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Recent progress on characterizing lattices $C(X)$ and $U(Y)$[1]

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Abstract. *Our effort to weaken algebraic assumptions led us to obtain characterizations of $C(X)$ as Riesz spaces, real ℓ-groups, semi-affine lattices and real lattices by using different techniques. We present a unified approach valid for any "convenient" category. By setting equivalent conditions to equi-uniform continuity, we obtain a characterization of the lattice $U(Y)$ in parallel with that of $C(X)$.*

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1. Introduction

In the early forties and starting mainly by Yosida [20], the topology community was very interested in obtaining internal conditions under which an object is isomorphic to the set $C(X)$ of all the real valued continuous functions on some topological space X.

The problem essentially depends on the algebraic structure in which we are interested, the weaker assumption the more difficult the answer. Whenever $X \neq \varnothing$, the set $C(X)$ endowed with its pointwise defined order becomes a distributive lattice containing all the constant functions into $\mathbb{R}$, thus a copy of $\mathbb{R}$ as a sublattice. Henceforth, our basic starting structure on $C(X)$ will be that of the real lattices (Definition 2.1).

At the crux of most attempts the following conditions on a real lattice L somehow are needed: (a) L embeds into some $C(X)$ and (b) the lattice of bounded elements L^* is isomorphic to $C^*(X)$. Without loosing generality we may assume that X is a Tychonoff space and even realcompact (since $C(X)$ is lattice-ordered algebra unit preserving isomorphic to $C(\upsilon X)$).

The only contribution appearing in the literature for the more general case is that of Jensen [15] as a refinement of that of Anderson [1], but by assuming

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richer compatible algebraic structures, namely for Φ-algebras. However, no effort was made to extend these results to more general situations.

Under this general context, Birkhoff [4] proposed explicitly in his venerable *Lattice Theory* the open problem 81 by asking for an internal characterization of $C(X)$ with X a compact Hausdorff space only as a lattice. The problem was solved by several authors, by making a special emphasis in the Anderson-Blair's solution [2]. Supported by this outstanding result (Lemma 4.3), verification of above condition (b) can be done by using the Urysohn's method on constructing a separating function (Definition 4.11), and embedding condition (a) can be established in any convenient subcategory of the real lattices, where among other requirements, morphisms should be defined by means of operations (Definition 2.3).

Still some conditions are needed to complete the characterization. We shall present different approaches to this aim, namely: 2-universal completeness (Definition 5.2), local uniform completeness (Definition 5.4) and pointwise completeness (Definition 5.8).

Similarly to $C(X)$, an internal characterization of the real lattice $U(Y)$ of real uniformly continuous functions on some uniform space Y will be obtained by determining equi-uniformly continuous sequences (Definition 4.19) and by setting equi-uniform completeness (Definition 5.10).

This paper has a survey character aiming recent contributions by the authors to the problem. All the technical proofs are avoided refering the readers to their respective original sources.

2. Representation in convenient categories

We start denoting by $\boldsymbol{T}$ the category of the topological Hausdorff spaces with their continuous maps $\mathrm{Hom}_{\boldsymbol{T}}$, and by $\boldsymbol{U}$ the category of Hausdorff uniform spaces with their uniformly continuous maps $\mathrm{Hom}_{\boldsymbol{U}}$.

As usual, $C(X) = \mathrm{Hom}_{\boldsymbol{T}}(X, \mathbb{R})$ and $U(Y) = \mathrm{Hom}_{\boldsymbol{U}}(Y, \mathbb{R}) \subseteq C(Y)$ are the sets of real continuous functions on $X \in \boldsymbol{T}$ and real uniformly continuous functions on $Y \in \boldsymbol{U}$ respectively. Our basic structure both on $C(X)$ and $U(Y)$ is that of a distributive lattice by assuming its pointwise defined order relationship:

$$f \leq g \text{ iff } f(x) \leq g(x), \text{ for all } x \in X \text{ or } Y \quad (f, g \in C(X) \text{ or } U(Y)).$$

Notice that whenever $X \neq \varnothing$, the set $\mathbb{R}$ of constant functions becomes a sublattice of $C(X)$. This requirement can be stated in terms of the lattice structure since every densely-ordered countable chain of a distributive lattice is isomorphic to the chain $\mathbb{Q}$ of the rational numbers (Birkhoff [4]).

Definition 2.1. *A real lattice is a distributive lattice containing the conditional completion R of a fixed densely-ordered countable chain by removing the first and the last element.*

In the sequel, we shall make no distinction between R and the chain $\mathbb{R}$ of the real numbers. We are denoting by $\boldsymbol{L}$ the category of the real lattices with their lattice homomorphisms $\mathrm{Hom}_{\boldsymbol{L}}$ being identity on $\mathbb{R}$.

One of the stronger reasons of starting with a real lattice L is that we may work with its real sublattice

$$L^* = \{f \in L : r \leq f \leq s \text{ for some } r, s \in \mathbb{R}\} \in \boldsymbol{L}$$

of *bounded elements*.

On the other hand, morphisms in $\boldsymbol{L}$ are defined by means of "operations". Let us formally generalize this framework: A *signature* is a nonempty set O endowed with a mapping $a : O \to \mathbb{Z}_+$ called *arity*.

Every signature O defines a category $\boldsymbol{O}$ called *universal algebra* whose objects L satisfy that for any $o \in O$, there are subsets $L_1^o, \ldots, L_{a(o)}^o \subseteq L$ and a mapping

$$o_L : L_1^o \times \cdots \times L_{a(o)}^o \to L, \quad (f_1, \ldots, f_{a(o)}) \mapsto o_L(f_1, \ldots, f_{a(o)})$$

called *$a(o)$-ary operation*.

Their homomorphisms $\mathrm{Hom}_{\boldsymbol{O}}$ are the mappings $x : L \to C$ preserving operations:

$$x\left(o_L(f_1, \ldots, f_{a(o)})\right) = o_C\left(x(f_1), \ldots, x(f_{a(o)})\right),$$

for every $o \in O$, $f_1 \in L_1^o, \ldots, f_{a(o)} \in L_{a(o)}^o$.

In the sequel the inclusion symbol among categories refers whenever to be a subcategory (for instance, $\boldsymbol{C} \subseteq \boldsymbol{L}$ means that the $\boldsymbol{C}$-objects and $\boldsymbol{C}$-morphisms becomes at least real lattices and real lattice morphisms). Some technical considerations will be required on setting up a suitable representation theory.

Definition 2.2. *A category $\boldsymbol{C}$ is said to be appropriate if it is a full subcategory of some universal algebra $\boldsymbol{O} \subseteq \boldsymbol{L}$, and L^* is a $\boldsymbol{C}$-subobject of L whenever $L \in \boldsymbol{C}$.*

Denote by $\boldsymbol{K}$ and $\boldsymbol{X}$ the full subcategories of $\boldsymbol{T}$ consisting of compact and realcompact Hausdorff spaces respectively.

Definition 2.3. *A subcategory $\boldsymbol{C} \subseteq \boldsymbol{L}$ is said to be convenient if it is appropriate and satisfies:*

(a) $\{C(X) : X \in \boldsymbol{T}\} \subseteq \boldsymbol{C}$;

(b) *$C(X)$ is $\boldsymbol{C}$-isomorphic to $C(\upsilon X)$ ($\upsilon X \in \boldsymbol{X}$ is the Hewitt-Nachbin realcompactification of X);*

(c) *if $X, Y \in \boldsymbol{X}$, then $T \in \mathrm{Hom}_{\boldsymbol{C}}(C(X), C(Y))$ iff there exists $t \in \mathrm{Hom}_{\boldsymbol{T}}(Y, X)$ such that $Tf = f \circ t$;*

(d) *$C^*(X)$ is $\boldsymbol{C}$-isomorphic to $C(\beta X)$ ($\beta X \in \boldsymbol{K}$ is the Čech-Stone compactification of X).*

Almost all the algebraic structures appearing in the literature regarding characterizations of $C(X)$ are convenient in the above sense, namely: Φ-algebras (see [6, 14, 18]), Archimedean Riesz spaces with a designated weak order unit (see [13, 17, 19]), real ℓ-groups (see [7, 20]) and semi-affine lattices ([9]).

In order to characterize $U(Y)$, first we must define uniform spaces topologically equivalent to realcompact spaces.

Definition 2.4. *A uniform space is called* realcomplete *if it is both complete and uniformly homeomorphic to a subspace of a power of $\mathbb{R}$. In the sequel $\boldsymbol{Y}$ denotes the full subcategory of $\boldsymbol{U}$ of realcomplete Hausdorff uniform spaces.*

Given $Y \in \boldsymbol{U}$, we set

$$cY = \big\{ \{f(x)\}_{f \in U(Y)} : x \in Y \big\} \subset \mathbb{R}^{U(Y)},$$

the *prerealcomplete modification* of Y, and $\gamma cY \in \boldsymbol{Y}$ its completion in $\mathbb{R}^{U(Y)}$.

Definition 2.5. *A subcategory $\boldsymbol{C} \subseteq \boldsymbol{L}$ is said to be* uniformly convenient *if it is appropriate and satisfies*

(a) *$\big\{U(Y) : Y \in \boldsymbol{U}\big\} \subseteq \boldsymbol{C}$;*

(b) *$U(Y)$ is $\boldsymbol{C}$-isomorphic to $U(\gamma cY)$ ($\gamma cY \in \boldsymbol{Y}$ is the* realcompletion *of Y);*

(c) *if $X, Y \in \boldsymbol{Y}$, then $T \in \mathrm{Hom}_{\boldsymbol{C}}(U(X), U(Y))$ iff there exists $t \in \mathrm{Hom}_{\boldsymbol{U}}(Y, X)$ such that $Tf = f \circ t$;*

(d) *$U^*(Y)$ is $\boldsymbol{C}$-isomorphic to $U(sY)$ ($sY \in \boldsymbol{K}$ is the* Samuel compactification *of Y).*

In the sequel $\boldsymbol{C}$ denotes either a convenient or uniformly convenient category according either to the topological or uniform case.

Definition 2.6. *The* spectrum (*resp.* uniform spectrum) *of a given object $L \in \boldsymbol{C}$ is the set $X_L^{\boldsymbol{C}} = \mathrm{Hom}_{\boldsymbol{C}}(L, \mathbb{R})$ equipped with the subspace topology* (*resp. $Y_L^{\boldsymbol{C}} = \mathrm{Hom}_{\boldsymbol{C}}(L, \mathbb{R})$ equipped with the subspace uniformity*) *of $\mathbb{R}^L$.*

It is not difficult to prove that $X_L^{\boldsymbol{C}} \in \boldsymbol{X}$, $Y_L^{\boldsymbol{C}} \in \boldsymbol{Y}$ and that both $X_{L^*}^{\boldsymbol{C}}, Y_{L^*}^{\boldsymbol{C}} \in \boldsymbol{K}$. Topological and uniform version of next theorem can be found in [11] and [10] respectively, and it is the key of our representation theory.

THEOREM 2.7. *The functors*

$$\boldsymbol{C} \to \boldsymbol{X},\ L \rightsquigarrow X_L^{\boldsymbol{C}}, \quad \textit{and} \quad \boldsymbol{X} \to \boldsymbol{C},\ X \rightsquigarrow C(X),$$
$$\boldsymbol{C} \to \boldsymbol{Y},\ L \rightsquigarrow Y_L^{\boldsymbol{C}}, \quad \textit{and} \quad \boldsymbol{Y} \to \boldsymbol{C},\ Y \rightsquigarrow U(Y),$$

form adjoint situations in convenient and uniformly convenient categories $\boldsymbol{C}$ *respectively.*

As a consequence $X \in \boldsymbol{X}$ iff $X_{C(X)}^{\boldsymbol{C}} = X$, and $Y \in \boldsymbol{Y}$ iff $Y_{U(Y)}^{\boldsymbol{C}} = Y$. Moreover, for $L \in \boldsymbol{C}$ there are reflections

$$\eta_L^{\boldsymbol{C}} \in \mathrm{Hom}_{\boldsymbol{C}}\left(L, C(X_L^{\boldsymbol{C}})\right),\ x \mapsto \eta_L^{\boldsymbol{C}}(f)(x) = x(f) \quad (x \in X_L^{\boldsymbol{C}}, f \in L),$$
$$\mu_L^{\boldsymbol{C}} \in \mathrm{Hom}_{\boldsymbol{C}}\left(L, U(Y_L^{\boldsymbol{C}})\right),\ y \mapsto \mu_L^{\boldsymbol{C}}(f)(y) = y(f) \quad (y \in Y_L^{\boldsymbol{C}}, f \in L),$$

called *spectral* and *uniform spectral representation* of L respectively.

QUESTION 1. *Which* $\boldsymbol{C}$*-stated conditions are required for an object* $L \in \boldsymbol{C}$ *in order to* $\eta_L^{\boldsymbol{C}} \in \mathrm{Iso}_{\boldsymbol{C}}\left(L, C(X_L^{\boldsymbol{C}})\right)$ *or* $\mu_L^{\boldsymbol{C}} \in \mathrm{Iso}_{\boldsymbol{C}}\left(L, U(Y_L^{\boldsymbol{C}})\right)$*?*

We shall proceed in three steps:

- *Embedding*: $L \subseteq C(X_L^{\boldsymbol{C}})$ or $L \subseteq U(Y_L^{\boldsymbol{C}})$;
- *Intermediateness*: $L^* = C^*(X_L^{\boldsymbol{C}})$ or $L^* = U^*(Y_L^{\boldsymbol{C}})$;
- *Completion*: $L = C(X_L^{\boldsymbol{C}})$ or $L = U(Y_L^{\boldsymbol{C}})$.

3. Embedding

The first task will consist on setting when the spectral representation is injective (we may use the notation $\eta_L^{\boldsymbol{C}}(L) = L$).

Let $\boldsymbol{V} \subset \boldsymbol{L}$ be the convenient category consisting of *Archimedean* (i.e. $nf \leq g$ for all $n \in \mathbb{N}$ implies $f \leq 0$) vector lattices with a designated *weak order unit* $e > 0$ (i.e. $f \wedge e > 0$ for every $f > 0$), and $\mathrm{Hom}_{\boldsymbol{V}}$ their vector lattices homomorphisms mapping weak order units in weak order units.

Luxemburg-Zaanen [17] showed that there is a one-to-one correspondence between $X_L^{\boldsymbol{V}}$ and the set of *real maximal ideals* of L (i.e. vector subspaces M of L not containing the weak order unit e, and which are maximal among those being *solid*, i.e. $|f| \leq |g|$ for $g \in M$ implies $f \in M$).

LEMMA 3.1. $\eta_L^{\boldsymbol{V}}(L) = L$ *iff the intersection of all the real maximal ideals of* L *is* $\{0\}$.

This condition is usually known as "semisimplicity" and can be generalized to any convenient category $\boldsymbol{C}$.

Definition 3.2. *L is said to be $\boldsymbol{C}$-semisimple if for every $r \in \mathbb{R}$*

$$\bigcap_{x \in X_L^{\boldsymbol{C}}} x^{-1}(r) = \{r\}$$

Since morphisms in convenient categories are defined by means of operations, and reasoning as in [11], we may correspond elements of $X_L^{\boldsymbol{C}}$ with certain subsets of the real lattice L.

Lemma 3.3. *Let $\boldsymbol{C}$ be a convenient category with signature O. There is a one-to-one correspondence between $X_L^{\boldsymbol{C}}$ and the set $R_L^{\boldsymbol{C}}$ consisting of real indexed families $R = \{R(r)\}_{r\in\mathbb{R}}$ of subsets of L having the following properties*

(a) $\bigcup R(r) = L$ *and* $R(r) \cap R(s) = \varnothing$ *if* $r \neq s$;

(b) $R(r) \cap \mathbb{R} = \{r\}$ *for every* $r \in \mathbb{R}$;

(c) *if* $o_L(f_1, \ldots, f_{a(o)}) \in R(r)$, *then* $f_1 \in R(r_1), \ldots, f_{a(o)} \in R(r_{a(o)})$ *for some* $r_1, \ldots, r_{a(o)} \in \mathbb{R}$ *such that* $o_{\mathbb{R}}(r_1, \ldots, r_{a(o)}) = r$, *for every* $r \in \mathbb{R}$.

Such families from $R_L^{\boldsymbol{C}}$ are called real-systems of L.

Recall that given $R \in R_L^{\boldsymbol{C}}$, the mapping

$$R^{-1} : L \to \mathbb{R},\ f \mapsto R^{-1}(f) = r, \text{ such that } f \in R(r)$$

belongs to $X_L^{\boldsymbol{C}}$, and conversely if $x \in X_L^{\boldsymbol{C}}$, then $\{x^{-1}(r)\}_{r\in\mathbb{R}} \in R_L^{\boldsymbol{C}}$.

Corollary 3.4. *L is $\boldsymbol{C}$-semisimple iff $\bigcap_{R\in R_L^{\boldsymbol{C}}} R(r) = \{r\}$ for every $r \in \mathbb{R}$.*

Unfortunately, $\boldsymbol{C}$-semisimplicity is not enough to ensure $\eta_L^{\boldsymbol{C}}(L) = L$. The well behavior of $\boldsymbol{V}$-semisimplicity responds to the embedding $\eta_{L^*}^{\boldsymbol{V}}(L^*) = L^*$ (from [20]), but without assuming linear structures this fact does not hold.

In [9] the convenient category $\boldsymbol{S}$ of semi-affine lattices is studied in details (roughly speaking, a semi-affine lattice of $C(X)$ is a sublattice which is closed under addition by $\mathbb{R}$ and multiplication by $\{0\} \cup \{w^n : n \in \mathbb{N}\}$ for some real number $w < -1$) where the following counterexample is produced:

Example 3.5. *Let*

$$L = \left\{ (a,b,i) \in \mathbb{R}^2 \times \{-1,0,1\} \ : \ \begin{array}{l} \text{if } a < b, \text{ then } i \in \{-1,0\}, \\ \text{if } a = b, \text{ then } i = 0, \\ \text{if } a > b, \text{ then } i \in \{0,1\}, \end{array} \right\}$$

endowed with the following order, addition and multiplication:

$$(a,b,i) \leq (c,d,j) \text{ iff } (a,b) \leq (c,d), \text{ and either } i \leq j \text{ or } a \geq b, c \leq d\,,$$

$$r + (a,b,i) = (a+r, b+r, i)\,, \quad r * (a,b,i) = (ra, rb, \mathrm{sign}(ri))\,.$$

Then L is $\boldsymbol{S}$-semisimple but $\eta_L^{\boldsymbol{S}}(L) \neq L$ (since $\eta_{L^*}^{\boldsymbol{S}}(L^*) \neq L^*$).

Next definition from [3] ensures injectivity of $\eta_{L^*}^{\boldsymbol{L}}$.

Definition 3.6. *$L \in \boldsymbol{L}$ is said to be special if*

(a) *for every $r, s \in \mathbb{R}$ and $f \in L$:*

  (a.1) *$f \vee r \geq s > r$ implies $f \geq s$;*

  (a.2) *$f \wedge r \leq s < r$ implies $f \leq s$;*

(b) *for every pair $f < g$ in L there exists $r < s$ in $\mathbb{R}$ and $h \in L$ such that $f \wedge h \leq r$ and $g \wedge h \nleq t$ for every $t < s$.*

Lemma 3.7. *L^* is special iff $\eta_{L^*}^{\boldsymbol{L}}(L^*) = L^*$.*

By adding speciality to semisimplicty, injectivity of the spectral representation yields in any convenient category (see [11]).

Theorem 3.8. *L is special and $\boldsymbol{C}$-semisimple iff $\eta_L^{\boldsymbol{C}}(L) = L$.*

4. Intermediateness

Yosida proved in [20] that L^* is uniformly dense in $C(X_{L^*}^{\boldsymbol{V}})$. However this fact does not work in weaker convenient categories, even by assuming speciality as one can see in the next counterexample extracted from [3].

Example 4.1. *Let $L = \{f \in C(\{0,1\}) : |f(0) - f(1)| < 1\}$. Then L is special and $\boldsymbol{L}$-semisimple, but L^* is not uniformly dense in $C(X_{L^*}^{\boldsymbol{L}})$.*

In order to solve this gap, Anderson-Blair introduced in [3] the notion of normality.

Definition 4.2. *$L \in \boldsymbol{L}$ is said to be normal if for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\beta < \gamma$ and for every $f \in L^*$, there exist $g, h, k \in L^*$ such that $g \wedge h \leq \alpha$, $\beta \leq h \vee f$, $f \wedge k \leq \gamma$ and $\delta \leq k \vee g$.*

This condition allowed them to obtain a Stone-Weierstrass-like theorem in the category $\boldsymbol{L}$.

Lemma 4.3 (Stone-Weierstrass-like). *L^* is special and normal iff L^* is uniformly dense in $C(X_{L^*}^{\boldsymbol{L}})$.*

Once arrived at this point, our interest focuses on stating a Kakutani-like theorem in $\boldsymbol{C}$, i.e. to obtain conditions under which L^* is uniformly dense in $C^*(X_L^{\boldsymbol{C}})$. In general, it is false (even by assuming semisimplicity, speciality and normality) as next example from [13] shows.

EXAMPLE 4.4. *Let* $L = \{f_{|\mathbb{R}} : f \in C(\overline{\mathbb{R}}, \overline{\mathbb{R}}),\ f(x) = \pm\infty \text{ iff } x = \pm\infty\}$. *Then* L *is* $\boldsymbol{V}$*-semisimple* (*of course* L^* *is both special and normal*) *but* L^* *is not uniformly dense in* $C^*(X_L^{\boldsymbol{V}})$.

Next lemma from [11] will be important to our aims.

LEMMA 4.5 (Kakutani-like). *Under the assumption of speciality and normality,* L *is* $\boldsymbol{C}$*-semisimple iff* $X_{L^*}^{\boldsymbol{C}}$ *is a compactification of* $X_L^{\boldsymbol{C}}$. *As a consequence, if* L *is special, normal and* $\boldsymbol{C}$*-semisimple, the following are equivalent:*

(i) L^* *is uniformly dense in* $C^*(X_L^{\boldsymbol{C}})$*;*

(ii) L^* *separates disjoint zero-sets of* $X_L^{\boldsymbol{C}}$.

Functionally separated subsets can be described by means of the method of the famous Urysohn's lemma. We shall start by determining closed subsets:

DEFINITION 4.6. *A real indexed family* $C = \{C(r)\}_{r\in\mathbb{R}}$ *of subsets of* L *is said to be a closed-system if there exists a class* $\{R_i\}_i \subseteq R_L^{\boldsymbol{C}}$ *of real-systems in* L (*defined as in Lemma 3.3*) *such that* $C(r) = \bigcap_i R_i(r)$ *for every* $r \in \mathbb{R}$.

If $F \neq \varnothing$ is a closed subset of $X_L^{\boldsymbol{C}}$, then the family $C_F = \{\bigcap_{x\in F} x^{-1}(r)\}_{r\in\mathbb{R}}$ becomes a closed-system, and conversely, if $C = \{\bigcap_i R_i(r)\}_{r\in\mathbb{R}}$ is a closed-system, then $F_C = \{R_i^{-1}\}_i$ becomes a closed subset of $X_L^{\boldsymbol{C}}$ (see in the comments below Lemma 3.3 how the morphisms R_i are constructed). Furthermore, $C_{X_L^{\boldsymbol{C}}} = \mathbb{R}$ and $F_\mathbb{R} = X_L^{\boldsymbol{C}}$.

Denote by $C_L^{\boldsymbol{C}}$ the set consisting of its closed-systems by adding L as one of its members under the assertion $C_\varnothing = L$ and $F_L = \varnothing$. As a consequence:

COROLLARY 4.7. *There is a one-to-one correspondence between* $C_L^{\boldsymbol{C}}$ *and the family of closed subsets of* $X_L^{\boldsymbol{C}}$.

Actually, $C_L^{\boldsymbol{C}}$ becomes a complete lattice with the first element $\mathbb{R}$ and the last element L, whenever we are setting for $B, D \in C_L^{\boldsymbol{C}}$ the lattice operations

$$B \wedge D = C_{(F_B \cup F_D)} \text{ and } B \vee D = C_{(F_B \cap F_D)}.$$

A description of cozero-sets was given by Kerstan [16].

PROPOSITION 4.8. *A subset of a topological space* X *is a cozero-set iff it belongs to a family* $\mathscr{V}$ *of open sets having the property that for every its member* U *there exist two sequences* $\{U_n\}_n, \{V_n\}_n$ *in* $\mathscr{V}$ *such that*

$$U = \bigcup U_n\,,\ U_n \subset X \setminus V_n \subset U,\ \textit{for each } n.$$

Above result will be helpful in order to determine zero-sets.

Definition 4.9. *A closed-system in L is said to be a zero-system provided it belongs to a class $\mathscr{Z}$ of closed-systems in L having the property that for every its member B there are countable sequences $\{B_n\}_n$, $\{D_n\}_n$ from $\mathscr{Z}$ such that*

$$B = \bigvee_n B_n,\ B_n \wedge D_n = \mathbb{R} \text{ and } D_n \vee B = L, \text{ for all } n.$$

Denote by $Z_L^{\boldsymbol{C}}$ the set consisting of zero-systems in L.

Lemma 4.10. *There is a one-to-one correspondence between $Z_L^{\boldsymbol{C}}$ and the family of zero-sets of $X_L^{\boldsymbol{C}}$.*

Proof. Every zero-system belongs to a class $\mathscr{Z}$ of closed-systems in L such that for every its member B there are sequences $\{B_n\}_n$,$\{D_n\}_n$ from $\mathscr{Z}$ such that $B = \bigvee_n B_n$, $B_n \wedge D_n = \mathbb{R}$ and $D_n \vee B = L$, for every n. One derives that $F_B = \bigcap_n F_{B_n}$, $F_B \subset X_L^{\boldsymbol{C}} \setminus F_{D_n} \subset F_{B_n}$. Thus, F_C becomes a zero-set whenever $C \in Z_L^{\boldsymbol{C}}$.

Conversely, from Proposition 4.8, the complementary of a given zero-set Z from $X_L^{\boldsymbol{C}}$ belongs to a family $\mathscr{V}$ of open sets in X_L having the property that that for every its member U there are sequences sequences $\{U_n\}_n, \{V_n\}_n$ from $\mathscr{V}$ such that $U = \bigcup U_n$, $U_n \subset X_L^{\boldsymbol{C}} \setminus V_n \subset U$ for each n. By taking $\mathscr{Z} = \{C_{X_L^{\boldsymbol{C}} \setminus U} : U \in \mathscr{V}\}$, the closed-systems $B_n = C_{X_L^{\boldsymbol{C}} \setminus U_n}$, $D_n = C_{X_L^{\boldsymbol{C}} \setminus V_n}$ satisfy $C_{X_L^{\boldsymbol{C}} \setminus U} = \bigvee_n B_n$, $B_n \wedge D_n = \mathbb{R}$ and $D_n \vee B = L$, for every n. As a consequence, $C_Z \in Z_L^{\boldsymbol{C}}$. □

Definition 4.11. *L is said to be $\boldsymbol{C}$-separating provided $C(r) \cap D(s) \neq \varnothing$ for every pair of distinct reals $r \neq s$, and for any pair $C, D \in Z_L^{\boldsymbol{C}}$ which satisfies $C \vee D = L$.*

On the one hand, $C \vee D = L$ for $C, D \in Z_L^{\boldsymbol{C}}$ is equivalent to assert that F_C, F_D are disjoint zero-sets of $X_L^{\boldsymbol{C}}$. On the other hand, if $C = \{\bigcap_i R_i(r)\}_{r\in\mathbb{R}}$ and $D = \{\bigcap_j S_j(r)\}_{r\in\mathbb{R}}$, $f \in C(r) \cap D(s)$ iff $R_i^{-1}(f) = r$ and $S_j^{-1}(f) = s$ for every i, j, equivalently $f(F_C) = r$ and $f(F_D) = s$. As a consequence:

Theorem 4.12 (Uryshon-like). *L is $\boldsymbol{C}$-separating iff L separates functionally separated subsets of $X_L^{\boldsymbol{C}}$.*

The isomorphism $L^* = C^*(X_L^{\boldsymbol{C}})$ can be currently obtained by assuming uniform completeness. However, to define this concept subtraction and absolute value operations are needed. A partial solution was proposed in [3].

Definition 4.13. *A continuous ideal is a solid subset I of L which is closed under finite suprema and such that for any $0 < r \in \mathbb{R}$ there exists $0 < \alpha < \beta < r$ in $\mathbb{R}$, $k_1, k_2 \in L$ and $g \in I$ such that $I \leq k_1 \vee k_2$, and if $h \in I$, $g \leq h$ and $k_i \wedge \alpha \nleq h$, then $k_i \wedge h \leq \beta$ $(i = 1, 2)$.*

Next theorem constitutes the Anderson-Blair's solution [3] to the problem 81 of Birkhoff.

THEOREM 4.14. *L is special, normal and every continuous ideal in L^* has a supremum in L^* iff $L^* = C(X_{L^*}^{\boldsymbol{L}})$.*

We shall say:

DEFINITION 4.15. *L is $\boldsymbol{C}$-intermediate if:*

(a) *L is $\boldsymbol{C}$-semisimple;*

(b) *L is $\boldsymbol{C}$-separating;*

(c) *L is special, normal and every continuous ideal in L^* has a supremum in L^*.*

From all above mentioned, we get the following result:

COROLLARY 4.16. *L is $\boldsymbol{C}$-intermediate iff $C^*(X_L^{\boldsymbol{C}}) \subseteq L \subseteq C(X_L^{\boldsymbol{C}})$.*

Once arrived at this point, we asked whether it would be possible to translate this intermediate situation to uniform spaces. Suppose $\boldsymbol{C}$ is a uniformly convenient category and $L \in \boldsymbol{C}$.

Given $\delta > 0$ and $g \in L$ we set

$$U_{\delta,g} = \big\{(R,S) \in R_L^{\boldsymbol{C}} \times R_L^{\boldsymbol{C}} : g \in R(r) \cap S(s) \text{ implies } |r-s| < \delta\big\}.$$

Notice that the definition is internal in character since the operation $|r-s| < \delta$ is in $\mathbb{R}$.

DEFINITION 4.17. *A sequence $\{f_n\}_n \subseteq L$ is said to be equi-uniformly $\boldsymbol{C}$-continuous if for any $\varepsilon > 0$ there are $g_1, \dots, g_m \in L$ and $\delta > 0$ such that for all n*

$$f_n \in \bigcap \left\{ \left[\bigcup \{R(r) \cap S(s) : |r-s| < \varepsilon\} \right] : (R,S) \in U_{\delta,g_1} \cap \dots \cap U_{\delta,g_m} \right\}.$$

If $Y \in \boldsymbol{Y}$, then $\{f_n\}_n \subseteq U(Y)$ is equi-uniformly $\boldsymbol{C}$-continuous iff for any $\varepsilon > 0$ there exists an entourage U of Y such that $|f_n(x) - f_n(y)| < \varepsilon$ for all n, $(x,y) \in U$.

We have recently obtained in [12] an equivalent condition to intermediateness by avoiding separation.

THEOREM 4.18. *Suppose L is special, normal and every continuous ideal in L^* has a supremum in L^* (recall from Theorem 4.14 that this is equivalent to $L^* = U(Y_{L^*}^{\boldsymbol{C}})$). The following are equivalent:*

(i) *$L^* = U^*(Y_L^{\boldsymbol{C}})$;*

(ii) *every equi-uniformly $\boldsymbol{C}$-continuous sequence $\{f_n\}_n \subset L_+$ bounded from above by a real number has a supremum f in L which satisfies $f \wedge g = 0$ whenever $f_n \wedge g = 0$ for all n.*

Now we may add a condition ensuring injectivity of $\mu_L^{\boldsymbol{C}}$ and furthermore determining uniform intermediateness (see [12]).

Definition 4.19. *L is uniformly $\boldsymbol{C}$-intermediate if:*

(a) *L is special, normal and every continuous ideal in L^* has a supremum in L^*;*

(b) *every equi-uniformly $\boldsymbol{C}$-continuous sequence $\{f_n\}_n \subset L_+$ bounded from above by a real number has a supremum f in L which satisfies $f \wedge g = 0$ whenever $f_n \wedge g = 0$ for all n;*

(c) *if $f \neq g$ from L, there exist $n, k \in \mathbb{N}$ such that $(f \wedge (-k) \vee n) \neq (g \wedge (-k) \vee n)$.*

Corollary 4.20. *L is uniformly $\boldsymbol{C}$-intermediate iff $U^*(Y_L^{\boldsymbol{C}}) \subseteq L \subseteq U(Y_L^{\boldsymbol{C}})$.*

5. Completion

We produced in [8] a $\boldsymbol{C}$-intermediate lattice not isomorphic to any $C(X)$.

Example 5.1. *Let $L = C^*(\mathbb{N}) \cup \{f \in C(\mathbb{N}) : |f(n)| \leq n$ starting from some $n_0 \in \mathbb{N}\}$. Then L is $\boldsymbol{L}$-intermediate, but L is not $\boldsymbol{L}$-isomorphic to $C(\mathbb{N})$.*

Next definition close to inversion closeness is due to Feldman-Porter [5].

Definition 5.2. *L is 2-universally $\boldsymbol{C}$-complete if any sequence $\{f_n\}_n \subseteq L_+$ (resp. in L_-) having some member $f_m \notin R(0)$ for every $R \in R_L^{\boldsymbol{C}}$ and satisfying that $f_n \wedge f_k \neq 0$ (resp. $f_n \vee f_k \geq 0$) for at most two indices k distinct from n, has a supremum (resp. infimum) f in L.*

Montalvo et al. obtained in [19] an internal characterization of $C(X)$ as a Riesz space.

Theorem 5.3. *The following are equivalent:*

(i) *L is $\boldsymbol{V}$-isomorphic to some $C(X)$;*

(ii) *L is $\boldsymbol{V}$-intermediate and 2-universally $\boldsymbol{V}$-complete.*

By taking into account that $|f - g| \leq \varepsilon$ on $\mathrm{coz}(h)$ iff $mh \wedge |f - g| \leq \varepsilon$ for all $m \in \mathbb{N}$, a "local uniform completeness" definition can be proposed.

Definition 5.4. *Let L be a $\boldsymbol{V}$-object with a designated weak order unit e. A sequence $\{f_n\}_n$ in L_+ is said to be* locally $\boldsymbol{V}$-Cauchy *if there exists a subset H of L_+ contained in no real maximal ideal and having the property: if $h \in H$, and $\varepsilon > 0$, then there exists $n_\varepsilon^h \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $mh \wedge |f_n - f_{n_\varepsilon^h}| \leq \varepsilon e$ whenever $n \geq n_\varepsilon^h$.*

L is said to be locally uniformly $\boldsymbol{V}$-complete *if for every locally $\boldsymbol{V}$-Cauchy sequence $\{f_n\}_n$ in L_+ there exists $f \in L_+$ such that for every $h \in H$ and $\varepsilon > 0$, $mh \wedge |f - f_{n_\varepsilon^h}| \leq \varepsilon e$ for all $m \in \mathbb{N}$.*

In [13] we have obtained recently an improvement of Theorem 5.3 by removing both uniform completeness and 2-universal completeness.

Theorem 5.5. *The following are equivalent:*

(i) *L is $\boldsymbol{V}$-isomorphic to some $C(X)$;*

(ii) *L is $\boldsymbol{V}$-semisimple, $\boldsymbol{V}$-separating and locally uniformly $\boldsymbol{V}$-complete.*

Furthermore, in the category $\boldsymbol{S}$ of semi-affine lattices, condition $mh \wedge |f - g| \leq \varepsilon$ for all $m \in \mathbb{N}$ is equivalent to both

$$(w^{2m} * h) \wedge g_+ - \varepsilon \leq (w^{2m} * h) \wedge f_+ \leq (w^{2m} * h) \wedge g_+ + \varepsilon, \text{ and}$$
$$(w^{2m-1} * h) \vee g_- - \varepsilon \leq (w^{2m-1} * h) \vee f_- \leq (w^{2m-1} * h) \vee g_- + \varepsilon.$$

By shifting the condition that H is contained in no real maximal ideal by that: for any $R \in R_L^{\boldsymbol{S}}$ there exists $f \in H$ such that $f \notin R(0)$, then local uniform $\boldsymbol{S}$-completeness yields, and we derive next characterization theorem (see [8]).

Theorem 5.6. *The following are equivalent:*

(i) *L is $\boldsymbol{S}$-isomorphic to some $C(X)$;*

(ii) *L is special, $\boldsymbol{S}$-semisimple, $\boldsymbol{S}$-separating and locally uniformly $\boldsymbol{S}$-complete.*

However, local uniform completeness can not be stated in $\boldsymbol{L}$, and Theorem 5.3 does not work, as next example from [8] shows.

Example 5.7. *Let $L = C^*(\mathbb{R}) \cup C_+(\mathbb{R}) \cup C_-(\mathbb{R})$. Then L is $\boldsymbol{L}$-intermediate and 2-universally $\boldsymbol{L}$-complete, but L is not $\boldsymbol{L}$-isomorphic to some $C(X)$.*

We develop a different approach.

Definition 5.8. *A sequence $\{f_n\}_n$ in L is said to be* pointwise $\boldsymbol{C}$-bounded *if for every $R \in R_L^{\boldsymbol{C}}$ there are $r < s$ in $\mathbb{R}$ with $f_n \wedge r \in R(r)$ and $f_n \vee s \in R(s)$ for each n.*

L is said to be pointwise $\boldsymbol{C}$-complete *if every increasing (resp. decreasing) pointwise $\boldsymbol{C}$-bounded sequence $\{f_n\}_n$ in L having the property that $f_n \wedge k = f_k$ (resp. $f_n \vee -k = f_k$) for every $n > k$, has a supremum (resp. infimum) f in L which satisfies $f \wedge n = f_n$ (resp. $f \vee -n = f_n$) for each n.*

In [11] we have obtained the most general characterization of $C(X)$ up to the date.

Theorem 5.9. *The following are equivalent:*

(i) *L is $\boldsymbol{C}$-isomorphic to some $C(X)$;*

(ii) *L is $\boldsymbol{C}$-intermediate and pointwise $\boldsymbol{C}$-complete.*

On realizing a uniform version of previous theorem we had to impose different requirements.

Definition 5.10. *$\{f_n\}_n$ is said to be* weakly pointwise $\boldsymbol{C}$-bounded from above *(resp. below) if for every $R \in R_L^{\boldsymbol{C}}$ there is n with $f_n \notin R(n)$ (resp. $f_n \notin R(-n)$).*

L is said to be equi-uniformly $\boldsymbol{C}$-complete *if every equi-uniformly $\boldsymbol{C}$-continuous and weakly pointwise $\boldsymbol{C}$-bounded from above (resp. from below) sequence $\{f_n\}_n$ in L having the property that $f_n \wedge k = f_k$ (resp. $f_n \vee -k = f_k$) whenever $n > k$, has an upper bound (or a lower bound) f in L which satisfies $f \wedge n = f_n$ (or $f \vee -n = f_n$, resp.) for all n.*

We obtained in [12]:

Theorem 5.11. *The following are equivalent:*

(i) *L is $\boldsymbol{C}$-isomorphic to some $U(Y)$;*

(ii) *L is uniformly $\boldsymbol{C}$-intermediate and equi-uniformly $\boldsymbol{C}$-complete.*

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