Strongly inessential elements of a perfect height 2 ideal

Giannina Beccari and Carla Massaza

ABSTRACT. *In this paper we expand on some results exposed in a previous one, in which we introduced the concept of inessential and strongly inessential generators in a standard basis of a saturated homogeneous ideal. The appearance of strongly inessential elements seemed to be a non generic situation; in this paper we analyze their presence in a perfect height 2 ideal with the greatest number of generators, according to Dubreil's inequality.*

1. Introduction

In a previous paper [4] we introduced the concept of *strongly inessential element* (briefly s.i.) in a homogeneous ideal $\mathfrak{I} \subset K[X_1, \ldots, X_n]$. Our first idea, when we started to think about essential and inessential elements of a standard basis (see [4], n.3), was that every homogeneous ideal should have a standard basis consisting of essential forms, but we very soon found many counterexamples. Therefore, our next conjecture was that the assertion might be true for a sufficiently general ideal. In this paper we thus investigate the structure of e-maximal bases ([4], Definition 5.1) and, as a consequence ([4], Theorem 5.1), the presence of s.i. elements, in what seemed to be the easiest situation, that is when $\mathfrak{I}$ is a perfect height 2 ideal. In this case, it is possible to associate to every $\mathcal{B}(\mathfrak{I})$ a Hilbert-Burch matrix ([13, 14]) and to decide the nature of the forms of $\mathcal{B}(\mathfrak{I})$, with respect to essentiality, just looking at the ideals generated by the entries of its columns ([4]).

We observe that, if the multiplicity $e(\mathfrak{I})$ ([10, 11, 15]) is low, our first idea was correct; more precisely, if $e(\mathfrak{I}) < 6$, then every standard basis consists of essential elements, while, if $6 < e(\mathfrak{I}) < 9$, $\mathfrak{I}$ has at least a standard basis whose elements are all essential.

To deal with the problem when the multiplicity is ≥ 9, we observe that strong inessentiality is preserved modulo a regular sequence (while essentiality

is not). So, the first case to be considered seems to be the one of zero depth. As the general case still appears hard to be analyzed, we replace the family of all perfect height 2 ideals with its subfamily $F = \bigcup_{n \geq 2} F[n]$, where $F[n]$ is the set of all perfect height two ideals in $S = K[X_1, \ldots, X_n], n \geq 2$, whose standard bases are of maximal cardinality with respect to Dubreil's inequality ([9]). In a previous paper [3], in fact, we found a description of F that is of help in dealing with the problem considered here. So, as we restrict our attention to the ideals of zero depth, we study $F[2]$. For every ideal $\mathfrak{I} \in F[2]$, we produce a *canonical* Hilbert matrix, with the property that its corresponding basis is e-maximal, which means that its inessential elements are s.i.. Using such a matrix, we prove that the number of s.i. elements appearing in an e-maximal basis is linked to the greatest common divisor Φ of its generators of minimal degree $\alpha(\mathfrak{I})$; in fact, it depends on the decomposition of Φ into linear factors (see Theorem 4.1). More precisely, we prove that $\mathfrak{I}$ has a basis of essential elements iff all the linear factors of Φ are distinct; therefore, the generic $\mathfrak{I} \in F[2]$ has this a property.

The description of the e-maximal bases is much more complicated when we pass from $F[2]$ to $F[3]$. The Hilbert-Burch matrix of any element $\mathfrak{I} \in F[3]$ can be obtained by lifting the Hilbert-Burch matrix of its image $\bar{\mathfrak{I}} \in F[2]$ modulo any linear form, regular for $S/\mathfrak{I}$ ([3]); however, it may happen that there exists some $\widetilde{\mathfrak{I}}$ with the same number of s.i. elements of $\bar{\mathfrak{I}}$ in any e-maximal basis, among the ideals of $F[3]$ lifting $\bar{\mathfrak{I}} \in F[2]$, but there are also cases in which no lifting of $\bar{\mathfrak{I}}$ preserves a s.i. element. We prove that the greatest expected number of s.i. generators in a standard basis of $\mathfrak{I} \in F[3]$ is $\alpha(\mathfrak{I})-2$ and that this number is attained. So, we focus on the set $\mathcal{S} \subset F[3]$ of the ideals with $\alpha(\mathfrak{I})-2$ s.i. generators in their e-maximal bases, finding some of their properties and giving examples. In particular, we completely describe the ideals $\mathfrak{I}$ generated in two different degrees, with $\alpha(\mathfrak{I}) = 3$ and a s.i. element in any e-maximal basis.

2. Background and Notation

Let $S = K[X_1, \ldots, X_n]$, where K is an algebraically closed field, be the co-ordinate ring of $\mathbb{P}^{n-1}$, $\mathfrak{I} = \bigoplus \mathfrak{I}_d, d \in \mathbf{N}$, a homogeneous ideal of S, and $\mathfrak{M} = (X_1, \ldots, X_n)$ be the irrelevant maximal ideal. We recall some basic definitions.

The Hilbert function of $S/\mathfrak{I}$ ([12]), denoted $H(S/\mathfrak{I}, -)$, is the function defined by

$$H(S/\mathfrak{I}, t) = dim_K(S/\mathfrak{I})_t.$$

It is well known that for $t \gg 0$ the function $H(S/\mathfrak{I}, t)$ is a polynomial, with rational coefficients, of degree $r(S/\mathfrak{I}) - 1$, where $r(S/\mathfrak{I})$ is the Krull dimension of $S/\mathfrak{I}$.

If Δ denotes the difference operator on maps from $\mathbb{Z}$ to $\mathbb{Z}$, defined by $\Delta\phi(t) = \phi(t) - \phi(t-1)$, the function

$$\Gamma(\mathfrak{I}, t) = \Delta^{r(S/\mathfrak{I})} H(S/\mathfrak{I}, t)$$

is called the Castelnuovo function of $\mathfrak{I}$, while $\Delta^{r(S/\mathfrak{I})} H(S/\mathfrak{I}, t)$ is, for large t, a natural number $e(\mathfrak{I})$, independent on t, which is called the *multiplicity* of $S/\mathfrak{I}$, or also of $\mathfrak{I}$.

DEFINITION 2.1 ([8]). *A standard basis $\mathcal{B}(\mathfrak{I})$ of $\mathfrak{I}$ is an ordered set of forms of S, generating $\mathfrak{I}$, such that its elements of degree d define a K-basis of $\mathfrak{I}_d/(\mathfrak{I}_{d-1}S_1)$ ([5, 7, 8]).*

It is well known ([8]) that the degree vector of $\mathcal{B}(\mathfrak{I})$, with non decreasing entries, does not depend on the basis; $\alpha(\mathfrak{I})$ denotes its first entry, $\nu(\mathfrak{I})$ the number of entries, $\nu(\mathfrak{I}, t)$ the number of entries equal to t. Moreover, if $ht(\mathfrak{I}) > 1$, $\beta(\mathfrak{I})$ is the minimal degree t such that $GCD(\mathfrak{I}_t) = 1$.

The following theorem links $\alpha(\mathfrak{I})$ to $\nu(\mathfrak{I})$.

THEOREM 2.2 (Dubreil, [7, 8, 9]). *Let $\mathfrak{I}$ be a homogeneous perfect height 2 ideal. Then $\nu(\mathfrak{I}) \leq \alpha(\mathfrak{I}) + 1$.*

According to [3], $\mathcal{F}[n]$ denotes the set of all the homogeneous perfect height 2 ideals of $S = K[X_1, \ldots, X_n]$ such that $\nu(\mathfrak{I}) = \alpha(\mathfrak{I}) + 1$; in this paper they are called Dubreil' s ideals. In the special case $n = 2$, Theorem 1.7 ii) of [3] gives a description of every ideal of $\mathcal{F}[2]$ involving the greatest common divisor Φ of its elements of degree $\alpha(\mathfrak{I})$ and a decomposition of Φ as a product of forms.

A refinement of Theorem 2.2 ([5]) says, in particular, that, for every perfect height 2 ideal $\mathfrak{I}$ in S

$$t \leq \beta(\mathfrak{I}) \Rightarrow \nu(\mathfrak{I}, t) \leq -\Delta\Gamma(\mathfrak{I}, t). \tag{1}$$

We say that $\nu(\mathfrak{I}, t)$ is maximal when equality holds in (1).

If $\mathfrak{I}$ is a perfect height 2 ideal, then a minimal resolution of $S/\mathfrak{I}$ is defined by a Hilbert-Burch (shortly H.B.) matrix $M(\mathfrak{I})$ which, in turn, is uniquely determined by a standard basis $\mathcal{B}(\mathfrak{I})$ and by a minimal basis of its module of syzygies $Syz\,\mathcal{B}(\mathfrak{I})$. Its corresponding degree matrix $\partial M(\mathfrak{I})$ is uniquely determined by $\mathfrak{I}$.

We need some results, widely explained in [1, 2], that we summarize as follows.

THEOREM 2.3. *Let $\mathfrak{I}$ be a perfect height 2 ideal, $p + 1$ a degree in which the number $\nu(\mathfrak{I}, p+1)$ of generators in degree $p+1$ satisfies the following relation of maximality with respect to Dubreil-Campanella inequality*

$$\nu(\mathfrak{I}, p+1) = \Gamma(\mathfrak{I}, p) - \Gamma(\mathfrak{I}, p+1), \tag{2}$$

D the greatest common divisor of $\mathfrak{I}_p$. Then $\mathfrak{I}$ admits a basis

$$\mathcal{B} = (DF_1, \ldots, DF_m, \ G_1, \ldots, G_n),$$

where $(DF_1, \ldots, DF_m)S = \mathfrak{I}_p$, so that $\mathfrak{I}$ splits into two ideals $\mathfrak{I}' = (F_1, \ldots, F_m)$ and $\mathfrak{I}'' = (D, G_1, \ldots, G_n)$, which are still perfect of height 2. Moreover, there is a H.B. matrix $M(\mathfrak{I})$ with respect to $\mathcal{B}$, with the following shape

$$M(\mathfrak{I}) = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where

i) *$A \in K^{(m-1) \times m}$ is a H.B. matrix of $\mathfrak{I}'$,*

ii) *A H.B. matrix of $\mathfrak{I}''$ is $(B'' \ C)$, where $B'' = B \, {}^t(F_1 \ldots F_m)$,*

iii) *$\det C = D$*

3. Strongly inessential elements of an ideal: recalls and complements

Let $\mathfrak{I} = \oplus \mathfrak{I}_d, d \in \mathbf{N}, \mathfrak{I}_d \subset S_d$ be a homogeneous ideal of $S = K[X_1, \ldots, X_n]$. We recall some definitions and results appearing in [4].

DEFINITION 3.1 ([4], Definition 3.1). *An element f of a standard basis $\mathcal{B}(\mathfrak{I})$ is called an* inessential *generator of $\mathfrak{I}$ with respect to $\mathcal{B}(\mathfrak{I})$ iff*

$$\exists t \in \mathbf{N}, f M^t \subseteq (\mathcal{B}(\mathfrak{I}) - \{f\})S.$$

Otherwise we say that f is an essential *generator of $\mathfrak{I}$ with respect to $\mathcal{B}(\mathfrak{I})$.*

In the special case of perfect height 2 ideals, the essentiality of the r-th element f_r of $\mathcal{B}(\mathfrak{I})$ can be read on the ideal $\mathfrak{I}_{C_r}$ generated by the entries of the r-th column of any matrix of $Syz\,\mathcal{B}(\mathfrak{I})$. In fact, in [4], Proposition 4.1 says what follows.

PROPOSITION 3.2. *Let $\mathfrak{I}$ be a perfect codimension 2 ideal of S. Then $f_r \in \mathcal{B}(\mathfrak{I})$ is inessential for $\mathcal{B}(\mathfrak{I})$ iff the condition*

$$(\exists t \in \mathbf{N}) \ M^t \subseteq \mathfrak{I}_{C_r}$$

is satisfied.

DEFINITION 3.3 ([4], Definition 3.2). *An element $f \in \mathfrak{I}_d$ is strongly inessential (s.i.) iff $f \notin (\mathfrak{I}_{d-1})S$ and it is inessential with respect to any standard basis containing it.*

DEFINITION 3.4 ([4], Definition 5.1). *A standard basis is called e-maximal iff it has, in every degree d, exactly $\nu_e(d)$ essential generators, where $\nu_e(d)$ is the greatest number of essential generators of degree d appearing in a standard basis of $\mathfrak{I}$.*

THEOREM 3.5 ([4], Theorem 5.1). *A standard basis is e-maximal iff its inessential elements are strongly inessential.*

Starting from Theorem 3.5 we can prove the following statement.

PROPOSITION 3.6. *The ideal $\mathfrak{I} \subset S$ admits a basis of essential elements iff none of its elements is s.i..*

Proof. Proposition 5.2 of [4] says that two different e-maximal bases contain the same number of inessential elements. So, $\mathfrak{I}$ has a basis of essential elements iff all its e-maximal bases do not contain inessential elements, and we know that they should be s.i., thanks to Theorem 3.5. Now, every s.i. element can be considered as an entry of a standard basis $\mathcal{B}(\mathfrak{I})$ and from any standard basis $\mathcal{B}(\mathfrak{I})$ it is possible to produce an e-maximal basis $\mathcal{B}_M(\mathfrak{I})$, containing as a subset all the s.i. elements appearing in $\mathcal{B}(\mathfrak{I})$ (see Proposition 5.4 in [4]). So, the e-maximal bases do not contain inessential elements iff s.i. elements do not exist in $\mathfrak{I}$.   □

In other words, $\mathfrak{I}$ admits a basis of essential elements iff one of its e-maximal basis (and, as a consequence, all of them) consists of essential elements and this is equivalent to say that $\mathfrak{I}$ does not contain s.i. forms.

Next proposition says that a s.i. element of $\mathfrak{I}$ preserves its property modulo a linear form, regular for $S/\mathfrak{I}$. We will use the following notation.

Notation If z is any element of $S = K[X_1, \ldots, X_n]$ and $\phi : S \longrightarrow S/zS$ is the canonical morphism, then we set : $\phi(s) = \bar{s}, \forall s \in S$ and $\phi(\mathcal{A}) = \bar{\mathcal{A}}$ for any subset $\mathcal{A} \subseteq S$, if the element z can be understood.

We need the following lemma.

LEMMA 3.7 ([7, 8]). *If $\mathcal{B}$ is a standard basis of $\mathfrak{I}$ and $z \in S$ is a linear form, regular for $S/\mathfrak{I}$, then $\bar{\mathcal{B}}$ is a standard basis of $\bar{\mathfrak{I}}$.*

PROPOSITION 3.8. *Let $s \in \mathfrak{I}$ be a s.i. element and z a linear form regular for $S/\mathfrak{I}$. Then $\bar{s} \in \bar{\mathfrak{I}}$ is s.i..*

Proof. Without any loss of generality we can suppose $z = X_1$. At first we notice that if s is inessential for $\mathcal{B}(\mathfrak{I}) = \mathcal{B}$, then $\bar{s}$ is inessential for the standard basis $\bar{\mathcal{B}}(\bar{\mathfrak{I}}) = \bar{\mathcal{B}}$ of $\bar{\mathfrak{I}}$. In fact we have:

$$s\,\mathfrak{M}^t \subseteq (\mathcal{B} - \{s\})S \Rightarrow \bar{s}\,\bar{\mathfrak{M}}^t \subseteq (\bar{\mathcal{B}} - \{\bar{s}\}).$$

Now, let us suppose s to be s.i. and consider a standard basis $\mathcal{B}$ containing it, say $\mathcal{B} = (b_1, b_2, \ldots, b_h)$, where $b_i = s$. Then $\bar{\mathcal{B}}$ is a standard basis of $\bar{\mathfrak{I}}$, containing $\bar{b}_i = \bar{s}$ and any other standard basis $\mathcal{C}$ of $\bar{\mathfrak{I}}$ is of the form $\mathcal{C} = \bar{\mathcal{B}}P$, where $P = (p_{ji})$ is an invertible matrix, whose entries are forms in $K[X_2, \ldots, X_n]$. Let us observe that $\bar{s} = \bar{b}_i \in \mathcal{C}$ iff $p_{ii} \neq 0$ and $p_{ij} = 0$ when $j \neq i$. As a consequence, $\mathcal{B}' = \mathcal{B}P$ is a standard basis containing $s = b_i$; in $\mathcal{B}'$ the element s is inessential, as it is so in every basis in which it appears. The first part of the proof allows to conclude that $\bar{s}$ is inessential for $\mathcal{C}$. $\qquad\square$

   In Section 5 we will see that the lifting of a s.i. element of $\bar{\mathfrak{I}}$ is not necessarily s.i. in $\mathfrak{I}$. (see Remark 5.4).
   A consequence of Proposition 3.8 is that if the image $\bar{\mathfrak{I}}$ of $\mathfrak{I} \subset K[X_1, \ldots, X_n]$ modulo a maximal regular sequence does not contain any s.i. element, the same property holds for $\mathfrak{I}$. So, it seems convenient to start considering the problem of the presence of s.i. elements when $depth\ (S/\mathfrak{I}) = 0$ (see Section 3).

   In the sequel we use the following statement (see Theorem 2.3 for notation).

THEOREM 3.9. *Let $\mathfrak{I} \subset S$ be a perfect height 2 ideal and $p+1$ a degree in which the maximality condition (2) is verified. The following statements hold.*

 i) *If a form $F \in \mathfrak{I}'$ is s.i. in $\mathfrak{I}'$, then also $DF \in \mathfrak{I}$ is s.i. in $\mathfrak{I}$.*

 ii) *$G \in \mathfrak{I}''$ is s.i. iff $G \in \mathfrak{I}_t, t > p$ and G is s.i. as an element of $\mathfrak{I}$.*

Proof. i) Let $F \in \mathfrak{I}'_u, u \leq p - d$, where d is the degree of D, be s.i.. Then $F \notin \mathfrak{I}'_{u-1}S_1$, because it is an element of a standard basis of $\mathfrak{I}'$. As a consequence $FD \in \mathfrak{I}_{d+u}$, $FD \notin \mathfrak{I}_{d+u-1}S_1$, so that FD can be an element of some standard basis of $\mathfrak{I}$. Let $\mathcal{B} = (DF_1, \ldots, DF_m, G_1, \ldots, G_n)$ be any basis of $\mathfrak{I}$ such that $F = F_i$. As $(F_1, \ldots, F_m)$ is a standard basis of $\mathfrak{I}'$, we have

$$(\exists t)\ F\mathfrak{M}^t \subset (F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_m).$$

So, for some t, the relation

$$(DF)\mathfrak{M}^t \subset (DF_1, \ldots, DF_{i-1}, DF_{i+1}, \ldots, DF_m, G_1, \ldots, G_n)$$

holds.

ii) Let G be a s.i. element of $\mathfrak{I}''$. Thanks to Proposition 3.4 in [4], stating that no element of degree $\alpha(\mathfrak{I})$ can be s.i., G cannot be of the form kD, $k \in K$, so that $t = \deg G \geq p + 1$. First we observe that, as an element of $\mathfrak{I}$, G can belong to a standard basis. In fact, as it is a form of a standard basis of $\mathfrak{I}''$, we have $G \notin (\mathfrak{I}''_{t-1})S_1 \supseteq (\mathfrak{I}_{t-1})S_1$, so that $G \notin \mathfrak{I}_{t-1}S_1$. Now, let $\mathcal{B} = (DF_1, \ldots, DF_m, G_1, \ldots, G_{i-1}, G, G_{i+1}, \ldots, G_n)$ be any standard basis of $\mathfrak{I}$ containing G. Then $(D, G_1, \ldots, G_{i-1}, G, G_{i+1}, \ldots, G_n)$ is a standard basis of $\mathfrak{I}''$. The hypothesis of inessentiality of G as an element of $\mathfrak{I}''$ implies that

$$(\exists t) \ G\mathfrak{M}^t \subset (D, G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_n).$$

In other words, for every form $P \in \mathfrak{M}^t$, we have

$$GP = DV + \sum_{j \neq i} V_j G_j,$$

so that $(V, V_1, \ldots, V_{i-1}, -P, V_{i+1}, \ldots, V_n) \in Syz\,\mathfrak{I}''$. From (c) of Theorem 3.1 ([2]) it follows $V \in \mathfrak{I}'$, so that $GP \in (DF_1, ..., DF_m, G_1, ..., G_{i-1}, G_{i+1}, ..., G_n)$. This means that G is s.i. also as an element of $\mathfrak{I}$.

Viceversa, let $G \in \mathfrak{I}_t, t > p$ be a s.i. element in $\mathfrak{I}$. If $\mathcal{B} = (DF_1, \ldots, DF_m, G_1, \ldots, G_{i-1}, G, G_{i+1}, \ldots, G_n)$ is a basis of $\mathfrak{I}$ containing G, then $\mathcal{B}'' = (D, G_1, \ldots, G_{i-1}, G, G_{i+1}, \ldots, G_n)$ is a basis of $\mathfrak{I}''$. Thanks to Proposition 5.1 in [4], it is enough to prove that G is inessential with respect to any basis

$$\tilde{\mathcal{B}}'' = (D, G_1 + A_1 G, \ldots, G_{i-1} + A_{i-1} G, G, G_{i+1} + A_{i+1} G, \ldots, G_n + A_n G)$$

for every (degree allowed) choice of $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$.
As $\tilde{\mathcal{B}} = (DF_1, ..., DF_m, G_1 + A_1 G, ..., G_{i-1} + A_{i-1} G, G, G_{i+1} + A_{i+1} G, ..., G_n + A_n G)$ is still a standard basis of $\mathfrak{I}$, G is inessential with respect to it. This means that

$$(\exists t \in \mathbf{N}) \ G\mathfrak{M}^t \subset (DF_1, \ldots, DF_m, G_1 + A_1 G, \ldots, G_{i-1} + A_{i-1} G,$$
$$G_{i+1} + A_{i+1} G, \ldots, G_n + A_n G)$$
$$\subset (D, G_1 + A_1 G, \ldots, G_{i-1} + A_{i-1} G,$$
$$G_{i+1} + A_{i+1} G, \ldots, G_n + A_n G).$$

As a consequence, G is inessential also with respect to the basis $\tilde{\mathcal{B}}''$.   □

REMARK 3.10. *Theorem 3.9 can also be proved by working on a suitable H.B. matrix of $\mathfrak{I}$, taking into account Proposition 1.2 in [1] and Corollary 4.1 in [4].*

REMARK 3.11. *It may happen that in $\mathfrak{I}$ there exist s.i. elements that do not produce s.i. elements in $\mathfrak{I}'$ (see Remark 4.18)*

REMARK 3.12. *For every $\mathfrak{I} \in F[n]$, the maximality condition required in Theorem 3.9 is verified at any degree.*

Proposition 3.2 suggests a situation in which all the elements of every basis of $\mathfrak{I}$ are essential because the columns of its H.B. matrix are "short", so that they cannot generate a power of $\mathfrak{M}$.

COROLLARY 3.13. *Let $\mathfrak{I}$ be a perfect height 2 ideal of $S = K[X_1, \ldots, X_n]$. Each of the following conditions is enough to guaranty that in any standard basis of $\mathfrak{I}$ all the elements are essential:*

i) $\nu(\mathfrak{I}) < n + 1$

ii) $\alpha(\mathfrak{I}) < n$

iii) $e(\mathfrak{I}) < \dfrac{n(n+1)}{2}$.

Proof. i) and ii) are the statement of Corollary 5.1 and Remark in [4]; iii) comes from the inequality $\dfrac{\alpha(\alpha+1)}{2} \le e(\mathfrak{I})$, where $\alpha = \alpha(\mathfrak{I})$, just observing that $e(\mathfrak{I}) < \dfrac{n(n+1)}{2}$ implies $\alpha < n$. $\square$

REMARK 3.14. *It is easy to find examples of ideals with $e(\mathfrak{I}) = \dfrac{n(n+1)}{2}$ containing inessential elements in some standard basis; see, for instance, Example 3.1 in [4], where $n = 3, e = 6$.*

Proposition 3.4 of [4] says that in degree $\alpha(\mathfrak{I})$ no element is s.i.. So, the existence of a basis of essential elements is assured if the generators of degree $> \alpha$ are essential. Such a condition is verified when in the degree matrix $\partial M(\mathfrak{I}) = (d_{ij}), i = 1, \ldots, \nu(\mathfrak{I}) - 1, j = 1, \ldots, \nu(\mathfrak{I})$ the inequality $d_{h,\nu(\mathfrak{I},\alpha)} \le 0$ is verified for $h = \nu(\mathfrak{I}) - n$ (and, as a consequence, for $h < \nu(\mathfrak{I}) - n$), because it assures that the columns $C_j, j \ge \nu(\mathfrak{I}, \alpha)$, have at most $n - 1$ elements different from zero. This justifies the following statement.

PROPOSITION 3.15. *Let $\mathfrak{I}$ be a perfect height 2 ideal of $S = K[X_1, \ldots, X_n]$, with degree matrix $\partial M(\mathfrak{I}) = (d_{ij}), i = 1, \ldots, \nu(\mathfrak{I}) - 1, j = 1, \ldots, \nu(\mathfrak{I})$. If $d_{\nu(\mathfrak{I})-n, \nu(\alpha,\mathfrak{I})} \le 0$, then $\mathfrak{I}$ has a basis of essential elements.*

A consequence of Proposition 3.15 is the following statement.

COROLLARY 3.16. *Let $\mathfrak{I}$ be a perfect height 2 ideal of $S = K[X_1, \ldots, X_n]$. If*

$$e(\mathfrak{I}) < \frac{n(n+3)}{2},$$

then $\mathfrak{I}$ has a standard basis whose elements are all essential.

Proof. Taking into account the inequality $\dfrac{\alpha(\alpha+1)}{2} \leq e(\mathfrak{I})$, we see that the hypothesis implies $\alpha \leq n$. In case $\alpha < n$ we apply Corollary 3.13 ii). In case $\alpha = n$ and $\nu = \nu(\mathfrak{I}) < \alpha + 1$ we apply Corollary 3.13 i). So, the only case to be considered is $\alpha = n, \nu = n + 1$. In this situation the degree matrix $\partial M(I) = (d_{ij})$ satisfies the conditions $d_{i,i+1} = 1$, $i = 1, \ldots, n$. Taking into account the rule of computation of $e(\mathfrak{I})$ starting from $\partial M(\mathfrak{I})$ (see [6]), it is easy to verify that the only values of d_{ii} compatible with the hypothesis are the following ones:

a) $d_{ii} = 1$, $i = 1, \ldots, n$

b) $d_{ii} = 1$, $i \neq i_0, d_{i_0 i_0} = 2$ for some $i_0 \neq 1$.

In case a) the ideal is generated in degree α, so that we apply Proposition 3.4 of [4].
In case b) we have necessarily $d_{i_0(i_0+1)} = 0$, so that Proposition 3.15 can be used. □

REMARK 3.17. *If the inequality of Corollary 3.16 is not satisfied, there exist examples of ideals with s.i. elements. For instance, let us consider in $S = K[X_1, \ldots, X_n]$ the ideal $\mathfrak{I}$, with H.B. matrix*

$$M(\mathfrak{I}) = \begin{pmatrix} X_2^2 & -X_1 & & & & \\ & X_2 & -X_1 & & & \\ & X_3 & X_2 & -X_1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & X_n & & & X_2 & -X_1 \end{pmatrix},$$

where the unwritten entries are zero forms.
$\mathfrak{I}$ satisfies the condition $e(\mathfrak{I}) = \dfrac{n^2 + 3n}{2}$ and its second generator is s.i..

We observe that the ideals, with multiplicity $e(\mathfrak{I}) = \dfrac{n^2 + 3n}{2}$, that do not admit a basis of essential elements must necessarily have as a degree matrix the one defined by

$$d_{11} = 2; \ d_{ii} = 1, \ i \neq 1; \ d_{i(i+1)} = 1, \ i = 1, \ldots, n,$$

so that they have only one generator in degree α.

On the other side, it is possible to produce ideals with a basis of essential elements and with no upper limit on $e(\mathfrak{I})$. For instance, every ideal $\mathfrak{I}$ whose $\partial M(\mathfrak{I})$ is defined by

$$d_{i(i+1)} = 1, i = 1, \ldots, n; \ d_{ii} = 1, \ i = 1, \ldots, n-1; \ d_{nn} = h \geq 2$$

satisfies the condition of Proposition 3.15 and has multiplicity $e(\mathfrak{I}) = \dfrac{n(n+1)}{2} + h - 1$, which is arbitrarily large if $h \gg 0$.

We see that if $e(\mathfrak{I}) \geq \dfrac{n^2 + 3n}{2}$ the situation is hard to be examined , also if $\mathfrak{I}$ is a perfect height 2 ideal. That is a reason why we restrict our attention to the subfamily $\mathcal{F}[n]$ (see Section 2), starting with $n = 2$.

4. An e-maximal basis of $\mathfrak{I} \in \mathcal{F}[2]$

Relation (1.10) in Remark 1 to Theorem 17 in [3] gives a good description of every $\mathfrak{I} \in \mathcal{F}[2]$. With some change of notation, we rewrite it as follows:

$$\mathfrak{I} = \sum_{i=0}^{r} \Phi_{i+1} \ldots \Phi_{r+1} S_{\beta_i} S, \tag{3}$$

where Φ_i is a form of degree δ_i, $\Phi_{r+1} = 1$ and S_t is the subset of $S = K[X, Y]$ consisting of the forms of degree t.

Let us denote $\Delta_0 = 0, \Delta_i = \delta_1 + \ldots + \delta_i, i = 1, \ldots, r$ and $\Delta_r = \delta$ the degree of $\Phi = \Phi_1 \ldots \Phi_r$, so that we have

$$\delta = \sum_{i=1}^{r} \delta_i, \quad \beta_i = \beta_{i-1} + \delta_i + t_i,$$

where $t_i = \alpha_i - \alpha_{i-1} > 0, \ i = 1, \ldots, r$ is the difference between two successive different degrees of the generators appearing in a standard basis and $\alpha_0 = \alpha(\mathfrak{I}) = \alpha$.

In (3), $r + 1$ is the number of distinct elements appearing in any degree vector $\mathbf{a}$ of a standard basis of $\mathfrak{I}$; moreover, we have

$$\mathbf{a} = ((\beta_0 + \delta)^{[\beta_0 + 1]}, \ldots, (\beta_i + \delta - \Delta_i)^{[\delta_i]}, \ldots, \beta_r^{[\delta_r]})$$
$$= (\alpha_0^{[\beta_0 + 1]}, \alpha_1^{[\delta_1]}, \ldots, \alpha_i^{[\delta_i]}, \ldots, \alpha_r^{[\delta_r]}),$$

where $c^{[n]}$ is the sequence $(c, \ldots, c)$, with c repeated n times.

The degree matrix $\partial M(\mathfrak{I}) = (d_{ij})$ is completely determined by its elements in position $(i, i+1)$ (which are necessarily 1, as $M(\mathfrak{I})$ is a $\alpha \times (\alpha + 1)$ matrix) and by $\mathbf{a}$, or, equivalently, by its elements in position (i, i), which are

$$d_{ii} = 1 \text{ if } i \neq \beta_0 + 1 + \Delta_j, \, j = 0, \ldots, r-1 \,,$$
$$d_{ii} = t_{j+1} + 1 \text{ if } i = \beta_0 + 1 + \Delta_j \,.$$

Our aim is to produce an e-maximal basis of $\mathfrak{I}$ (see Definition 3.4), that allows to prove the following theorem.

THEOREM 4.1. *Let $\mathfrak{I}$ be as in (3) and let*

$$\Phi = \Phi_1 \ldots \Phi_r = H_1^{\mu_1} \ldots H_v^{\mu_v}, \quad \sum_{i=1}^{v} \mu_i = \delta \geq 1 \qquad (4)$$

be a factorization of Φ as a product of linear forms pairwise linearly indepen-dent. The number of s.i. elements appearing in every e-maximal basis of $\mathfrak{I}$ is $\delta - v$. If $\delta = 0$, then $\mathfrak{I} = S_\alpha S$ does not contain s.i. elements.

In order to prove Theorem 4.1 it is convenient (and possible) to produce an e-maximal basis $\mathcal{B}(\mathfrak{I})$ satisfying the following condition.

(*) *There is a basis of its module of syzygies linking only couples of adjacent elements* .

The Hilbert matrix corresponding to such a basis of $Syz(\mathcal{B}(\mathfrak{I}))$ will be called *the canonical matrix* of $\mathcal{B}(\mathfrak{I})$ or *a canonical matrix* of $\mathfrak{I}$.
Condition (*) will be of help in checking that $\mathcal{B}(\mathfrak{I})$ is an e-maximal basis.

Let us consider first two special cases, useful to face the general situation.

Case 1. $\mathfrak{I} = S_\alpha S$.

Thanks to Proposition 3.4 of [4], we know that an ideal generated in minimal degree cannot have s.i. elements. However, in the sequel we need an e-maximal basis, satisfying condition (*), constructed according to the following Proposi-tion. The notation $\hat{L}$ will always mean that the element L is omitted.

PROPOSITION 4.2. *Let $\mathfrak{I} = S_\alpha S$. If $\{L_0, \ldots, L_\alpha\}$ is a set of linear forms, pair-wise linearly independent, then $\mathcal{B}(\mathfrak{I}) = (F_i), i = 0, \ldots, \alpha, \, F_i = L_0 \ldots \hat{L}_i \ldots L_\alpha$, is a standard basis, consisting entirely of essential elements, whose canonical matrix M looks as follows:*

$$M = (m_{ij}), \quad i = 1, \ldots, \alpha, \, j = 1, \ldots, (\alpha + 1),$$
$$\text{where: } m_{ii} = L_{i-1}, \, m_{i(i+1)} = -L_i, \, m_{ij} = 0 \text{ otherwise.}$$

Proof. It is immediate to verify that the F_i's are a set of $\alpha + 1$ linearly independent elements of S_α, so that they are a basis of it as a K-space. The rows of M are syzygies linking adjacent elements; as they are linearly independent, they are a basis of $Syz\,(\mathcal{B}(\mathfrak{I}))$ (see Hilbert-Burch Theorem, [13]), so that M is a matrix of syzygies of $\mathfrak{I}$. The entries of every column C_i generate a principal ideal $\mathfrak{I}_{C_i}$; so, Proposition 3.2 says that all the elements of $\mathcal{B}(\mathfrak{I})$ are essential. $\qquad\square$

Case 2. $\mathfrak{I}$ is generated in two different degrees and in the lower one there is just one generator, so that

$$\mathfrak{I} = \Phi S + S_b S, \ \deg \Phi = \delta = \alpha(\mathfrak{I}), \ b = \beta(\mathfrak{I}) = \delta + t, \ t > 0. \qquad (5)$$

Let us consider the decomposition of Φ as in (4), with $r = 1$. We prove first the following lemma.

LEMMA 4.3. *Let* $\Phi = H_1^{\mu_1} \ldots H_v^{\mu_v}$ *be any form of degree* δ *in* $S = K[X, Y]$. *The K-space* S_b, $b = \delta + t, t \geq 0$, *admits a decomposition*

$$S_b = \Phi S_t \bigoplus T, \quad T = \bigoplus_{i=1}^{v} T_i, \qquad (6)$$

where a K-basis of T_i *is the ordered set* $\mathcal{B}_i = (F_{ij}), j = 1, \ldots, \mu_i$, *described as follows:*

$$F_{ij} = A_{ij} C_i, \qquad (7)$$

with

$$A_{ij} = H_i^{\mu_i - j} H_{i+1}^{\mu_{i+1}} \ldots H_v^{\mu_v} U^{j-1}, \ GCD(U, H_h) = 1, \ h = 1, \ldots, v, \qquad (8)$$

and C_i *any form of degree* $t + \mu_1 + \ldots + \mu_{i-1} + 1, (\mu_0 = 0)$, *satisfying the relation*

$$GCD(C_i, H_i) = 1. \qquad (9)$$

Proof. We use induction on v.

For $v = 1$ we have $\Phi = H^\mu, \delta = \mu$ and the statement becomes $T = T_1$, with basis $\mathcal{B}_1 = (F_{1j})$, $j = 1, \ldots, \mu$, where

$$F_{1j} = F_j = A_j C = H^{\mu - j} U^{j-1} C, \ \deg C = t + 1, \ GCD(C, H) = 1. \qquad (10)$$

It is immediate to prove that $F_1, \ldots, F_\mu$ are linearly independent, so we only have to show that $\Phi S_t \bigcap T = (0)$.

For $\Lambda \in S_t$, let us suppose $\Lambda \Phi = \sum_{j=1}^{\mu} a_j F_j = (\sum_{j=1}^{\mu} a_j H^{\mu-j} U^{j-1})C$. This implies that H^μ must divide $A = \sum_{j=1}^{\mu} a_j H^{\mu-j} U^{j-1}$. For degree reason, A must be zero, so that $a_j = 0$, $j = 1, \ldots, \mu$.

Let us suppose the statement true until $v - 1$ and prove it for v. We set $\Phi = \Psi H_v^{\mu_v}$ and use the decomposition of case $v = 1$ with H^μ replaced by $H_v^{\mu_v}$, so obtaining

$$S_b = H_v^{\mu_v} S_{b-\mu_v} \bigoplus T_v,$$

where $T_v = (F_{v1}, \ldots, F_{v\mu_v})$, with $F_{vj} = A_{vj} C_v$, $A_{vj} = H_v^{\mu_v - j} U^{j-1}$, $GCD(C_v, H_v) = 1$, $\deg C_v = b - \mu_v + 1$, according to (10).

Using induction, we have $S_{b-\mu_v} = \Psi S_t \bigoplus T'$, $T' = \bigoplus_{i=1}^{v-1} T_i'$, where T_i' has the basis (F_{ij}') described in the statement of Lemma 4.3, that is $F_{ij}' = H_i^{\mu_i - j} H_{i+1}^{\mu_{i+1}} \ldots H_{v-1}^{\mu_{v-1}} U^{j-1} C_i$. So, we finally obtain

$$S_b = H_v^{\mu_v} (\Psi S_t \bigoplus T') \bigoplus T_v = \Phi S_t \bigoplus T,$$

where $T = \bigoplus H_v^{\mu_v} T' \bigoplus T_v = \bigoplus_{i=1}^{v-1} H_v^{\mu_v} T_i' \bigoplus T_v$. It is immediate to check that $(F_{ij}) = (H_v^{\mu_v} F_{ij}')$, $j = 1, \ldots, \mu_i$, is the required basis of $T_i = H_v^{\mu_v} T_i'$, $i = 1, \ldots, (v-1)$. $\square$

REMARK 4.4. *Each space T_i depends on the choice of the form C_i, with the link (9). So, there are infinitely many decompositions of the type described in (6). Later on, we will use some of them, properly chosen accordingly to the situation.*

REMARK 4.5. *The basis of S_b, $b = \alpha$, used in Proposition 4.2 is obtained accordingly to Lemma 4.3, with the choice $\Phi = L_0 \ldots L_b$, $t = -1$, $C_i = H_1 \ldots H_{i-1}$. In this situation, the first summand of (6) is empty, so that $S_b = T$.*

PROPOSITION 4.6. *Let us consider the ideal*

$$\mathfrak{I} = \Phi S + S_b S, \ b = \delta + t, \ t > 0, \quad \Phi = H_1^{\mu_1} \ldots H_v^{\mu_v}, \deg \Phi = \delta.$$

i) $\mathfrak{I}$ has as a standard basis the set

$$\mathcal{B}(\mathfrak{I}) = (\Phi, F_{ij}), \ i = 1, \ldots, v, \ j = j(i) = 1, \ldots, \mu_i,$$

where:

$$F_{ij} = A_{ij} C_i, \tag{11}$$

$$A_{ij} = H_i^{\mu_i - j} H_{i+1}^{\mu_{i+1}} \ldots H_v^{\mu_v} U^{j-1}, \quad GCD(H_i, U) = 1 \tag{12}$$

$$C_1 = U^{t+1}, C_i = H_1 H_2 \ldots H_{i-1} U^{\nu_i}, \\ \nu_i = t + \mu_1 + \ldots + \mu_{i-1} - i + 2, \ i > 1, \tag{13}$$

ii) The basis $\mathcal{B}(\mathfrak{I})$ satisfies condition (*). Its canonical matrix $M(\mathfrak{I})$ has as rows the basis of syzygies $\{s_{ij}\}$, $i = 1, \ldots, v$, $j = j(i) = 1, \ldots, \mu_i$, with the lexicographic order, where:

$$s_{11} = (U^{t+1}, -H_1, 0, \ldots, 0),$$
$$s_{i1} = (0, \ldots, 0, H_{i-1}, -H_i, 0, \ldots 0), \ i = 2, \ldots, v,$$
$$-H_i \ in \ position \ \mu_1 + \mu_2 + \ldots + \mu_{i+2},$$
$$s_{ij} = (0, \ldots, 0, U, -H_i, 0, \ldots, 0), \ i = 1, \ldots, v, \ j = 2, \ldots, \mu_i,$$
$$-H_i \ in \ position \ \mu_1 + \ldots + \mu_{i-1} + j + 1, \ \mu_0 = 0.$$

So, $M(\mathfrak{I})$ looks as follows:

$$M(\mathfrak{I}) = \begin{pmatrix} U^{t+1} & \\ & \mathcal{A} \\ \mathcal{O} & \end{pmatrix},$$

where $\mathcal{A} = (a_{ij})$ is a square $\delta \times \delta$ matrix, whose elements different from $a_{ii}, a_{(i+1)i}$ are zero, and $(a_{11}, \ldots, a_{\delta\delta}) = ([-H_1]^{\mu_1}, [-H_2]^{\mu_2}, \ldots, [-H_v]^{\mu_v})$, $a_{(i+1),i} = -a_{ii}$ if $a_{ii} \neq a_{jj}, j > i$ and $a_{(i+1),i} = U$ otherwise.

iii) The essential elements of $\mathcal{B}(\mathfrak{I})$ are : $\Phi, F_{(i,\mu_i)}, i = 1, \ldots, v$. All the other $\delta - v$ elements of $\mathcal{B}(\mathfrak{I})$ are s.i..

Proof. i) This assertion is an immediate consequence of Lemma 4.3. In fact, thanks to the inequality $\mu_1 + \mu_2 + \ldots + \mu_{i-1} \geq i - 1$, we can choose $C_i = H_1 \ldots H_{i-1} U^{\nu_i}$, so obtaining F_{ij} as a basis of the K-space T complementary to ΦS_t in S_b.

ii) The fact that the $\{s_{ij}\}$'s are syzygies can be verified with an easy direct computation. Moreover, they are clearly linearly independent, of the expected degree and their number δ is the rank of the module of syzygies, according to Hilbert theorem. It is easy to verify that the first maximal minor of $M(\mathfrak{I})$ is Φ and (apart from a sign) the other maximal minors are the F_{ij}'s. Using Proposition 3.2, we see immediately that the essential columns of $M(\mathfrak{I})$ (that is the columns corresponding to essential elements, see [4]) are the first one and the $(\mu_1 + \mu_2 + \ldots + \mu_i + 1) - th, i = 1 \ldots v$; so, the essential elements of $\mathcal{B}$ are $\{\Phi, F_{i,\mu_i}, i = 1 \ldots v\}$.

iii) The proof that all the inessential elements are s.i. is a consequence of the following Lemma 4.8, stated in a form sufficiently general to be used later in a more general situation. In fact, the submatrix $\mathcal{A}$ appearing in $M(\mathfrak{I})$ satisfies the hypothesis of Lemma 4.8 .

$\square$

It is convenient to generalize the notion of inessential and strongly inessential columns of a matrix, as considered in [4].

DEFINITION 4.7. *Let A be a matrix whose entries a_{ij} are forms of $K[X_1, ..., X_n]$ such that $\deg a_{ij} - \deg a_{i(j+1)}$ is independent from i. A column C_j is inessential when the ideal $\mathfrak{I}_{C_j}$ generated by its entries contains a power of the irrelevant ideal. C_j is strongly inessential when every column $C_j' = \sum_i \lambda_i C_i = (a_{ij}'),\ \lambda_i \in K[X_1, \ldots, X_n],\ \lambda_j = 1,\ \deg a_{ij}' = \deg a_{ij}$, replacing C_j, is still inessential.*

LEMMA 4.8. *Let $A = (a_{ij}), a_{ij} \in K[X, Y]$, be a square $m \times m$ matrix such that $\deg a_{ij} - \deg a_{i(j+1)} \le 0$ is independent from i and satisfying the conditions:*
    i) $a_{ij} = 0$ if $i \ne j, j+1$
    ii) a_{ii} is a linear form L_i,
    iii) $a_{(i+1)i}$, is any form G_i, such that L_j is not a factor of G_i if $L_j \ne L_i$ and G_i is a multiple of L_i iff every $L_j, j > i$ is different from L_i.
    Then the inessential columns of A are s.i.

Proof. The inessential column we are considering is of the form

$$C_j = {}^t (0, \ldots, 0, L_j, G_j, 0, \ldots, 0),$$

where L_j does not divide G_j, so that no L_i divides G_j. Let us replace such a C_j with $C_j' = \sum_i C_i,\ \lambda_j = 1$ and prove that C_j' is still inessential. If h is the first index for which $\lambda_h \ne 0$, we point our attention on the column C_h (clearly, $h \ge j$). Let us distinguish two possible situations.

i) C_h *is essential*, so that $C_h = {}^t (0, \ldots, 0, -L_h, aL_h, 0, \ldots, 0)$. In this case we have $C_h \ne C_j$, so that $h < j$ and the entries of C_h must have the same degree of the corresponding entries of C_j (in particular, $a \in K^*$.)

   - If $\lambda_{h+1} \ne 0$, let us consider $C_{h+1} = {}^t (0, \ldots 0, L_{h+1}, G_{h+1}, 0, \ldots, 0)$. The entries of C_j' in position (h, j) and $(h+1, j)$ are respectively $c_{hj}' = \lambda_h L_h$ and $c_{(h+1)j}' = a\lambda_h L_h + \lambda_{h+1} L_{h+1}$, so that $\mathfrak{I}_{C_j'} = (L_h, L_{h+1}) = \mathfrak{M}$, as L_h is independent from L_{h+1}.

   - If $\lambda_{h+1} = \lambda_{h+2} = \ldots \lambda_{h+u-1} = 0, \lambda_{h+u} \ne 0, u > 1$, then necessarily $h + u \ge j$, so that C_j' has as elements $c_{hj}' = \lambda_h L_h$ and $c_{h+u}' = \lambda_{h+u} L_{h+u},\ \lambda_{h+u} \in K^*$; as a consequence, also in this case $\mathfrak{I}_{C_j'} = \mathfrak{M}$.

ii) C_h *is inessential*, so that $C_h = {}^t (0, \ldots, 0, -L_h, aL_h, 0, \ldots, 0)$, where G_h is not divisible for $L_q, q = 1, \ldots, m$. (As a special case, C_h might coincide with C_j.) Let us denote $h + u$ the least integer v for which $L_v = L_h$.

   1- If $u = 1$, then $c_{h,j}' = \lambda_h L_h,\ c_{(h+1)j}' = \lambda_h G_h + \lambda_{h+1} L_h$, so that $\mathfrak{I}_{C_j'} \supseteq (L_h, G_h) \supseteq \mathfrak{M}^t$, for some $t \in \mathbf{N}$.

   2- If $u \ne 1$ but $\lambda_{h+1} = 0$, then $c_{hj}' = L_h,\ c'(h+1)j = G_h$, so that $\mathfrak{I}_{C_j'} \supseteq (L_h, G_h)$, as in the previous case.

3- If $u \neq 1$, $\lambda_{h+1} \neq 0$, then $c'_{hj} = \lambda_h L_h$, $c'_{(h+1)j} = \lambda_h G_h + \lambda_{h+1} L_{h+1}$, where $\lambda_h \in K^*$ (as $h \leq j$).

If λ_{h+1} is such that $c'_{(h+1)j}$ is not a multiple of L_h, we get $C'_j \supseteq \mathfrak{M}^t$, for some $t \in \mathbf{N}$. However, for some choice of λ_{h+1} it may happen $c'_{(h+1)j} = L_h P$. In fact, if $G_h = M_1 \ldots M_s$ is a decomposition of G_h into linear factors, there exists $a \in K^*$ such that $\lambda_{h+1} = a M_1 \ldots M_{s-1}$ gives $c'_{(h+1)j} = M_1 \ldots M_{s-1}(\lambda_h M_s + a L_{h+1})$, where $\lambda_h M_s + a L_{h+1} = b L_h$, as M_s, L_{h+1} are linearly independent linear forms. Let us observe that such a λ_{h+1} cannot be a multiple of L_h, as G_h is not so. If we replace C_h with $C_h^* = \lambda C_h + \lambda_{h+1} C_{\lambda+1} C_{h+1}$ and consider $C'_j = C_h^* + \sum_{i=h+2}^m \lambda_i C_i$, we have a situation very similar to the previous one. In fact $c*_{hj} = \lambda_h L_h$, $c^*_{(h+1)j} = P L_h$, $c^*_{h+2,j} = \lambda_{h+1} G_{h+1}$, so that G_h is replaced with $\lambda_{h+1} G_{h+1}$, which is not a multiple of L_h. Now, we can repeat the same reasoning until when we find either case 2, if $\lambda_i = 0$ for some i with $h + 1 < i \leq h + u$, or case 1, for $i = h + u$.

$\square$

REMARK 4.9. *The essential generators of $\mathcal{B}(\mathfrak{I}) - \{\Phi\}$ are exactly the ones that do not contain as their factors all the linear factors of Φ; more precisely, $F_{i\mu_i}$ does not contain H_i, while it contains as factors $H_j, j \neq i$.*

In the sequel we will need also bases slightly different from the one produced in Proposition 4.6. We introduce them in the following Remarks.

REMARK 4.10. *If, in the definition of $F_{ij}, i > k$, $\mathbb{C}_i$ is replaced by $\tilde{C}_i = H_1 \ldots \hat{H}_k \ldots H_{i-1} U^{\nu_i + 1}$ (that is, if H_k is replaced with U), then $\tilde{\mathcal{B}}$, obtained from $\mathcal{B}$ by replacing F_{ij} with $\tilde{F}_{ij} = A_{ij} \tilde{C}_i$, is still a standard basis, whose Hilbert matrix $\tilde{M}(\mathfrak{I})$ differs from the $M(\mathfrak{I})$ described in Proposition 4.6 just in the column corresponding to $\tilde{F}_{k\mu_k}$, which becomes $\tilde{C}_k = {}^t (0, ..., -H_k, U, 0, ...0)$. The consequence is that the generator $\tilde{F}_{k\mu_k} = F_{k\mu_k}$ now is inessential, while the other generators are changed but remain with unchanged nature. $\tilde{\mathcal{B}}$ is not an e-maximal basis, but it will erase in a splitting (see Remark 4.18).*

REMARK 4.11. *Let us observe that the F_{ij}'s have U^{t+1} as a common factor. If we replace U^{t+1} by any form η, of degree $t + 1$, such that $G.C.D.(\eta, \Phi) = 1$, the matrix $M^*(\mathfrak{I})$ corresponding to the new basis $\mathcal{B}^*$ differs from $M(\mathfrak{I})$ only in the first column. In particular, $\mathcal{B}^*$ is still an e-maximal basis.*

REMARK 4.12. *Let us produce other H.B. canonical matrices of $\mathfrak{I}$, relative to standard bases different from the one described in Proposition 4.6 . They are defined as follows:*

$$M'(\mathfrak{I}) = \begin{pmatrix} U^{t+1} & \\ & \mathcal{A}' \\ \mathcal{O} & \end{pmatrix},$$

where $\mathcal{A}' = (a'_{ij})$ is a square $\delta \times \delta$ matrix, whose elements different from $a'_{ii}, a'_{(i+1)i}$ are zero, and

- $(a'_{11}, \ldots, a'_{\delta\delta}) = (-H_{\sigma(1)}, \ldots, -H_{\sigma(\delta)})$, with σ any permutation of the sequence $([1]^{\mu_1}, ([2]^{\mu_2}, \ldots, ([v]^{\mu_v}),$

- $a'_{(i+1),i} = -a'_{ii}$ if $a'_{ii} \neq a'_{jj}, j > i$ and $a'_{(i+1),i} = U$ otherwise.

In fact, Lemma 4.8 guaranties that all the inessential columns of $M'(\mathfrak{I})$ are s.i. and it is a matter of computation to check that the maximal minors of the new matrix are still the basis of a subspace T such that $\Phi S_t \bigoplus T = S_b$. The maximal minors of $M'(\mathfrak{I})$, different from Φ, apart from a sign are: $(\mathcal{B}'_i = U^{t+1} G_1 \ldots G_i \hat{H}_{\sigma(i)} H_{\sigma(i+1)} \ldots H_{\sigma(\delta)})$, $i = 1, \ldots, \delta$. A reasoning analogous to the one in the proof of Lemma 4.3 shows that they are linearly independent. In fact the relation $\lambda_1 H_{\sigma(1)} \ldots H_{\sigma(\delta)} + \sum_{i=2}^{\delta-1} = 0$, $(\lambda_1, \ldots, \lambda_\delta) \neq (0, \ldots, 0)$ implies that G_1 divides $H_{\sigma(1)} \ldots H_{\sigma(\delta)}$, against the hypothesis.

Moreover, let us denote T' the K-space generated by $(\mathcal{B}'_1, \ldots, \mathcal{B}'_\delta)$. Then $\Phi S_t \bigcap T' = (0)$, because $\Lambda \Phi = \sum a_i U^{t+1} G_1 \ldots G_i \hat{H}_{\sigma(i)} H_{\sigma(i+1)} \ldots H_{\sigma(\delta)})$, $\Lambda \neq 0$, implies that U must divide Φ, for degree reason, against the hypothesis.

EXAMPLE 4.13. *Let us consider the ideal*

$$\mathfrak{I} = (H_1^3 H_2^2 H_3)S + S_8 S,$$

where H_1, H_2, H_3 are linearly independent linear forms. The basis considered in Proposition 4.6 is $\mathcal{B}(\mathfrak{I}) = (\Phi, F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, F_{31})$, where $\Phi = H_1^3 H_2^2 H_3$, $F_{11} = H_1^2 H_2^2 H_3 U^3$, $F_{12} = H_1 H_2^2 H_3 U^4$, $F_{13} = H_2^2 H_3 U^5$, $F_{21} = H_1 H_2 H_3 U^5$, $F_{22} = H_1 H_3 U^6$, $F_{31} = H_1 H_2 U^6$.

The corresponding H.B. matrix is

$$M(\mathfrak{I}) = \begin{pmatrix} U^3 & -H_1 & & & & \\ & U & -H_1 & & & \\ & & U & -H_1 & & \\ & & & H_1 & -H_2 & \\ & & & & U & -H_2 \\ & & & & & H_2 & -H_3 \end{pmatrix},$$

where the unwritten entries are zero forms. The essential elements are: Φ, F_{13}, F_{22}, F_{31}. All the other elements are s.i..

- If in each generator of degree 8 we replace U^3 by any degree 3 form η, with $G.C.D.(\eta, \Phi) = 1$, we obtain a new e-maximal basis.

- If we replace F_{21}, F_{22}, F_{31} respectively by $\tilde{F}_{21} = H_2 H_3 U^6$, $\tilde{F}_{22} = H_3 U^7$, $\tilde{F}_{31} = H_2 U^7$, then in the new matrix $\tilde{M}(\mathfrak{I})$ the H_1 in $(4,4)$ position is replaced by U. As a consequence, $\tilde{F}_{11} = F_{11}$ and $\tilde{F}_{12} = F_{12}$ are s.i., while $\tilde{F}_{13} = F_{13}$ is inessential, but not strongly and $\tilde{F}_{21} \neq F_{21}$ is s.i..

- If we replace F_{31} by $\tilde{F}_{31} = H_1 U^7$ (or, equivalently, in $M(\mathfrak{I})$ the form H_2 in position $(6,6)$ is replaced by U), then F_{22} becomes inessential (but not strongly), while the nature of the other generators does not change.

The two special cases just considered suggest us to afford the general case pointing our attention on the H.B. matrix, more than on the standard basis. We need a decomposition of the Φ's appearing in (3) into pairwise independent linear forms, as follows

$$\Phi_k = H_{k1}^{\mu_{k1}} \dots H_{k2}^{\mu_{k2}} \dots H_{kv_k}^{\mu_{kv_k}}, \ k = 1, \dots, r. \tag{14}$$

Moreover, it is convenient to choose a set of linear forms $\{U, L_0, \dots, L_{\beta_0}\}$ such that the elements of the set $\{U, L_i, H_{kj}\}$, $i = 0, \dots, \beta_0$, $k = 1, \dots, r$, $j = 1, \dots, v_k$ are pairwise linearly independent and define

$$\Phi_0 = L_0 \dots L_{\beta_0}. \tag{15}$$

With this notation we can state the following proposition.

PROPOSITION 4.14. *A canonical matrix of the ideal $\mathfrak{I}$ of (3) is the following one*

$$M(\mathfrak{I}) = \begin{pmatrix} \mathcal{B} & \mathcal{O} \\ \mathcal{C} & \mathcal{A} \end{pmatrix},$$

where:

i) $\mathcal{B} \in S^{\beta_0 \times (\beta_0+1)}$, $\mathcal{A} \in S^{\delta \times \delta}$, $\mathcal{C} \in S^{\delta \times (\beta_0+1)}$, $\mathcal{O}$ *is a zero matrix, whose elements are of degree ≤ 0.*

ii) $\mathcal{B} = (b_{ij})$, *where:* $b_{ii} = L_{i-1}$; $b_{i(i+1)} = -L_i$; $b_{ij} = 0$ *if $j \neq i, i+1$.*

iii) $\mathcal{C} = (c_{ij})$, *where:* $c_{1(\beta_0+1)} = U^{t_1+1}$; $c_{ij} = 0$ *if $(ij) \neq (1(\beta_0 + 1))$*

iv) $\mathcal{A} = (a_{ij})$, *where:*

-   $a_{ij} = 0$ *if $j \neq i, j \neq i-1$,*
-   $(a_{11}, \dots, a_{\delta,\delta}) = ((-H_{11})^{[\mu_{11}]}, (-H_{12})^{[\mu_{12}]}, \dots, (-H_{1v_1})^{[\mu_{1v_1}]}, \dots$
    $\dots, (-H_{k1})^{[\mu_{k1}]}, (-H_{k2})^{[\mu_{k2}]}, \dots, ((-H_{kv_k})^{[\mu_{kv_k}]}, \dots$
    $\dots, (-H_{r1})^{[\mu_{r1}]}, (-H_{r2})^{[\mu_{r2}]}, \dots, (-H_{rv_r})^{[\mu_{rv_r}]}),$

- $a_{(i+1)i} = -a_{ii}$ *if* $a_{ii} \neq a_{jj}, j > i, \ i \neq \Delta_k,$

     $a_{(i+1)i} = -a_{ii}U^{t_k}$ *if* $a_{ii} \neq a_{jj}, j > i, \ i = \Delta_k,$

     $a_{(i+1)i} = -U$ *if* $(\exists j > i) \ a_{ii} = a_{jj}, \ j \neq \Delta_k, \ k < r,$

     $a_{(i+1)i} = U^{t_k+1}$ *if* $(\exists j > i) \ a_{ii} = a_{jj}, \ i = \Delta_k, \ k < r.$

Moreover, the inessential columns of $M(\mathfrak{I})$ are s.i..

Proof. We first observe that the degree matrix of $M(\mathfrak{I})$ is the expected one. Let us denote $\mathfrak{I}$ the ideal generated by the maximal minors of $M(\mathfrak{I})$ and prove that $\mathfrak{I}$ is the one described in (3). As $\mathcal{B}$ is the matrix considered in Proposition 4.2, it is immediate to see that $\mathfrak{I}_{\alpha_0} = \Phi S_{\beta_0}$ and that the minors of $\mathcal{B}$ are linearly independent.

The minors in degree α_1 have as a common factor Φ/Φ_1. So, it is enough to prove that, divided by their common factor, they are a basis of a subspace T_1 of S_{β_1} such that $S_{\beta_1} = \Phi_1 S_{\beta_1-\delta_1} \bigoplus T_1$. But we are in the situation described in Lemma 4.3, where:

$$t = t_1, \ b = \beta_1, \ \Phi = \Phi_1, \ H_i = H_{1i},$$
$$C_i = \Phi_0 L_{\beta_0}^{-1} U^{t_1+1} a_{21} \ldots a_{(j+1)j}, \ j = \mu_{11} + \mu_{12} + \ldots + \mu_{1(i-1)}, \ i = 1, \ldots, v_1.$$

So, let us suppose the statement true until the degree α_{k-1} and prove it for α_k. Just as in the case $k = 1$, we see that all the minors have as a common factor $\Phi_{k+1} \ldots \Phi_r = \Phi/\Phi_1 \ldots \Phi_k$. So, it is enough to show that, divided by this factor, they are a basis of a subspace T_k of S_{β_k} such that $S_{\beta_k} = \Phi_k S_{\beta_k-\delta_k} \bigoplus T_k$. We are again in the situation of Lemma 4.3, with:

$$t = t_k, \ b = \beta_k, \ \Phi = \Phi_k, \ H_i = H_{ki}$$
$$C_i = \Phi_0 L_{\beta_0}^{-1} U^{t_1+1} a_{21} \ldots a_{(\Delta_{k-1}+j+1)(\Delta_{k-1}+j)},$$
$$j = \mu_{k1} + \mu_{k2} + \ldots + \mu_{k(i-1)}, \ i = 1, \ldots, v_k.$$

Thanks to Proposition 3.2, we immediately see that the inessential columns are exactly the ones in which $a_{(i+1)i}$ is not a multiple of a_{ii} or, equivalently, the ones whose element a_{ii} is equal to some a_{jj}, with $j > i$. The proof that they are s.i. is a consequence of Lemma 4.8. $\qquad\square$

Extending the notation used in Proposition 4.6, we denote the basis linked to the canonical matrix of Proposition 4.14 as follows:

$$\mathcal{B}(\mathfrak{I}) = (\mathcal{B}^0, \mathcal{B}^1, \ldots, \mathcal{B}^k \ldots, \mathcal{B}^r), \quad \text{where}$$
$$\mathcal{B}^0 = (F_j^0), \ j = 0, \ldots, \beta_0, \ \mathcal{B}^k = (F_{ij}^k),$$
$$k = 1, \ldots, r, \ i = 1, \ldots, v_k, \ j = 1, \ldots, \mu_{ki}.$$

With this notation we can state the following corollary.

COROLLARY 4.15. *i) All the elements of $\mathcal{B}^0$ are essential. The generator $F_{ij}^k \in \mathcal{B}^k$ is essential iff $j = \mu_{ki}$ and H_{ki} is not a factor of it.*

ii) $\mathcal{B}(\mathfrak{I})$ is an e-maximal basis and the number of its essential elements in degree bigger then $\alpha(\mathfrak{I})$ is equal to the number v of the distinct linear factors appearing in a factorization of Φ.

iii) In any e-maximal basis the essential generators appearing in degree α_k are as many as the linear factors of Φ_k that do not divide $\Phi_{k+1} \dots \Phi_r$.

iv) $\mathfrak{I}$ admits a basis of essential elements iff Φ is a product of distinct linear factors.

Proof. i) From Proposition 4.14 iv) we easily see that the essential columns of $\mathcal{A}$ are the ones whose entry a_{hh} is different from every $a_{jj}, j > h$. This happens iff $a_{hh} = -H_{ki}$, where H_{ki} does not appear any more in the diagonal of $\mathcal{A}$, in position $(j,j), j > h$. A necessary condition for such a situation is that the generator corresponding to that column is of the kind $F_{i\mu_{ki}}^k$. In this case we have: $\prod_{j>h} a_{jj} = R\Phi_{k+1} \dots \Phi_r$, where H_{ki} is not a factor of R. So, the condition characterizing the essential $F_{i\mu_{ki}}^k$'s is that $\Phi_{k+1} \dots \Phi_r$ is not a multiple of H_{ki}. From the equality $F_{i\mu_{ki}}^k = \prod_{j>h} a_{jj} \prod_{j<h} a_{(j+1)j}$ we see that the previous condition is equivalent to say that H_{ki} does not divide $F_{i\mu_{ki}}^k$.

ii) $\mathcal{B}(\mathfrak{I})$ is an e-maximal basis, because its inessential elements are s.i. (Theorem 3.5). Moreover, the H_{ki} appearing in an essential column corresponding to $F_{i\mu_{ki}}^k$ is a linear factor of Φ, making there its last appearing as an element of the diagonal of $\mathcal{A}$. So, the essential columns of $\mathcal{A}$ are as many as the distinct linear factors of Φ.

iii) As the number of essential elements in an e-maximal basis does not depend on the e-maximal basis chosen, it is enough to verify the statement on the basis $\mathcal{B}(\mathfrak{I})$ of Proposition 4.14. In the proof of i) we observed that the essential elements of $\mathcal{B}^k$ are as many as the linear factors H_{ki} of Φ_k that are not divisors of $\Phi_{k+1} \dots \Phi_r$.

iv) This is an obvious consequence of ii).

$\square$

Corollary 4.15 completes the proof of Theorem 4.1.

COROLLARY 4.16. *Let $\mathfrak{I}$ be represented as in (3), with $\Phi_k = H_{k1}^{\mu_{k1}} \dots H_{kv_k}^{\mu_{kv_k}}$. If τ_k is the number of distinct linear factors that Φ_k has in common with $\Phi_{k+1} \dots \Phi_r$, then any e-maximal basis of $\mathfrak{I}$ has exactly $\sum_{j=1}^{v_k} (\mu_{kj} - 1) + \tau_k$ strongly inessential generators in degree α_k.*

Corollary 4.16 implies that it is possible to find $\mathfrak{I} \in F[2]$ with a prescribed number of strongly inessential elements in a prescribed number of sufficiently high degree, as we see in the following proposition.

PROPOSITION 4.17. *Let $(d_1 < d_2 < \ldots < d_s)$ and $(r_1, r_2, \ldots, r_s)$ be sequences of natural numbers. There exist ideals $\mathfrak{I} \in F[2]$ with exactly r_i s.i. elements in degree d_i, $i = 1, \ldots, s$, iff*

$$d_1 > \sum_{i=1}^{s} r_i + 1. \tag{16}$$

Proof. Let us observe that the minimal degree δ of a form Φ satisfying the condition $\delta - v = \sum_{i=1}^{s} r_i$ is obtained with $v = 1$, so that Φ looks as $\Phi = H^{m+1}$, where $m = \sum_{i=1}^{s} r_i$ and H is any linear form. So, condition (16) is necessary. It is also sufficient, because the ideal

$$\begin{aligned} \mathfrak{I} \;=\; & H^{m+1}S + H^{m+1-r_1}S_{d_1-(m+1-r_1)} + \cdots \\ & \cdots + H^{m+1-\sum_{i=1}^{j} r_i}S_{d_j-(m+1-\sum_{i=1}^{j} r_i)} + \ldots + S_{d_s} \end{aligned} \tag{17}$$

obtained with the choice $\Phi_j = H^{r_j}, j = 1, \ldots, (s-1)$, $\Phi_s = H^{r_s+1}$, satisfies the required condition. If $d_1 = \sum_{i=1}^{s} r_i + 2$, then (17) is the unique ideal satisfying the condition. If $d_1 > \sum_{i=1}^{s} r_i + 2$, there are many other possibilities. In fact, the set of the ideals satisfying the required condition increases with the degree $\delta = v + m$, or, equivalently, with the number v of different linear factors of Φ. Let us observe that the degree vector of the ideal $\mathfrak{I}$ considered in (17) is the least compatible with the required condition. $\qquad\square$

REMARK 4.18. *Every $\mathfrak{I} \in F[2]$ satisfies condition (2) (maximality with respect to Dubreil-Campanella inequality) in each degree α_i. So, for every j, $\mathfrak{I}$ splits into two ideals, $\mathfrak{I}' = (\mathfrak{I} : (\Phi_{j+1} \ldots \Phi_r))$ and $\mathfrak{I}'' = (\mathfrak{I}, \Phi_{j+1} \ldots \Phi_r)$, both elements of $F[2]$. The first $\beta_0 + 1 + \Delta_j$ rows and $\beta_0 + \Delta_j$ columns of the matrix $M(\mathfrak{I})$ produced in Proposition 4.14 form a H.B. matrix of $\mathfrak{I}'$, whose inessential columns are not necessarily s.i.. In fact, it may happen that a linear factor of $\Phi_i, i \leq j$ does not divide $\Phi_{i+1} \ldots \Phi_j$ but divides $\Phi_{j+1} \ldots \Phi_r$; so the assertion of Remark 3.11 is justified.*

EXAMPLES 4.19. *In the following examples U, H, K, L_0, L_1, L_2 are linear forms, pairwise linearly independent.*

   1- $\mathfrak{I} = H^3 K^2 S_2 S + K^2 S_6 S + S_{10} S.$

     In this case we have: $\Phi = H^3 K^2, \Phi_1 = H^3, \Phi_2 = K^2$, $GCD(\Phi_1, \Phi_2) = 1$. According to Proposition 4.14, we get

$$M(\mathfrak{I}) = \begin{pmatrix} L_0 & -L_1 & & & & & \\ & L_1 & -L_2 & & & & \\ & & L_2U & -H & & & \\ & & & U & -H & & \\ & & & & U & -H & \\ & & & & & HU^2 & -K \\ & & & & & & U & -K \end{pmatrix}.$$

The corresponding canonical basis is

$$\mathcal{B}(\mathfrak{I}) = (H^3K^2(L_1L_2, L_0L_2, L_0L_1); K^2L_0L_1L_2U(H^2, HU, U^2);$$
$$L_0L_1L_2HU^5(K, U)).$$

There are 3 s.i. generators, according to the fact that $\delta = 5, v = 2$. Let us observe that in this example a s.i. generator gives rise to a s.i. generator in any splitting.

2- $\mathfrak{I} = H^3K^2S_2S + HKS_6S + S_{10}S.$

In this case we have: $\Phi = H^3K^2, \Phi_1 = H^2K, \Phi_2 = HK$, so that all the linear factors of Φ_1 are also divisors of Φ_2. According to Proposition 4.14, we get

$$M(\mathfrak{I}) = \begin{pmatrix} L_0 & -L_1 & & & & & \\ & L_1 & -L_2 & & & & \\ & & L_2U & -H & & & \\ & & & U & -H & & \\ & & & & U & -K & \\ & & & & & U^3 & -H \\ & & & & & & H & -K \end{pmatrix}.$$

The corresponding canonical basis is

$$\mathcal{B}(\mathfrak{I}) = (H^3K^2(L_1L_2, L_0L_2, L_0L_1); HKL_0L_1L_2U(HK, UK, U^2);$$
$$L_0L_1L_2U^6(K, H)).$$

There are 3 s.i. generators, according to the fact that $\delta = 5, v = 2$.

Let us observe that in this case the splitting in degree $p = 8$ gives rise to the matrices

$$M(\mathfrak{I}') = \begin{pmatrix} L_0 & -L_1 & & & \\ & L_1 & -L_2 & & \\ & & L_2U & -H & \\ & & & U & -H & \\ & & & & U & -K \end{pmatrix},$$

and, respectively,

$$M(\mathfrak{I}'') = \begin{pmatrix} U^9 & -H & \\ & H & -K \end{pmatrix}.$$

The 6-th column of $M(\mathfrak{I}')$ is inessential, but not s.i.. A H.B. matrix of $\mathfrak{I}'$, whose corresponding inessential generators are s.i., can be obtained from $M(\mathfrak{I}')$ just by replacing U with H in its 6-th column.

3- $\mathfrak{I} = H^3 S_2 + H^2 S_4 + H S_8 + S_{10}$.

In this case we have: $\Phi = H^3, \Phi_1 = \Phi_2 = \Phi_3 = H$. According to Proposition 4.14, we get:

$$M(\mathfrak{I}) = \begin{pmatrix} L_0 & -L_1 & & & \\ & L_1 & -L_2 & & \\ & & L_2 U & -H & \\ & & & U^4 & -H \\ & & & & U^2 & -H \end{pmatrix}.$$

There are 2 s.i. generators and the corresponding canonical basis is:

$$\mathcal{B}(\mathfrak{I}) = (H^3(L_1 L_2, L_0 L_2, L_0 L_1); H^2 L_0 L_1 L_2 U; H L_0 L_1 L_2 U^5; L_0 L_1 L_2 U^7)$$

The splittings in degrees 8 and 9 produce a new inessential, but not s.i., element in $\mathfrak{I}'$.

5. Behaviour of $\mathfrak{I} \in F[3]$ with respect to essentiality: special cases and examples

According to Theorem 1.5 of [3], every element $\mathfrak{I} \in F[3]$ has a shape very similar to the one described in (3) for the elements of $F[2]$. The difference is that S_{β_i} is replaced by a linear subspace $T_{\beta_i} \subseteq S_{\beta_i}$ of $S = K[X, Y, Z]$, where $\dim T_{\beta_i} = \beta_i + 1$. The subspaces T_{β_i} are characterized by Theorem 3.4 of [3]. That theorem says that, up to a change of cohordinates, every element $\mathfrak{I} \in F[3]$ is generated by the maximal minors of an $\alpha \times (\alpha+1)$ matrix, obtained by lifting to $K[X, Y, Z]$ a H.B. matrix of its image $\bar{\mathfrak{I}} \subset K[X, Y] = \bar{S}$, modulo a regular linear form Z. So, $\mathfrak{I}$ looks like

$$\mathfrak{I} = \Phi_1 \ldots \Phi_r T_{\beta_0} S + \ldots + \Phi_i \ldots \Phi_r T_{\beta_{i-1}} S + \Phi_{i+1} \ldots \Phi_r T_{\beta_i} S + \ldots$$

$$\ldots + \Phi_r T_{\beta_{r-1}} S + T_{\beta_r} S = \sum_{i=0}^{r} \Phi_{i+1} \ldots \Phi_{r+1} T_{\beta_i} S,, \quad (18)$$

where Φ_i is a form in S and $\Phi_{r+1} = 1$, and its image modulo Z becomes

$$\bar{\mathfrak{I}} = \bar{\Phi}_1 \dots \bar{\Phi}_r \bar{S}_{\beta_0} \bar{S} + \dots + \bar{\Phi}_i \dots \bar{\Phi}_r \bar{S}_{\beta_{i-1}} \bar{S} + \bar{\Phi}_{i+1} \dots \bar{\Phi}_r \bar{S}_{\beta_i} \bar{S} + \dots$$

$$\dots + \bar{\Phi}_r \bar{S}_{\beta_{r-1}} \bar{S} + \bar{S}_{\beta_r} \bar{S} = \sum_{i=0}^{r} \bar{\Phi}_{i+1} \dots \bar{\Phi}_{r+1} \bar{S}_{\beta_i} \bar{S}, \ \bar{\Phi}_{r+1} = 1. \quad (19)$$

The problem of stating if a lifting of $\bar{\mathfrak{I}} \in F[2]$ to $\mathfrak{I} \in F[3]$ preserves the strong inessentiality of the entries of $\mathcal{B}(\bar{\mathfrak{I}})$ becomes a lifting problem of H.B. matrices, which seems not easy to be solved. So, we start to consider a very special case. More precisely, we focus our attention on the ideals $\mathfrak{I} \in F[3]$ with the largest number of s.i. generators in any e-maximal basis. If $\alpha = \alpha(\mathfrak{I}) = \alpha(\bar{\mathfrak{I}})$ is the minimal degree of the generators of $\mathfrak{I}$, we will see that the maximal expected number is $\alpha - 2$; we'll prove that such a number is reached. Let us first state a property for every homogeneous saturated ideal $\mathfrak{I}$ of $S = K[X_1, \dots, X_n]$.

PROPOSITION 5.1. *Let $\mathcal{B} = (b_1, \dots, b_h, c_1, \dots, c_k)$, $k \geq 1$, be an e-maximal basis of the saturated homogeneous ideal $\mathfrak{I} \subset S = K[X_1, \dots, X_n]$, where $b_1, \dots, b_h$ are essential and $c_1, \dots, c_k$ are s.i. elements. The condition depth $\mathfrak{I} = r$ implies $h > r$.*

Proof. Thanks to Corollary 5.3 of [4], $(c_1, \dots, c_k)$ is an inessential set ([4], Definition 5.2), so that $\mathfrak{I} = (b_1, \dots, b_h)^{sat}$. As depth $\mathfrak{I} = $ depth $(b_1, \dots, b_h)$, the hypothesis implies $h \geq r$; however, the equality holds iff $(b_1, \dots, b_h)$ is a c.i. and, as a consequence, a saturated ideal, against the hypothesis $k \geq 1$. $\square$

Choosing $h = 2$, we get immediately the following statement.

COROLLARY 5.2. *The largest possible number of s.i. generators in an e-maximal basis of an ideal $\mathfrak{I} \in F[3]$ is $\alpha(\mathfrak{I}) - 2$.*

If $\alpha(\mathfrak{I}) = 2$, then $\mathfrak{I} \in F[3]$ has 3 generators and Corollary 5.2 says that in every e-maximal basis they are essential. Let us point our attention on the case $\alpha(\mathfrak{I}) > 2$.

We will use the following Notation

$$\mathcal{S} = \{\mathfrak{I} \in F[3] : \nu_e(\mathfrak{I}) = 3, \ \alpha(\mathfrak{I}) > 2\},$$

where $\nu_e(\mathfrak{I})$ denotes the number of essential elements of any e-maximal basis of $\mathfrak{I}$ (see [4], Definition 5.1).

Let us observe that any dehomogenization $\mathfrak{I}_*$ with respect to a regular linear form of an ideal $\mathfrak{I} \in \mathcal{S}$ has just 3 generators (that is the least number for a non complete intersection), while the number of generators of $\mathfrak{I}$ is the maximum allowed by Dubreil's inequality.

PROPOSITION 5.3. *For every $\mathfrak{I} \in \mathcal{S}$ the form Φ appearing in (18) is of degree $\delta = \alpha(\mathfrak{I})$ and Φ has necessarily one of the following shapes:*

i) $\Phi = H^\delta$,

ii) $\Phi = H^r K^s$, $r + s = \delta$,

iii) $\Phi = C^\gamma, 2\gamma = \delta$,

where H and K are independent linear forms and C is a quadratic irreducible form in $K[X, Y, Z]$.

Proof. An immediate consequence of Proposition 3.8 is that if $\mathfrak{I} \subset K[X, Y, Z]$ has $\alpha - 2$ s.i. generators, then the number of s.i. generators of its quotient $\bar{\mathfrak{I}}$ modulo a regular linear form is either $\alpha - 2$ or $\alpha - 1$. Applying Theorem 4.1 to $\bar{\mathfrak{I}}$, we immediately get

$$\delta - v = \alpha - 1, \ \delta \le \alpha, \tag{20}$$

or

$$\delta - v = \alpha - 2, \ \delta \le \alpha. \tag{21}$$

Relation (20) is equivalent to $\delta = \alpha$, $v = 1$, while relation (21) gives two possible situations:

$$\delta = \alpha, \ v = 2 \tag{22}$$

and

$$\delta = \alpha - 1, \ v = 1. \tag{23}$$

Let us verify that (23) is not realized. In fact in this case we have $(\bar{\mathfrak{I}})_\alpha = \bar{H}^{\alpha-1} \bar{S}_1$ and the s.i. generators lie all in degree bigger than α; so, a splitting in degree α (see Theorem 3.9) gives rise to an ideal $\mathfrak{I}''$, with $\alpha(\mathfrak{I}'') = \alpha - 1$ and $\alpha - 2$ s.i. generators, against Corollary 5.2.

So, Φ must be a form, of degree α, whose quotient modulo any regular linear form splits into a product of powers of at most two different linear factors. This means that the curve $\Phi = 0$ meets a generic line in at most two different points, so that Φ is necessarily as described in i), ii), iii). $\square$

REMARK 5.4. 1- *We do not have examples in which the situation iii) appears. Let us observe that it requires every $\nu(\mathfrak{I}, j)$, $j = 1, \dots, r$, to be a power of 2.*

2- *Proposition 5.3 says that the schemes corresponding to ideals with $\alpha - 2$ s.i. generators lie necessarily either on a multiple line or on two multiple lines or (may be) on a multiple irreducible conic. However this condition is not sufficient. For instance, by lifting the canonical matrix M of an ideal $\mathfrak{J}$ of $K[X, Y]$ with $\alpha - 1$ s.i. generators with M itself, we obtain a basis for an ideal $\mathfrak{I} \subset K[X, Y, Z]$ without inessential elements.*

3- In case i), $\bar{\Phi}$ has the same structure of Φ, for every regular form L, while in case ii) and iii) we have generically $\bar{\bar{\Phi}} = \bar{H}^r \bar{K}^s$, but we get $\bar{\bar{\Phi}} = \bar{M}^{r+s}$, M a linear form in $K[X,Y]$, iff the line $L = 0$ either is tangent to $\Phi = 0$ (case iii)) or meets it in its singular point (case ii)). So, it is possible to represent an element $\mathfrak{I} \in \mathcal{S}$ as a lifting of ideals $\tilde{\mathfrak{I}} \subset k[X,Y]$ such that $\bar{\bar{\Phi}} = \bar{M}^\delta$, except for the case in which $\Phi = H^r K^s$ and the intersection between the two lines $H = 0$ and $K = 0$ is in the support of the corresponding scheme.

PROPOSITION 5.5. *If $\mathfrak{I}$ is an element of $\mathcal{S}$, let us consider its splitting into $\mathfrak{I}'$ and $\mathfrak{I}''$ (see Theorem 2.3), in degree α_k, with $k < r$ if $\deg \Phi_r \geq 2$ and $k < r-1$ if $\deg \Phi_r = 1$. Then $\mathfrak{I}''$ is still an element of $\mathcal{S}$.*

Proof. Theorem 3.9 says that if $\mathcal{B}(\mathfrak{I}) = (\Phi, \mathcal{B}^1, \mathcal{B}^2, \ldots, \mathcal{B}^r)$ is an e-maximal basis of $\mathfrak{I}$, then $\mathcal{B}(\mathfrak{I}'') = (\Phi'', \mathcal{B}^{k+1}, \ldots, \mathcal{B}^r)$, where $\Phi'' = \Phi_{k+1} \ldots \Phi_r$, is an e-maximal basis of $\mathfrak{I}''$ and the forms of $\mathcal{B}(\mathfrak{I}'')$, different from Φ'', maintain the same nature they had in $\mathcal{B}(\mathfrak{I})$. So, as $\mathcal{B}(\mathfrak{I})$ has two essential elements different from Φ, $\mathcal{B}(\mathfrak{I}'')$ cannot contain more then two essential elements different from Φ''; the hypothesis on the choice of k guaranties that it has at least 3 elements; then, Corollary 5.2 says that $\mathcal{B}(\mathfrak{I}'')$ must have exactly 3 essential elements, so that $\mathfrak{I}'' \in \mathcal{S}$. $\square$

COROLLARY 5.6. *In any e-maximal basis of an ideal $\mathfrak{I} \in \mathcal{S}$ the degree of the three essential generators are: $\alpha = \alpha(\mathfrak{I})$, $\alpha_r(\mathfrak{I})$ and either $\alpha_r(\mathfrak{I})$ or $\alpha_{r-1}(\mathfrak{I})$. The latter possibility takes place iff in degree $\alpha_r(\mathfrak{I})$ there is just one generator.*

Proof. It is enough to apply Proposition 5.5, with $k = r-1$ if in degree α_r the basis $\mathcal{B}(\mathfrak{I})$ contains at least two forms and with $k = r-2$ otherwise. In fact the two essential elements of $\mathcal{B}(\mathfrak{I}'')$, not in minimal degree, must be essential also in $\mathcal{B}(\mathfrak{I})$. $\square$

Now, let us produce examples of ideals of $\mathcal{S}$.

PROPOSITION 5.7. *Every $\bar{\mathfrak{I}} \subset K[X,Y]$ with $\alpha - 1$ s.i. generators in any e-maximal basis has at least a lifting in $\mathcal{S}$.*

Proof. After a possible change of coordinates, a H.B. matrix of $\bar{\mathfrak{I}}$ is

$$M(\bar{\mathfrak{I}}) = \begin{pmatrix} Y^{t_0} & -X & & & & \\ & Y^{t_1} & -X & & & \\ & & Y^{t_2} & -X & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & Y^{t_{\delta-2}} & -X \\ & & & & & Y^{t_{\delta-1}} & -X \end{pmatrix} \tag{24}$$

A lifting $M(\mathfrak{I})$ of $M(\bar{\mathfrak{I}})$, with $\alpha - 2$ s.i. columns, is

$$
\begin{pmatrix}
Y^{t_0} & -X & & & & & \\
ZP_1 & Y^{t_1} & -X & & & & \\
ZP_2 & Z^{t_1+t_2-1} & Y^{t_2} & -X & & & \\
ZP_3 & & Z^{t_2+t_3-1} & Y^{t_3} & & & \\
& \cdots & \cdots & \cdots\,\cdots & \cdots & \cdots & \\
ZP_{\delta-2} & & & & Y^{t_{\delta-2}} & -X & \\
ZP_{\delta-1} & & & & Z^{t_{\delta-2}+t_{\delta-1}-1} & Y^{t_{\delta-1}} & -X
\end{pmatrix}, \quad (25)
$$

where $P_i \in K[X,Y,Z]$ is a form of degree $\sum_{j=0}^{i} t_j - i - 1$ and, as usual, the unwritten entries are zero. $M(\mathfrak{I})$ is obtained from $M(\bar{\mathfrak{I}})$ by leaving unchanged the last two columns and replacing the zero entries in position $(i+1, i), i = 2, \ldots, \delta - 1$, with $Z^{t_{i-1}+t_i-1}$ and the ones in position $(i, 1)$ with ZP_{i-1}. It is immediate to verify that the second column is s.i. and Lemma 5.3 of [4] says that the same reasoning can be repeated for the following ones with three non zero entries. $\qquad\square$

Finally, let us point our attention on the ideals $\mathfrak{I} \in \mathcal{S}$ with the smallest $\alpha(\mathfrak{I})$ allowing the presence of s.i. generators. Corollary 5.2 implies that if $\mathfrak{I}$ has a s.i. generator then $\alpha(\mathfrak{I}) \geq 3$. So, let us look for the ideals with $\alpha(\mathfrak{I}) = 3$ and just one s.i. generator in every e-maximal basis; they are the elements of $\mathcal{S}$ with the smallest number of generators. Let us consider the special case of generators in two different degrees. Proposition 5.3 says that, apart from a coordinates change, they can be obtained by lifting an ideal of one of the following types

$$
\bar{\mathfrak{I}}_1 = X^3 S + S_\beta S, \ \beta > 3, \tag{26}
$$

$$
\bar{\mathfrak{I}}_2 = X^2 Y S + S_\beta S, \ \beta > 3. \tag{27}
$$

Let us first consider all the required liftings of $\bar{\mathfrak{I}}_1$ or, equivalently, all the liftings $M(\mathfrak{I}_1)$ of the matrix

$$
M(\bar{\mathfrak{I}}_1) = \begin{pmatrix} Y^t & -X & 0 & 0 \\ 0 & Y & -X & 0 \\ 0 & 0 & Y & -X \end{pmatrix}, \quad t = \beta - 2, \tag{28}
$$

having a s.i. column. $M(\mathfrak{I})$ has the following shape (see [3])

$$
M(\mathfrak{I}_1) = \begin{pmatrix} Y^t + ZP_1(X,Y,Z) & -X + a_{11}Z & a_{12}Z & a_{13}Z \\ ZP_2(X,Y,Z) & Y + a_{21}Z & -X + a_{22}Z & a_{23}Z \\ ZP_3(X,Y,Z) & a_{31}Z & Y + a_{32}Z & -X + a_{33}Z \end{pmatrix}, \tag{29}
$$

where $a_{ij} \in K$, $\deg P_i = t - 1$. The forms P_1, P_2, P_3 can be chosen arbitrarily among the ones of degree $t - 1$, so that we just have to characterize the matrix $\mathcal{A} = (a_{ij})$, $i, j = 1, 2, 3$. As the first column of $M(\mathfrak{I}_1)$ is essential for every choice of the a_{ij}'s, let us consider the second and third columns. The second column is s.i. iff the forms $L_1 = -X + a_{11}Z + \lambda_2 a_{12}Z + \lambda_3 a_{13}Z$, $L_2 = Y + a_{21}Z + \lambda_2(-X + a_{22}Z) + \lambda_3 a_{23}Z$, $L_3 = a_{31}Z + \lambda_2(Y + a_{23}Z) + \lambda_3(-X + a_{33}Z)$ are linearly independent, for every choice of λ_2, λ_3 or, equivalently, iff the matrix

$$\mathcal{B} = \begin{pmatrix} -1 & 0 & a_{11} + \lambda_2 a_{12} + \lambda_3 a_{13} \\ -\lambda_2 & 1 & a_{21} + \lambda_2 a_{22} + \lambda_3 a_{23} \\ -\lambda_3 & \lambda_2 & a_{31} + \lambda_2 a_{32} + \lambda_3 a_{33} \end{pmatrix}$$

has determinant different from zero. Such a condition gives the relation

$$-a_{12}\lambda_2^3 - a_{13}\lambda_2^2\lambda_3 + (a_{22} - a_{11})\lambda_2^2 + a_{13}\lambda_3^2 + (a_{12} + a_{23})\lambda_2\lambda_3$$
$$+(-a_{32} + a_{21})\lambda_2 + (-a_{33} + a_{11})\lambda_3 - a_{31} \neq 0. \quad (30)$$

An easy computation shows that the matrices $\mathcal{A}$ for which this condition is satisfied are

$$\mathcal{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ a_{31} & a_{21} & a_{11} \end{pmatrix}, \qquad a_{31} \neq 0. \quad (31)$$

Considerations very similar to the previous ones lead to the conclusion that the second column is s.i. iff the matrix $\mathcal{A}$ has the following shape

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{11} & -a_{12} \\ 0 & a_{21} & a_{11} \end{pmatrix}, \qquad a_{12} \neq 0. \quad (32)$$

In case (31), after the coordinate change $-a_{11}Z + X = X'$, $a_{21}Z + Y = Y'$, $a_{31}Z = Z'$ and dropping the apostrophes, the required matrix can be written

$$M(\mathfrak{I}_{11}) = \begin{pmatrix} Y^t + ZQ_1 & -X & 0 & 0 \\ ZQ_2 & Y & -X & 0 \\ ZQ_3 & Z & Y & -X \end{pmatrix} \quad (33)$$

In case (32), after the coordinate change $-a_{11}Z + X = X'$, $-a_{21}Z + Y = Y'$, $a_{12}Z = Z'$ and dropping the apostrophes, the required matrix can be written

$$M(\mathfrak{I}_{12}) = \begin{pmatrix} Y^t + ZQ_1 & -X & Z & 0 \\ ZQ_2 & Y & -X & -Z \\ ZQ_3 & 0 & Y & -X \end{pmatrix}, \quad t = \beta - 2. \quad (34)$$

Let us observe that both schemes relative to $\mathfrak{I}_{11}$ and $\mathfrak{I}_{12}$ are supported at at most $t + 1$ points lying on a triple line ($X = 0$ in our basis) and that their multiplicity is $e(\mathfrak{I}) = 3 + 3t$ ([6]).

With a very similar computation it is possible to see that, apart from a coordinate change, a lifting $\mathfrak{I}_2$ of $\bar{\mathfrak{I}}_2$ belongs to $\mathcal{S}$ iff its H.B. matrix has the following shape

$$M(\mathfrak{I}_2) = \begin{pmatrix} (X+Y)^t + ZQ_1 & -X & 0 & 0 \\ ZQ_2 & X+Y & -X & 0 \\ ZQ_3 & Z & X & -Y \end{pmatrix}, \qquad (35)$$

where P_1, P_2, P_3 are forms of degree $t-1$ in $K[X, Y, Z]$. The corresponding schemes, still of multiplicity $e(\mathfrak{I}) = 3t + 3$, are all supported at two different lines ($X = 0$ and $Y = 0$). The intersection of the two lines is one of the points in the support of the scheme; as a consequence, the ideals cannot be obtained by lifting an ideal of type (28).

The characterization of the elements of $\mathcal{S}$ with $\alpha(\mathfrak{I}) > 3$ is more difficult to be faced, also for ideals generated in two degrees. In fact, the request of (28) (and the analogous for the lifting of $\bar{\mathfrak{I}}_2$) are replaced by the requirement that a system of non linear equations $\{E_u(a_{ij}, \lambda_v) = 0\}$, in a set $\{\lambda_v, v = 1, \ldots, \alpha - 1\}$ of variables, admits no solutions. Such a condition defines the entries a_{ij}'s of the matrix $\mathcal{A}$ as the elements for which the ideal generated by the E_u's is the whole ring $K[\lambda_1, \ldots, \lambda_{\alpha-1}]$.

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Authors' addresses:

Giannina Beccari
Dipartimento di Scienze Matematiche
Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail: `giannina.beccari@polito.it`

Carla Massaza
Dipartimento di Scienze Matematiche
Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail: `carla.massaza@polito.it`