore $G[p]$ is infinite as well. So $G[p] \cong \bigoplus_{r_p(G[p])} \mathbb{Z}(p)$, consequently $K/pK \cong \mathbb{Z}(p)^{r_p(G[p])}$ and hence $r_p(K/pK) = 2^{r_p(G[p])}$. □

Since pK contains the connected component $c(K)$, which is divisible, K/pK is a quotient of $K/c(K)$ and it is worth to compare their p-ranks. Note that in general $\rho_p(K) = r_p(K/pK)$ does not coincide with $r_p(K/c(K))$ for a compact abelian group K; indeed, take for example $K = \mathbb{J}_p$. On the other hand, one can easily prove the following properties of the invariant ρ_p for compact abelian groups.

Lemma 3.2. *Let p be a prime and K be a compact abelian group. Then:*

(a) $\rho_p(K) \geq \rho_p(K_1)$ if K_1 is a quotient of K;

(b) $\rho_p(K) = \rho_p(K/c(K))$;

(c) $\rho_p(K^n) = n\rho_p(K)$.

The next lemma proves in particular the equivalence of conditions (b), (c) and (d) in Theorem A of the Introduction.

Lemma 3.3. *Let K be an infinite compact abelian group and $n \in \mathbb{N}_+$. Then the following conditions are equivalent:*

(a) $w(K) \leq \mathfrak{c}$ and $K/c(K)$ is a quotient of M^n;

(b) $w(K) \leq \mathfrak{c}$ and $\rho(K) \leq n$ (i.e., $\rho_p(K) \leq n$ for every prime p);

(c) K is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M^n$.

Proof. (a)⇒(b) By Lemma 3.2 we have $\rho_p(K) = \rho_p(K/c(K)) \leq \rho_p(M^n) = n\rho_p(M) = n$ for every prime p.

(b)⇒(c) Let $G = \widehat{K}$ be the discrete dual of K and denote by $D(G)$ the divisible hull of G. Since $|G| = w(K) \leq \mathfrak{c}$ and $\widehat{c(K)} \cong G/t(G)$, in particular $r(D(G)) = r(G) = r(G/t(G)) \leq \mathfrak{c}$. On the other hand, for every prime p, the p-rank of $D(G)$ is $r_p(D(G)) = r_p(G) = r(G[p]) \leq n$ by Lemma 3.1. Then $D(G)$ is contained in $\bigoplus_{\mathfrak{c}} \mathbb{Q} \times (\mathbb{Q}/\mathbb{Z})^n$, hence by Pontryagin duality we have the condition in (c).

(c)⇒(a) Since the weight is monotone under taking quotients, we have $w(K) \leq \mathfrak{c}$. By hypothesis we have that K is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times (\widehat{\mathbb{Q}/\mathbb{Z}})^n$, so by Pontryagin duality G admits an injective homomorphism in $\bigoplus_{\mathfrak{c}} \mathbb{Q} \oplus (\mathbb{Q}/\mathbb{Z})^n$ and in particular $t(G)$ is contained in the subgroup $(\mathbb{Q}/\mathbb{Z})^n$. Applying again Pontryagin duality we conclude that $K/c(K) \cong \widehat{t(G)}$ is a quotient of $(\widehat{\mathbb{Q}/\mathbb{Z}})^n \cong M^n$. □

The following result on monothetic groups is known, we give it here as a consequence of the previous lemma noting that an infinite quotient of a monothetic group is monothetic as well.

PROPOSITION 3.4. *Let K be an infinite compact abelian group. If $w(K) \leq \mathfrak{c}$ and $\rho_p(K) \leq 1$ for every prime p, then K is monothetic.*

Proof. By Lemma 3.3 the group K is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M$, which is monothetic. □

We are now in position to prove the next characterization of compact abelian groups admitting a dense free abelian subgroup of finite rank, i.e., with finite topological free rank. Along with Lemma 3.3, this concludes the proof of Theorem A of the Introduction.

THEOREM 3.5. *Let K be an infinite compact abelian group and $n \in \mathbb{N}_+$. Then the following conditions are equivalent:*

(a) $K \in \mathcal{F}$ and $r_t(K) \leq n$;

(b) $w(K) \leq \mathfrak{c}$ and $\rho(K) \leq n$ (i.e., $\rho_p(K) \leq n$ for every prime p);

Proof. (a)⇒(b) By hypothesis in particular $d(K) \leq \omega$, so $w(K) \leq \mathfrak{c}$ as $d(K) = \log w(K)$ (see Remark 2.7).

Let p be a prime. Let $q : K \to K/pK$ be the canonical projection and let F be a dense free abelian subgroup of K of finite rank $r(F) \leq n$. Since $pK \cap F \supseteq pF$ and $\ker q = pK$, we have that $q(F) \cong F/\ker q$ is finite, being a quotient of the finite group F/pF. Now the density of F in K implies the density of the finite subgroup $q(F)$ in K/pK, which has exponent p. Therefore $K/pK = q(F)$ has at most n many generators, in other words $r_p(K/pK) \leq n$. This proves $\rho_p(K) \leq n$.

(b)⇒(a) By Lemma 3.3 we know that $K/c(K)$ is a quotient of $M^n \cong \prod_p \mathbb{J}_p^n$. In particular this implies that $K/c(K)$ is a product of at most n monothetic subgroups. Indeed, for a prime p, a quotient of $\mathbb{J}_p^n$ is always of the form $C_{p,1} \times \ldots \times C_{p,n}$ where $C_{p,i}$ is either $\mathbb{J}_p$ or a cyclic p-group for every $i = 1, \ldots, n$, or 0. Therefore, one can write $K/c(K) = M_1 \times \ldots \times M_n$, letting $M_i = \prod_p C_{p,i}$ for every $i = 1, \ldots, n$; observe that M_i is monothetic for every $i = 1, \ldots, n$.

Let now $q : K \to K/c(K)$ be the canonical projection and $K_1 = q^{-1}(M_1)$. Then $c(K_1) = c(K)$ and $K_1/c(K) \cong M_1$ is monothetic, so K_1 is monothetic as well by Proposition 3.4. Since $K/K_1 \cong M_2 \times \ldots \times M_n$ is topologically $(n-1)$-generated, Proposition 2.4 implies that K contains a dense free abelian subgroup of rank $\leq n$. □

One can easily see that if K is a totally disconnected compact abelian group with a dense finitely generated subgroup, then $K = M_1 \times \ldots \times M_n$, where M_i are compact monothetic groups (see the proof of the implication (b)⇒(a) in Theorem 3.5). We do not know whether this factorization in direct product of compact monothetic groups remains true without the additional restraint of total disconnectedness.

4. Compact abelian groups with infinite topological free rank

We recall the following known result in terms of the divisible weight.

THEOREM 4.1. [14, Theorem 2.6] *Let κ be a cardinal. A discrete abelian group G admits a dense embedding into $\mathbb{T}^\kappa$ if and only if $|G| \leq 2^\kappa$ and $\log \kappa \leq w_d(G)$.*

Recall that the bimorphisms (i.e., monomorphisms that are also epimorphisms) in the category $\mathcal{L}$ of LCA groups are precisely the continuous injective homomorphisms with dense image. Therefore, applying the Pontryagin duality functor $\widehat{\ }\colon \mathcal{L} \to \mathcal{L}$, we deduce that for a cardinal κ the following conditions are equivalent:

(a) there exists a bimorphism $\bigoplus_\kappa \mathbb{Z} \to K$;

(b) there exists a bimorphism $\widehat{K} \to \mathbb{T}^\kappa$.

In equivalent terms:

LEMMA 4.2. *Let K be a compact abelian group and let κ be a cardinal. Then the following conditions are equivalent:*

(a) K admits a dense free abelian subgroup of rank κ (in particular, $K \in \mathcal{F}$);

(b) $\widehat{K}$ admits a dense embedding in $\mathbb{T}^\kappa$.

The following easy relation between the divisible weight of a compact abelian group and that of its discrete Pontryagin dual group was already observed in [9].

LEMMA 4.3. *Let K be a compact abelian group. Then $w_d(K) = w_d(\widehat{K})$.*

Consequently, K is w-divisible if and only if $\widehat{K}$ is w-divisible.

Proof. Let $G = \widehat{K}$. Since $nG \cong \widehat{nK}$ for every $n \in \mathbb{N}_+$, one has $|nG| = w(nK)$, hence the conclusion follows. $\square$

We recall now a fundamental relation given in [9] between the divisible weight and the rank of a compact abelian group. It is worth to note that the rank is a purely algebraic invariant, while the divisible weight is a topological one.

THEOREM 4.4. [9, Corollary 3.9] *Let K be a compact abelian group. Then $r(K) = 2^{w_d(K)}$.*

Applying these observations we can give now the proof of Theorem B. Note that in the proof of the implication (c)⇒(b) we apply both Theorem 4.4 and Theorem 4.1.

Proof of Theorem B. We have to prove that if K is an infinite compact abelian group and κ is an infinite cardinal, then the following conditions are equivalent:

(a) $K \in \mathcal{F}$ and $r_t(K) \leq \kappa$;

(b) $\widehat{K}$ admits a dense embedding in $\mathbb{T}^\lambda$ for some $\lambda \leq \kappa$;

(c) $d(K) \leq r(K)$ and $d(K) \leq \kappa$.

The equivalence (a)$\Leftrightarrow$(b) is contained in Lemma 4.2 and (a)$\Rightarrow$(c) is clear.

(c)$\Rightarrow$(b) Let $G = \widehat{K}$. Put $\lambda = d(K)$. Since $d(K) = \log w(K)$ and $w(K) = |G|$, we have that $\lambda = \log |G|$, and so $2^\lambda \geq |G|$. On the other hand,

$$\log \lambda = \log \log |G| = \log d(K) \leq \log r(K) \leq w_d(K),$$

where the last inequality follows from Theorem 4.4. So $\log \lambda \leq w_d(G)$ by Lemma 4.3, and Theorem 4.1 guarantees that G admits a dense embedding into the power $\mathbb{T}^\lambda$. $\square$

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