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Katětov order, Fubini property and Hausdorff ultrafilters[1]

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Abstract. *We study the Fubini property of ideals on ω and prove that the Solecki's ideal $\mathcal{S}$ is critical for this property in the Katětov order. We show that a well-known F_σ-ideal is critical for Hausdorff ultrafilters in the Katětov order and, by investigating the position of this ideal in the Katětov order, we show some of the known properties of this class of ultrafilters, including the Fubini property.*

Keywords: Hausdorff ultrafilter, Katětov order, Fubini property
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1. Introduction

An ultrafilter $\mathcal{U}$ on an infinite set is *Hausdorff* if the ultrapower of $\mathbb{N}$ modulo $\mathcal{U}$, equipped with the S-topology, is Hausdorff. The S-topology is defined for non-standard models ${}^*X$ of a topological space X, as the generated by the ${}^*A$ sets, for open sets $A \subseteq X$. In the particular case of the ultrapower $\mathbb{N}^\mathbb{N}/\mathcal{U}$ as a non-standard model for the first-order arithmetic, we consider $\mathbb{N}$ equipped with the discrete topology, and then, the S-topology on $\mathbb{N}^\mathbb{N}/\mathcal{U}$ is Hausdorff if and only if, for every $f, g \in \mathbb{N}^\mathbb{N}$ there exists $U \in \mathcal{U}$ such that either $f \restriction U = g \restriction U$ or $f''U \cap g''U = \emptyset$ (see Proposition 2.1).

Hausdorff ultrafilters have been studied recently by several authors, see e.g. by M. di Nasso and M. Forti [6]. The main question about them is their existence, that is, does ZFC prove the existence of a Hausdorff ultrafilter? In this note we characterize this class of ultrafilters by using the Katětov order and an F_σ-ideal on the integers that we call $\mathcal{G}_{fc}$.

The *Katětov order* is defined as follows: for any two ideals I, J on countable sets X and Y respectively, $\mathsf{I} \leq_K \mathsf{J}$ if there is a function f from Y to X so that $f^{-1}[I] \in \mathsf{J}$ for all $I \in \mathsf{I}$. We write $\mathsf{I} \leq_{KB} \mathsf{J}$ (the *Katětov-Blass order*) when f is a finite-to-one function. An introduction to the Katětov order can be found in [8].

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Katětov order is closely connected to Baumgartner's notion of I-ultrafilter. An ultrafilter $\mathcal{U}$ is an I-*ultrafilter* if and only if $\mathsf{I} \not\leq_K \mathcal{U}^*$. Several classes of ultrafilters are characterized as I-ultrafilters, for example, selective ideals are exactly the $\mathcal{ED}$-ultrafilters (see [7, 10, 14]).

Information about the position of ideals in the Katětov order provides information about belonging to classical families of ultrafilters, like P-points, Q-points and selective ultrafilters, since the I-ultrafilters (in the sense of Baumgartner [1]) are exactly the ultrafilters $\mathcal{U}$ such that $I \not\leq_K \mathcal{U}^*$.

We also study a property that Kanovei and Reeken [12] call the Fubini property. It concerns ideals (and filters) in general. For simplicity, we use a common notation: for any $A \subseteq \omega \times 2^\omega$, $n \in \omega$ and $x \in 2^\omega$ we denote $(A)_n = \{y \in 2^\omega : (n, y) \in A\}$ and $(A)^x = \{k \in \omega : (k, x) \in A\}$.

Definition 1.1. I *satisfies the* Fubini property *if for any Borel subset A of $\omega \times 2^\omega$ and any $\varepsilon > 0$, $\{n < \omega : \lambda((A)_n) > \varepsilon\} \in \mathsf{I}^+$ implies $\lambda^*(\{x \in 2^\omega : (A)^x \in \mathsf{I}^+\}) \geq \varepsilon$ (here λ^* means the Lebesgue outer measure on 2^ω).*

Particularly relevant for this work are the following ideals:

1. $\mathcal{ED} = \{A \subseteq \mathbb{N}^2 : \exists n \forall m > n |A \cap (\{m\} \times \mathbb{N})| \leq n\}$ is critical for selective ultrafilters in the Katětov order.

2. Let us denote by Δ the set $\{(n, m) : m \leq n\}$. Then, the ideal $\mathcal{ED}_{fin} = \{I \cap \Delta : I \in \mathcal{ED}\}$ on Δ is critical for Q-point ultrafilters in the Katětov-Blass order.

3. The *Solecki's ideal* $\mathcal{S}$ on the countable set Ω of all the clopen subsets of $2^{\mathbb{N}}$ with Lebesgue-measure equal to $\frac{1}{2}$, is generated by the family $\{A \subseteq \Omega : \bigcap A \neq \emptyset\}$. It was defined in [16], where the author proved that $\mathcal{S}$ is critical for the Fatou's property.

4. $\mathcal{G}_{fc} = \{A \subseteq [\mathbb{N}]^2 : ch(A) < \infty\}$, the ideals of graphs with finite chromatic number,[1] was used by Solecki in [16], where he asked if this ideal is critical for the Fatou property. This question was answered in the negative in [11].

5. $\mathcal{G}_c = \{A \subseteq [\mathbb{N}]^2 : \forall B \in [\mathbb{N}]^{\aleph_0}([B]^2 \setminus A \neq \emptyset)\}$, the ideal of graphs without infinite complete subgraphs.

The first four ideals are F_σ while the last is co-analytic.

[1]The *chromatic number* $ch(A)$ of a graph A on ω is defined as the minimal cardinal number κ for which there is a *coloring* $c : \omega \to \kappa$ so that $c(a) \neq c(b)$ for all $\{a, b\} \in A$.

2. Hausdorff ultrafilters and $\mathcal{G}_{fc}$

We now prove that $\mathcal{G}_{fc}$ is critical for Hausdorff ultrafilters in the Katětov order, i.e. $\mathcal{U}$ is Hausdorff if and only if $\mathcal{G}_{fc} \not\leq_K \mathcal{U}$. First we prove the following easy characterizations of Hausdorff ultrafilters. Note that f and g are $\mathcal{U}$-equivalent if and only if there is $U \in \mathcal{U}$ such that $f \restriction U = g \restriction U$.

PROPOSITION 2.1 ([6]). *The following conditions are equivalent, for any ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

1. *$\mathcal{U}$ is Hausdorff,*
2. *for every $f, g \in \mathbb{N}^{\mathbb{N}}$, f and g are $\mathcal{U}$-equivalent or $f''U \cap g''U = \emptyset$ for some $U \in \mathcal{U}$, and*
3. *for every $f, g \in \mathbb{N}^{\mathbb{N}}$, if $f(\mathcal{U}) = g(\mathcal{U})$ then there is $U \in \mathcal{U}$ such that $f \restriction U = g \restriction U$.*

Proof. We denote by $[h]$ the equivalence class of $h \in \mathbb{N}^{\mathbb{N}}$ modulo $\mathcal{U}$. $(1 \Rightarrow 2)$ If f and g are not $\mathcal{U}$-equivalent then there is $A \subseteq \mathbb{N}$ such that $[f] \in {}^*A$ and $[g] \in {}^*(N \setminus A)$, which means that there are V and W in $\mathcal{U}$ so that $f''V \subseteq A$ and $g''W \subseteq \mathbb{N} \setminus A$. Let $U = V \cap W$.
$(2 \Rightarrow 3)$ Assume $f \restriction X \neq g \restriction X$ for all $X \in \mathcal{U}$, and take U as in (2). From $f''(U) \in f(\mathcal{U})$ and $g''(U) \in g(\mathcal{U})$ follows $f(\mathcal{U}) \neq g(\mathcal{U})$.
$(3 \Rightarrow 1)$ If f and g are non-$\mathcal{U}$-equivalent then by (3) there is $A \in f(\mathcal{U}) \setminus g(\mathcal{U})$, and then $[f] \in {}^*A$ and $[g] \in {}^*(\mathbb{N} \setminus A)$. □

Now we describe a useful characterization of the ideal $\mathcal{G}_{fc}$. For each ordered pair $\langle A, B\rangle$ of nonempty disjoint subsets of $\mathbb{N}$, we define de set

$$I_{\langle A,B\rangle} = \{\{n, m\} : n \in A,\ m \in B,\ n < m\}$$

PROPOSITION 2.2. *$\mathcal{G}_{fc}$ is generated by the sets $I_{\langle A,B\rangle}$.*

Proof. On the one hand, it is clear that $ch(I_{\langle A,B\rangle}) \leq 2$. On the other hand, note that bipartite graphs are a base for $\mathcal{G}_{fc}$, since if $ch(G) = n$ then pick a coloring $c : \omega \to n$ so that $\{a, b\} \in G$ implies $c(a) \neq c(b)$, and for each pair $0 \leq i < j < n$ define $G_{i,j} = \{\{a, b\} : c(a) = i, c(b) = j\}$. Then, $G \subseteq \bigcup_{0 \leq i<j<n} G_{i,j}$. Finally, note that $I_{\langle A,B\rangle} \cup I_{\langle B,A\rangle}$ is the bipartite graph defined by A and B. □

We now prove the characterization of Hausdorff ultrafilters in the Katětov order, and additionally two graph-theoretic characterizations.

THEOREM 2.3. *The following conditions are equivalent for any ultrafilter $\mathcal{U}$ on $\mathbb{N}$*

1. *$\mathcal{U}$ is Hausdorff,*

2. *for every graph (G, E) and every $\varphi : \mathbb{N} \to E$, there exists $U \in \mathcal{U}$ such that $\varphi''U$ is contained in a bipartite graph.*

3. *for every graph (G, E) on $\mathbb{N}$ and every $\varphi : \mathbb{N} \to E$, there exists $U \in \mathcal{U}$ such that $ch(\varphi''U) < \infty$, and*

4. $\mathcal{G}_{fc} \not\leq_K \mathcal{U}^*$,

Proof. $(1 \to 2)$ Let us assume $\mathcal{U}$ is Hausdorff, and let φ be a function from $\mathbb{N}$ to $[\mathbb{N}]^2$. Define $f(n) = \min(\varphi(n))$ and $g(n) = \max(\varphi(n))$. It is clear that $f \neq g$ mod $\mathcal{U}$. By 2.1 there is $U \in \mathcal{U}$ such that $f''U \cap g''U = \emptyset$. Clearly, $I_{\langle f''U, g''U \rangle}$ is contained in a bipartite graph, and $\varphi''U \subseteq I_{\langle f''U, g''U \rangle})$.

$(2 \to 3)$ and $(3 \to 4)$ are immediate.

$(4 \to 1)$ Let us assume $\mathcal{G}_{fc} \not\leq \mathcal{U}^*$, and let f and g two non $\mathcal{U}$-equivalent functions. Since $\{n : f(n) = g(n)\} \notin \mathcal{U}$, either $\{n : f(n) > g(n)\} \in \mathcal{U}$ or $\{n : f(n) < g(n)\} \in \mathcal{U}$. Let us assume the first case (the other one is analogous), and define $\varphi(n) = \{g(n), f(n)\}$ if $g(n) < f(n)$, and $\varphi(n) = \{0, 1\}$ if not. Since there is $V \in \mathcal{U}$ such that $\varphi''V \in \mathcal{G}_{fc}$, and each element in $\mathcal{G}_{fc}$ is covered by a finite family of basic sets, there exist disjoint sets A and B so that for some $W \subseteq \{n \in V : g(n) < f(n)\}$ in $\mathcal{U}$, $\varphi''W \subseteq I_{\langle A,B \rangle}$, but this implies $f''W \subseteq A$ and $g''W \subseteq B$. □

About the position of $\mathcal{G}_{fc}$ some results are known: The identity function in $[\mathbb{N}]^2$ witnesses $\mathcal{G}_{fc} \leq_K \mathcal{G}_c$. Solecki proved in [16] that $\mathcal{S} \leq_K \mathcal{G}_{fc}$.

Lemma 2.4. *[14]* $\mathcal{G}_{fc} \geq_{KB} \mathcal{ED}_{fin}$

Proof. Define f from $[\mathbb{N}]^2$ to $\mathbb{N} \times \mathbb{N}$ by

$$f(\{n, m\}) = (\max\{m, n\}, \min\{m, n\}).$$

This f witnesses the Katětov relation since the chromatic numbers of the f-preimages of sets $\{k\} \times \mathbb{N}$ are equal to 2, and the chromatic numbers the f-preimages of sets $H = \{(n, h(n)) : n \in \omega\}$ $(h \in \mathbb{N}^{\mathbb{N}})$ are also equal to 2, since we can construct recursively a coloring c by letting $c(0) = 0$, $c(1) = 1$ and for $n \geq 2$, $c(n) = 1 - c(h(n))$ if $h(n) < n$. Hence, if $\{n < m\} \in f^{-1}[H]$ then $n = h(m)$ and then $c(n) \neq c(m)$. □

Since $\mathcal{ED} \leq_{KB} \mathcal{ED}_{fin}$ (inclusion of Δ into $\omega \times \omega$ witnesses the Katětov-Blass relation), we get immediately the following corollary.

Corollary 2.5 (Daguenet-Teissier [5]). *Every selective ultrafilter is Hausdorff.*

3. Fubini property

In [12, Proposition 24], Kanovei and Reeken claimed without a proof that Fubini property is equivalent to the validity of Fatou's lemma. We will prove this as a corollary of the following Theorem, which is obtained by mimicking Solecki's proof of [16, Theorem 2.1].

THEOREM 3.1. *Let I be an ideal on ω. Then, there exists an I-positive set X such that $\mathsf{I} \restriction X \geq_K \mathcal{S}$ if and only if I does not satisfy the Fubini property.*

Proof. Let $f : X \to \Omega$ be a witness of $\mathsf{I} \restriction X \geq_K \mathcal{S}$, and define $A = \{(n, x) : x \in f(n)\}$. Note that $(A)_n = f(n)$ and then $\lambda((A)_n) = \frac{1}{2}$ for all $n \in X$. For any $x \in 2^\omega$, $\{S \in \Omega : x \in S\} \in \mathcal{S}$ and then $\{n < \omega : x \in (A)_n\} \in \mathsf{I}$ for all $x \in 2^\omega$.

On the other hand, assume that I does not satisfy the Fubini property, and take a Borel set $A \subseteq \omega \times 2^\omega$ such that for some $\varepsilon > 0$, the set $X := \{n < \omega : \lambda((A)_n) > \varepsilon\}$ is I-positive, and if $R := \{x \in 2^\omega : (A)^x \in \mathsf{I}^+\}$ then $\lambda^*(R) < \varepsilon$.

First, we can assume that (1) $R = \emptyset$, (2) for any $n \in X$, A_n is closed and (3) for any $n \in X$, $\lambda(A_n) = \varepsilon$. If it is not the case, we could replace (a) ε with $\varepsilon' = \varepsilon - \lambda^*(R)$ and (b) for each n, A_n with a closed subset A'_n of $A_n \setminus R'$, so that $\lambda(A'_n) = \varepsilon'$, where R' is a G_δ-set so that $R' \supseteq R$ and $\lambda(R') = \lambda^*(R)$.

Let $k < \omega$ be so that $(1 - \varepsilon)^k < \frac{1}{3}$. Recall that the power of Cantor space $(2^\omega)^k$ endowed with the product measure λ^k is isomorphic to the Cantor space 2^ω endowed with the Lebesgue measure λ, via a homeomorphism between those spaces. For any $n < \omega$, we define a subset B_n of $(2^\omega)^k$ by $B_n = \bigcup_{i=1}^k proj_i^{-1}[A_n]$. Then $(2^\omega)^k \setminus B_n = \prod_{i=1}^k (2^\omega \setminus A_n)$ and then $\lambda^k(B_n) > \frac{2}{3}$. Note that the family $\{B_n : n \in X\}$ fulfils that $R'' := \{x \in (2^\omega)^k : \{n < \omega : x \in B_n\} \in \mathsf{I}^+\} = \emptyset$, since if $x = \langle x_i : 1 \leq i \leq k\rangle$ then $\{n < \omega : x \in B_n\} = \bigcup_{i=1}^k \{n < \omega : x_i \in A_n\} \in \mathsf{I}$.

Now, for $n \in X$ choose a clopen subset U_n of $(2^\omega)^k$ such that $\lambda^k(U_n) \geq \frac{7}{12}$ and $\lambda^k(U_n \setminus B_n) < \frac{1}{3 \cdot 2^{n+2}}$. If $S := \{x \in (2^\omega)^k : \{n \in \omega : x \in U_n\} \in \mathsf{I}^+\}$ then $S \subseteq \bigcap_{m<\omega} \bigcup_{n \geq m} (U_n \setminus B_n)$, proving that $\lambda^k(S) = 0$. Let $\{C_n : n < \omega\}$ be an increasing family of clopen sets such that $S \subseteq \bigcup_{n<\omega} C_n$ and $\lambda^k(\bigcup_{n<\omega} C_n) \leq \frac{1}{12}$. Finally, by taking for any $n \in X$ a clopen subset $f(n)$ of $U_n \setminus C_n$ with $\lambda^k(f(n)) = \frac{1}{2}$ we get the Katětov function f wanted, since for any $x \in 2^\omega = (2^\omega)^k$, if $\{n \in X : x \in f(n)\}$ is infinite then $x \notin \bigcup C_n$ and then $x \notin S$. Hence $\{n \in X : x \in f(n)\} \in \mathsf{I}$ for all $x \in 2^\omega$. $\square$

From Solecki's [16, Theorem 2.1] and the previous theorem we get:

COROLLARY 3.2. *If I is a universally measurable ideal on ω then I has the Fubini property if and only if I fulfils Fatou's lemma.* $\square$

EXAMPLE 3.3. **Fin** *and $\mathcal{Z}$ have the Fubini property.*

Proof. (**Fin**) Since $\mathcal{S}$ is a tall ideal and **Fin** is K-uniform we have that $\mathcal{S} \not\leq_K$ **Fin** $\restriction X$, for all infinite subset X of ω.

($\mathcal{Z}$) Let $f : \omega \to \Omega$ be a function. By the classical Fubini's Theorem, for every $n < \omega$, there is $A_n \in \Omega$ such that for all $x \in A_n$,

$$|\{m \in [2^n, 2^{n+1}) : x \in f(m)\}| \geq 2^{n-1}.$$

Since **Fin** has the Fubini property, there is $x \in 2^\omega$ and there is an increasing sequence $\langle n_k : k \in \omega \rangle$ such that $x \in A_{n_k}$. Then, for any $k < \omega$,

$$\limsup_{n\to\infty} \frac{|f^{-1}[I_x] \cap [2^n, 2^{n+1})|}{2^n} \geq \lim_{k\to\infty} \frac{|f^{-1}[I_x] \cap [2^{n_k}, 2^{n_k+1})|}{2^{n_k}} \geq \frac{1}{2}$$

proving that f is not a witness for $\mathcal{S} \leq_K \mathcal{Z}$. □

4. Fubini and Hausdorff ultrafilters

Let $\mathcal{U}$ be an ultrafilter on ω, and A_n a Borel subset of Cantor space 2^ω, for all $n < \omega$. The $\mathcal{U}$-limit of the sequence of sets is the set defined by

$$\mathcal{U}\text{-}\lim A_n = \{x \in 2^\omega : \{n \in \omega : x \in A_n\} \in \mathcal{U}\}.$$

If $\langle x_n : n < \omega \rangle$ is a sequence of real numbers then $l \in \mathbb{R}$ is the $\mathcal{U}$-limit of the sequence provided that $\{n < \omega : |x_n - l| < \varepsilon\} \in \mathcal{U}$ for all $\varepsilon > 0$.

As usual, an $\mathcal{S}$-ultrafilter is a free ultrafilter $\mathcal{U}$ whose dual ideal is not Katětov above the Solecki's ideal $\mathcal{S}$.

Theorem 4.1. *Let $\mathcal{U}$ be a free ultrafilter. Then the following are equivalent:*

1. *$\mathcal{U}$ is an $\mathcal{S}$-ultrafilter,*
2. *$\mathcal{U}^*$ satisfies the Fubini property and*
3. *for any sequence $\langle A_n : n < \omega \rangle$ of Borel subsets of 2^ω, if $\mathcal{U}\text{-}\lim \lambda(A_n) > 0$ then $\mathcal{U}\text{-}\lim A_n \neq \emptyset$.*

Proof. Theorem 3.1 claims that the ideals I which do not have I-positive sets X such that $\mathsf{I} \restriction X \geq_K \mathcal{S}$, are exactly those ideals satisfying the Fubini property, and since every maximal ideal is Katětov equivalent to all its restrictions to positive sets, we have that dual ideals of $\mathcal{S}$-ultrafilters are exactly the maximal ideals with the Fubini property. Now, Fubini property among maximal ideals (or ultrafilters) means: for any sequence $\langle A_n : n < \omega \rangle$ of Borel subsets of 2^ω and any $\varepsilon > 0$, if $\{n < \omega : \lambda(A_n) > \varepsilon\} \in \mathcal{U}$ then $\lambda^*(\{x \in 2^\omega : \{n < \omega : x \in A_n\} \in \mathcal{U}\}) \geq \varepsilon$. Hence, if $\mathcal{S} \not\leq_K \mathcal{U}^*$ and $\mathcal{U}\text{-}\lim \lambda(A_n) > 0$ then $\lambda^*(\mathcal{U}\text{-}\lim A_n) > 0$ and then $\mathcal{U}\text{-}\lim A_n \neq \emptyset$. On the other hand, let suppose that

$\mathcal{U}\text{-}\lim \lambda(A_n) > \varepsilon$ and $\lambda^*(\mathcal{U} \lim A_n) = \delta < \varepsilon$, for some sequence $\langle A_n : n < \omega \rangle$ and some $\varepsilon > 0$. For any $k < \omega$ let us choose a Borel set $A'_k \subseteq A_k \setminus \mathcal{U} \lim A_n$, with $\lambda(A'_k) = \varepsilon - \delta$. Then, $\mathcal{U}\text{-}\lim \lambda(A'_n) \geq \varepsilon - \delta$ but $\mathcal{U}\text{-}\lim A'_n = 0$, since for any $x \in 2^\omega$, $\{n : x \in A_n\} \in \mathcal{U}^*$. $\square$

Corollary 4.2 (Benedikt). *Every Fubini ulfrafilter is a Hausdorff ultrafilter.*

Proof. Solecki proved in [16] that $\mathcal{G}_{fc} \geq_K \mathcal{S}$ and if $\mathcal{U}$ is Fubini then by 4.1 $\mathcal{U}^* \not\geq \mathcal{S}$. Hence, $\mathcal{U}^* \not\geq_K \mathcal{G}_{fc}$. $\square$

5. Final remarks and questions

The known Katětov relations are displayed in the following diagram:[2]

$$\begin{tikzcd}
 & \mathcal{G}_c & \\
\begin{array}{c}\mathsf{nwd}\\ \text{(Nowhere dense)}\end{array} & \begin{array}{c}\mathcal{G}_{fc}\\ \text{(Hausdorff)}\end{array} \arrow[u] & \\
\begin{array}{c}\mathcal{S}\\ \text{(Fubini)}\end{array} \arrow[u] \arrow[ur] & \begin{array}{c}\mathcal{ED}_{fin}\\ \text{(Q-points}\\ \text{in RB-order)}\end{array} \arrow[u] & \begin{array}{c}\mathbf{Fin} \times \mathbf{Fin}\\ \text{(P-Points)}\end{array} \\
 & & \begin{array}{c}\mathcal{ED}\\ \text{(Selective)}\end{array} \arrow[ul] \arrow[u]
\end{tikzcd}$$

Of course, the main question about Hausdorff ultrafilters is if ZFC implies their existence. As a consequence of the fact that $\mathcal{S} \leq_K \mathsf{nwd}$ ([11, Theorem 5.10]), every Fubini ultrafilter is a nowhere dense ultrafilter. This fact was proved by Shelah ([15, Proposition 26]). The same does not hold for nowhere dense and Hausdorff ultrafilters since in [11] it was proved that $\mathcal{G}_{fc} \not\leq_K \mathsf{nwd}$, which is a consequence of 2.4 and the following

Proposition 5.1. $\mathcal{ED} \not\leq \mathsf{nwd}$.

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[2] An ultrafilter $\mathcal{U}$ is:
(1) *nowhere dense* if for each function f from $\mathbb{N}$ to $\mathbb{R}$, there is $U \in \mathcal{U}$ such that $f''U$ is nowhere dense.
(2) Q-*point* if for each partition $\{A_n : n < \omega\}$ of $\mathbb{N}$ such that each A_n is finite, there is $U \in \mathcal{U}$ such that $|U \cap A_n| \leq 1$ for all n.
(3) P-*point* if for each partition $\{A_n : n < \omega\}$ of $\mathbb{N}$, there is $U \in \mathcal{U}$ such that $|U \cap A_n| < \aleph_0$ for all n.

Proof. Let f be an arbitrary function from $\mathbb{Q}$ to $\omega \times \omega$ and let $\{U_n : n < \omega\}$ be a base for the open sets of $\mathbb{Q}$. Assume that $f^{-1}(n \times \omega) \in \mathsf{nwd}$ for all $n < \omega$ (if it is not the case we finished). Choose q_0 arbitrarily and recursively, choose $q_n \in U_n$ so that $proj_1(f(q_n)) > \max\{proj_1(f(q_j)) : j < n\}$. This is possible by our assumption. Then, $\{f(q_n) : n < \omega\} \in \mathcal{ED}$ but $\{q_n : n < \omega\}$ is dense in $\mathbb{Q}$. □

Di Nasso and Forti proved that if $\mathcal{U}$ and $\mathcal{V}$ are two isomorphic ultrafilters then $\mathcal{U} \times \mathcal{V}$ is not Hausdorff. On the other hand, it is easy to prove that if $\mathcal{U}$ is nowhere dense and $\mathcal{V}$ is P-point then $\mathcal{U} \times \mathcal{V}$ is a nowhere dense ultrafilter. Since every P-point is nowhere dense, for any P-point $\mathcal{U}$ we have that $\mathcal{U} \times \mathcal{U}$ is nowhere dense but not Hausdorff. Hence, from the consistency of the existence of a P-point ultrafilter it follows that there is a nowhere dense non Hausdorff ultrafilter. Consequently, a natural question is:

Problem 5.2: Are there consistently Hausdorff ultrafilters that are not nowhere dense?

It is well known that there is no P-point ultrafilter extending the filter nwd^*, however we would like to know if (consistently) there is a Hausdorff ultrafilter extending nwd^*, which is clearly a little stronger than Problem 5.2.

Di Nasso and Forti [6] asked about a set-theoretic hypothesis weaker than those providing selective ultrafilters, which implies the existence of Hausdorff ultrafilters, e.g. an equality or inequality between cardinal invariants of the continuum. We think it would be interesting to understand generic existence of Hausdorff ultrafilters[3]. For some classes of ideals this cardinal conditions are well known, for example, Canjar [3] proved that $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ is equivalent to generic existence of selective ultrafilters, and Benedikt [2] proved that $\mathsf{cov}(\mathcal{E}) = \mathfrak{c}$ is equivalent to generic existence of Fubini ultrafilters. The natural question is

Problem 5.3: Is there a suitable cardinal condition which is equivalent to generic existence of Hausdorff ultrafilters?

Finally, we want to ask about the existence of $\mathcal{G}_c$ ultrafilters.

Problem 5.4: Does ZFC prove that there exists a $\mathcal{G}_c$-ultrafilter?

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