Rend. Istit. Mat. Univ. Trieste
Volume 44 (2012), 313–348

Solvable (and unsolvable) cases of the decision problem for fragments of analysis

Domenico Cantone, Eugenio G. Omodeo and Gaetano T. Spartà

Dedicated to Fabio Zanolin on the occasion of his 60th birthday

Abstract. *We survey two series of results concerning the decidability of fragments of Tarksi's elementary algebra extended with one-argument functions which meet significant properties such as continuity, differentiability, or analyticity. One series of results regards the initial levels of a hierarchy of prenex sentences involving a single function symbol: in a number of cases, the decision problem for these sentences was solved in the positive by H. Friedman and Á. Seress, who also proved that beyond two quantifier alternations decidability gets lost. The second series of results refers to merely existential sentences, but it brings into play an arbitrary number of functions, which are requested to be, over specified closed intervals, monotone increasing or decreasing, concave, or convex; any two such functions can be compared, and in one case, where each function is supposed to own continuous first derivative, their derivatives can be compared with real constants.*

Keywords: decidable theories, Tarski's elementary algebra, one-variable functions
MS Classification 2010: 03B25, 26A06

Introduction

We will address the decidability issue for various fragments of real analysis.

In the background, we have the fundamental decidability result proved by Tarski in [17] about the theory, named *elementary algebra*, where real *numbers* only—not functions—come into play. This result refers to the entire first-order language whose signature consists of the numerical constants 0, 1, -1, the operators $+, -, \cdot$, and the comparators $>, <, =$. As usual, an adequate basis of propositional connectives (e.g., $\wedge, \vee, \neg$) is also available, together with a denumerable infinity of variables: these are assumed to range over the reals and can be quantified by means of the symbols $\exists, \forall$, without restraints. Tarski produced an algorithm which, given any formula Φ devoid of free variables in this language, provides the yes/no answer as whether Φ is true or false.

Note that in elementary algebra each variable represents a generic real number. If there were means to impose that some variables range over integers, then one would be able to recast in elementary algebra all sentences of elementary arithmetic, and could thereby decide which of these sentences are true: an impossible situation, as shown by Church in [4].

A decision algorithm for elementary algebra could become part of a *proof assistant*, to wit, of a computerized system offering support to scholars either by way of autonomous theorem-proving abilities or through verification that proposed proofs are impeccable [9]. Anyway, for applications of this nature one must necessarily take into account the computational cost of the algorithm.

It turns out, in particular, that although the procedure proposed by Collins [5] has doubly exponential complexity relative to the number of variables occurring in the sentence (or just exponential, if the endowment of variables is finite and fixed), its computational cost is considerably lower than in case of Tarski's algorithm. A refinement of this result is achieved with Grigoriev's algorithm [12] applicable to sentences in prenex normal form, whose complexity is doubly exponential relative to the number of quantifier alternations.

Even when we merely consider the *existential* theory of reals,[1] consisting of those sentences $\exists x_1 \cdots \exists x_n \vartheta$ in Tarksi's algebra, where ϑ is a quantifier-free formula (involving no variables distinct from $x_1, \ldots, x_n$), the known decision algorithms have a complexity at best exponential relative to the number n of variables [8]; however, if one fixes the number of variables that can be used, then an algorithm of polynomial complexity becomes available [14].

As observed by Tarski himself [17], the decidability of elementary algebra entails decidability of various other first-order theories regarding complex numbers or n-dimensional vectors, as well as decidability of elementary geometries of the plane, of 3-, or of n-dimensional space; of analogous non-Euclidean geometries, and of projective geometry. It is in fact possible to translate statements of these systems into statements about real numbers, thereby reducing their decision problems to the analogous problem for elementary algebra.

For instance, a first-order system of *elementary plane geometry* can be instructed [17, 18] over a language endowed with a denumerable infinity of variables (ranging over the points of Euclidean space), with the familiar dyadic sign $=$ (*identity* of points in the plane), with the 3-adic *betweenness* predicate

[1] As seen here, we are taking the liberty of calling 'theory' a fragment of the language of a theory proper (cf. [7])—usually of a complete one, so that the distinction between valid and true sentence becomes immaterial. Such a fragment, to wit, a syntactically delimited family Θ, does not comprise exclusively true sentences; so, when saying that a 'theory' is decidable, we will actually mean that its true sentences form a decidable subset of Θ.

symbol $B(x, y, z)$, interpreted as "y lies between x and z on the straight line xz", and with the 4-adic *equidistance* predicate symbol $D(x, y; z, t)$, interpreted as "the distance from x to y equals the distance from z to t". To get a decision method for this system:

- one associates with each sentence Φ of elementary plane geometry a sentence Φ^* of elementary algebra, by mapping each variable x of Φ into two real-valued variables $\overline{x}, \overline{\overline{x}}$ which represent its coordinates, so that to any two distinct point-variables x and y there correspond four distinct real variables $\overline{x}, \overline{\overline{x}}, \overline{y}, \overline{\overline{y}}$;
- one translates $B(-,-,-)$ and $D(-,-;-,-)$, inside Φ^*, into algebraic relations involving the coordinates of points.

One can achieve that the sentence Φ be true if and only if Φ^* is true; thus a decision problem for geometry gets reduced to elementary algebra. (Tarski proposes also a complete axiomatization for elementary plane geometry and, more generally, for n-dimensional Euclidean geometries [18, 19]).

A first limitation to extensions of Tarski's theories by real functions stems from the fact that by extending elementary algebra with the function $\sin x$ one disrupts its decidability [17] (in fact, by resorting to the periodicity of that function, one can define within Tarski's theory the predicate "x is an integer").

The existential theory of reals, extended with the numbers $\log 2$, π and with the functions e^x and $\sin x$ turns out to be, by itself, undecidable (Richardson, [15]).

In fact, let E^* be a set of real-valued functions (at least partially defined) of one real argument, which is closed relative to addition, subtraction, multiplication, and function composition, and which contains the identity function and all rational numbers (seen, here, as constant functions). Moreover, let E be a set of formal expressions, each one representing a function belonging to E^* so that every function in E^* is represented by at least one expression in E (if $A \in E$, we indicate by $A(x)$ the corresponding function in E^*). Suppose, also, that through an effective procedure one can, given expressions A and B in E, find expressions in E which represent the functions $A(x) + B(x)$, $A(x) - B(x)$, $A(x) \cdot B(x)$, and $A(B(x))$. Richardson proves that if E^* comprises the functions e^x, $\sin x$ and the constant functions $\log 2$, π, then the *negative value* problem "given an expression A in E, determine whether or not there is a real number $\boldsymbol{x}$ such that $A(\boldsymbol{x}) < 0$" is undecidable. Let us suppose, for the sake of contradiction, that the existential theory of reals extended with the numbers $\log 2$, π and with the functions e^x, $\sin x$ is decidable. Then, in particular, one could decide of any given sentence $(\exists x)\vartheta$, where ϑ is a quantifier-free formula of elementary algebra extended with the numbers $\log 2$, π and with the functions e^x, $\sin x$, whether $(\exists x)\vartheta$ is true or false. This could be done, in particular, for sentences of the form $(\exists x) f(x) < 0$, where f is a real function of the real variable x, built from $x, \log 2, \pi, e^x, \sin x$ and rational constants, by means of addition, subtraction, multiplication, and function composition. In other words, the negative-value problem would be decidable that refers to the smallest collection E^* including $\{x, \log 2, \pi, e^x, \sin x\} \cup \mathbb{Q}$ and closed relative to addition, subtraction, multiplication, and function composition; but this would conflict with what was stated earlier.

Richardson also proves, under suitable assumptions about E^*, that the *identity* problem "given an expression A in E, establish whether or not $A(x) \equiv \mathbf{0}$" (where $\mathbf{0}$ is the everywhere null function over $\mathbb{R}$) and the *integration* problem "given an expression A in E, establish whether or not there is a function f in E^* such that $f'(x) \equiv A(x)$" are undecidable (the symbol $\equiv$ indicates that the functions coincide, i.e., they share the same domain, over which they take, corresponding to the same value for the argument, equal value).

In order to prove the undecidability of these problems, Richardson exploits the existence [6] of a function of type

$$P(y, x_1, \ldots, x_n) = ay + b_1 x_1 + \ldots + b_n x_n + c_1 2^{x_1} + \ldots + c_n 2^{x_n} + d\,,$$

with $a, b_1, \ldots, b_n, c_1, \ldots, c_n, d \in \mathbb{Z}$, such that the problem "given $y \in \mathbb{N}$, establish whether or not there exist $x_1, \ldots, x_n \in \mathbb{N}$ such that $P(y, x_1, \ldots, x_n) = 0$" turns out to be undecidable. In fact, arguing by contradiction, he shows that if the negative value problem, the identity problem, or the integration problem were decidable, then through the construction of suitable "intermediate problems" the said problem could be decided too.

In what follows we will present two series of decidability (and undecidability) results about fragments of real analysis, one series having been obtained by Friedman and Seress [10, 11] (concerning what we will simply designate as FS theory), and the other by Cantone, Cincotti, Ferro, Gallo, Omodeo, and Schwartz in [2, 3] (RMCF, RMCF^+, and RDF theories).

The FS theory consists of sentences of type $(\forall f \in \mathscr{F})\varphi$, where $\mathscr{F}$ is a family of monadic functions from $\mathbb{R}$ to $\mathbb{R}$ (respectively, from $\mathbb{I} = [0, 1]$ to $\mathbb{I}$) and φ is a first-order sentence involving, besides the function symbol f, variables ranging over $\mathbb{R}$ (resp., over $\mathbb{I}$), the comparison signs $>, <$, and $=$, the usual connectives $\wedge, \vee, \neg$, and $\exists/\forall$–quantifiers.

As for RMCF, RMCF^+, and RDF, these are unquantified theories involving real-valued variables (and constants), additional variables (and constants) to be interpreted as real-valued functions of a real argument, also involving operations between numbers and between functions, the ordering relations and predicate symbols for comparing functions, for comparing function derivatives and real numbers, predicates stating (strict and non-strict) function monotonicity, and predicates stating (strict and non-strict) convexity and concavity of functions over real intervals.

The style of our presentation will be rather casual; in the sense that it will privilege conceptual aspects over technical ones—without neglecting the latter whenever deemed necessary. We will strive to bring into evidence the expressiveness of the theories presented by casting inside them various theorems of elementary analysis; thus, in the case of decidable theories, our examples will entail the possibility of proving certain theorems automatically.

1. The FS theory

To begin our discussion on the FS theory, we must recall a common classification of quantified sentences (i.e., formulae devoid of free variables) in a first-order theory. One defines a sentence φ to be Σ_k when it is either of the *prenex* type

$$(\exists x_{1,1}\cdots\exists x_{1,m_1})(\forall x_{2,1}\cdots\forall x_{2,m_2})\cdots$$
$$\cdots(\forall x_{k-1,1}\cdots\forall x_{k-1,m_{k-1}})(\exists x_{k,1}\cdots\exists x_{k,m_k})\varphi_0$$

(where φ_0 is quantifier-free) with k an odd number, or of the prenex type

$$(\exists x_{1,1}\cdots\exists x_{1,m_1})(\forall x_{2,1}\cdots\forall x_{2,m_2})\cdots$$
$$\cdots(\exists x_{k-1,1}\cdots\exists x_{k-1,m_{k-1}})(\forall x_{k,1}\cdots\forall x_{k,m_k})\varphi_0$$

(where φ_0 is devoid of quantifiers again) with k an even number; that is, if the prenex normal form of φ, in which all quantifiers have been brought to the beginning, alternates $k-1$ times between batches of existential and universal quantifiers and shows an $\exists$-quantifier at its very start. The definition of Π_k sentences is analogous, but in this case a $\forall$-quantifier occurs first.

1.1. Decidability of Σ_1 sentences, of Π_1 sentences, and of Π_2 separated sentences of FS

As already recalled, the sentences in the FS theory are of type

$$(\forall f\in\mathscr{F})\varphi\,,$$

where $\mathscr{F}$ is a family of functions from $\mathbb{R}$ to $\mathbb{R}$ (respectively, from $\mathbb{I}$ to $\mathbb{I}$) and φ is a first-order sentence involving the monadic function symbol f, individual variables ranging over $\mathbb{R}$ (resp., over $\mathbb{I}$), the dyadic comparators $>$, $<$, $=$, the propositional connectives $\wedge, \vee, \neg$, and $\exists/\forall$–quantifiers.

In our study on decidability, we first address the case in which φ is Σ_1 (to wit, φ is of type $\exists x_1\cdots\exists x_n\varphi_0$, where φ_0 is quantifier-free). We will see, in particular, that if $\mathscr{F}$ is formed by all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ (or from $\mathbb{I}$ to $\mathbb{I}$), then the Σ_1 sentences are decidable; but the same is known to hold for the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ (or from $\mathbb{I}$ to $\mathbb{I}$) which are differentiable, for those which are of class C^∞, and for the analytic functions.

Observe, in the first place, that the Σ_1 sentences admit an equivalent normalized form, according to the following lemma:

LEMMA 1.1 ([10, Section 1, Lemma 1.1]). *Let φ be the Σ_1 sentence $\exists x_1 \cdots \exists x_n \varphi_0$, where φ_0 is quantifier-free. Then φ is equivalent to a sentence ψ of the form*

$$\exists x_1 \cdots \exists x_p \left(\bigvee_{i=1}^{m} \left[\left(\bigwedge_{j=1}^{k_i - 1} (x_j < x_{j+1}) \right) \wedge \psi_i \right] \right) ,$$

where each ψ_i has the form $\bigwedge_{j=1}^{\ell_i} (f(x_{a_j}) = x_{b_j})$ with

(a) $1 \leqslant a_j \leqslant k_i$ and $1 \leqslant b_j \leqslant k_i$ for each j,

(b) every variable x_c $(1 \leqslant c \leqslant k_i)$ occurs at least once as either x_{a_j} or x_{b_j},

(c) every variable x_c occurs at most once as x_{a_j}.

Moreover, by means of a suitable algorithm it is possible to get ψ from φ in a finite number of steps. The case $m = 0$ reflects the impossibility of having a coherent ordering for the variables of φ.

The algorithm is based on techniques such as transformation into disjunctive normal form, introduction of new variables, review of all possible orderings of the variables, and renumbering of variables.

As regards complexity, let us observe that, at least in principle, the application of this lemma could lead to a combinatorial explosion. Suffice it to say that, given r variables $x_1, ..., x_r$, the number of possible chains with the ordering $<$, with possible identifications of some variables through the equivalence relation $=$, is of order $r! \cdot r \cdot e^r$ ([2, p. 775]).

The following holds for the sentences on which we are focusing, when $\mathscr{F}$ is the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$:

PROPOSITION 1.2 (Characterization theorem, cf. [10, Section 1, Theorem 1.3]). *Let $\mathscr{F}$ be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and let φ be a Σ_1 sentence. Let, moreover, ψ be a Σ_1 sentence, equivalent to φ, of the form*

$$\exists x_1 \cdots \exists x_p \left(\bigvee_{i=1}^{m} \left[\left(\bigwedge_{j=1}^{k_i - 1} (x_j < x_{j+1}) \right) \wedge \psi_i \right] \right)$$

meeting all conditions stated in Lemma 1.1. Then $(\forall f \in \mathscr{F})\varphi$ is true if and only if each one of the following types of formula occurs among the ψ_i's:

(1) $\bigwedge_{j=1}^{k} (f(x_j) = x_j)$;

(2) a subset of $\bigwedge_{j=1}^{k} (f(x_j) = x_{k+1-j})$ meeting condition (b) of Lemma 1.1 (here and below, if Y is a conjunction of literals, by the locution "subset of Y" we informally refer to a conjunction of some of the literals in Y);

(3) $\bigwedge_{j=1}^{\ell}(f(x_{a_j}) = x_{b_j})$ *meeting, in addition to (b) and (c) of Lemma 1.1, the conditions*

(3.a) *if* $f(x_{a_j}) = x_{b_j}$ *then* $x_{a_j} < x_{b_j}$,

(3.b) *if* $f(x_{a_j}) = x_{b_j}$, $f(x_{a_{j'}}) = x_{b_{j'}}$, *and* $x_{a_j} < x_{a_{j'}}$, *then* $x_{b_j} < x_{b_{j'}}$;

(4) $\bigwedge_{j=1}^{\ell}(f(x_{a_j}) = x_{b_j})$ *meeting, in addition to (b) and (c) of Lemma 1.1, the conditions*

(4.a) *if* $f(x_{a_j}) = x_{b_j}$ *then* $x_{a_j} > x_{b_j}$,

(4.b) *if* $f(x_{a_j}) = x_{b_j}$, $f(x_{a_{j'}}) = x_{b_{j'}}$, *and* $x_{a_j} < x_{a_{j'}}$, *then* $x_{b_j} < x_{b_{j'}}$;

(5) *either one of the types* $\bigwedge_{j=1}^{k}(f(x_j) = x_n)$, $\bigwedge_{j=1, j\neq n}^{k}(f(x_j) = x_n)$, *for some* n *with* $1 \leqslant n \leqslant k$;

(6) *a subset of* $\bigwedge_{j=1}^{k}(f(x_j) = x_{g_j})$ *meeting condition (b) of Lemma 1.1 along with the following conditions: for some* n, *with* $1 \leqslant n \leqslant k$,

(6.a) *either* $g_n = n$ *and*

$$\forall j[((1 \leqslant j \leqslant n-1) \Rightarrow (n+1 \leqslant g_j \leqslant k)) \\ \wedge ((n+1 \leqslant j \leqslant k) \Rightarrow (1 \leqslant g_j \leqslant n-1))]$$

hold, or

$$\forall j[((1 \leqslant j \leqslant n) \Rightarrow (n+1 \leqslant g_j \leqslant k)) \\ \wedge ((n+1 \leqslant j \leqslant k) \Rightarrow (1 \leqslant g_j \leqslant n))]$$

holds,

(6.b) *if* $1 \leqslant j < h \leqslant k$ *then* $g_h < g_j$,

(6.c) *if* $1 \leqslant j \leqslant n < s \leqslant g_j$ *and* $f(x_s) = x_\ell$, *then* $j < \ell$,

(6.d) *if* $g_j \leqslant s \leqslant n < j \leqslant k$ *and* $f(x_s) = x_\ell$, *then* $j > \ell$;

(7) *a subset of* $\bigwedge_{j=1}^{k}(f(x_j) = x_{g_j})$ *meeting condition (b) of Lemma 1.1 along with the following conditions: for some* n, *with* $1 \leqslant n \leqslant k$,

(7.a) *either* $g_n = n$ *and*

$$\forall j[((1 \leqslant j \leqslant n-1) \Rightarrow (n+1 \leqslant g_j \leqslant k)) \\ \wedge ((n+1 \leqslant j \leqslant k) \Rightarrow (1 \leqslant g_j \leqslant n-1))]$$

hold, or

$$\forall j[((1 \leqslant j \leqslant n) \Rightarrow (n+1 \leqslant g_j \leqslant k)) \\ \wedge ((n+1 \leqslant j \leqslant k) \Rightarrow (1 \leqslant g_j \leqslant n))]$$

holds,

(7.b) *if* $1 \leqslant j < h \leqslant k$ *then* $g_h < g_j$,

(7.c) *if* $1 \leqslant j \leqslant n$, $g_j \leqslant s \leqslant k$, *and* $f(x_s) = x_\ell$, *then* $j \geqslant \ell$ *(where equality can hold only if* $j = g_j = s = \ell = n$*),*

(7.d) *if* $n+1 \leqslant j \leqslant k$, $1 \leqslant s \leqslant g_j$, *and* $f(x_s) = x_\ell$, *then* $j < \ell$*;*

(8) *for some* n *with* $1 \leqslant n \leqslant k$*, a subset of*

$$\bigwedge_{j=1}^{n}(f(x_j) = x_n) \wedge \bigwedge_{j=n+1}^{k}(f(x_j) = x_{g_j})$$

meeting condition (b) of Lemma 1.1 along with the conditions

(8.a) *if* $n+1 \leqslant j \leqslant k$ *then* $1 \leqslant g_j < n$,

(8.b) *if* $n+1 \leqslant j < h \leqslant k$ *then* $g_j > g_h$*;*

(9) *for some* n *with* $1 \leqslant n \leqslant k$*, a subset of*

$$\bigwedge_{j=1}^{n-1}(f(x_j) = x_{g_j}) \wedge \bigwedge_{j=n}^{k}(f(x_j) = x_n)$$

meeting condition (b) of Lemma 1.1 along with the conditions

(9.a) *if* $1 \leqslant j \leqslant n-1$ *then* $n < g_j \leqslant k$,

(9.b) *if* $1 \leqslant j < h \leqslant n-1$ *then* $g_j > g_h$.

Notice that a ψ_i can belong to more than one type. For instance, the formula $f(x_1) = x_1$ is of types (1), (2), (5), (6), (7), (8), (9).

Here we offer some clues about the necessity of the above conditions. If φ is true of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, then, since ψ is equivalent to φ, ψ is true of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Therefore ψ will be satisfied, in particular, by the function $f(x) = x$; this implies that there must be a ψ_i of type (1). Likewise ψ must be true, in particular, of the function $f(x) = -x$; this implies that there must be a ψ_i of type (2). By choosing suitable functions for the remaining types, in the same fashion, one proves that the ψ_i's must include at least one formula of each type.

The proof that the above conditions are also sufficient is more intricate. To show that if ψ encompasses all nine types then ψ is true of all continuous f (from $\mathbb{R}$ to $\mathbb{R}$), one takes into account all possibilities about the number of fixpoints which a given f can own (none, exactly one, a finite number greater than one, infinitely many). One proves that in each case f falls under at least one of the nine types, and hence it satisfies ψ. Consider, e.g., the simplest case, namely the one of an f with infinitely many fixpoints: then, given a positive integer k, there must exist $x_1, \cdots, x_k \in \mathbb{R}$ such that $x_1 < \cdots < x_k$ and $f(x_1) = x_1, \ldots, f(x_k) = x_k$; therefore f satisfies the ψ_i's of type (1) and the sentence ψ.

Let us observe that through application of the preceding lemma and proposition one can decide by means of an algorithm whether each given sentence $(\forall f \in \mathscr{F})\varphi$ is true or false; otherwise stated, these results provide an automatic proof-procedure for statements of this nature.

To illustrate application of the preceding proposition, let us examine a simple example:

Example 1.3. *Consider the sentence*

$$(\forall f \in \mathscr{F})\exists x\, \exists y(f(x) = y)\,,$$

which can be interpreted as claiming "for every continuous function f from $\mathbb{R}$ to $\mathbb{R}$ there exist $x, y \in \mathbb{R}$ such that $f(x) = y$". In this case φ is the Σ_1 sentence

$$\exists x\, \exists y(f(x) = y)\,,$$

equivalent to

$$\exists x_1\, \exists x_2[(x_1 < x_2 \wedge f(x_1) = x_2) \vee (f(x_1) = x_1) \vee (x_1 < x_2 \wedge f(x_2) = x_1)]\,.$$

The formula $(x_1 < x_2 \wedge f(x_1) = x_2)$ matches type (3), the formula $(f(x_1) = x_1)$ matches types (1), (2), (5), (6), (7), (8), (9), and the formula $(x_1 < x_2 \wedge f(x_2) = x_1)$ matches type (4). Hence all of the nine types are encompassed, which amounts to saying that the sentence $(\forall f \in \mathscr{F})\exists x, y(f(x) = y)$ is true.

The following example formalizes another lemma expressible by means of a Σ_1 sentence.

Example 1.4. *Consider the claim "for each continuous function f from $\mathbb{R}$ to $\mathbb{R}$ there exist $x, y, z \in \mathbb{R}$, with $x < y < z$, such that either $f(x) \leqslant f(y) \leqslant f(z)$ or $f(x) \geqslant f(y) \geqslant f(z)$ holds". This can be formalized as*

$$(\forall f \in \mathscr{F})\exists x\exists y\exists z(x < y < z \wedge (f(x) \leqslant f(y) \leqslant f(z) \vee f(x) \geqslant f(y) \geqslant f(z)))$$

and hence it can be proved automatically thanks to the preceding results.

The above-seen characterization theorem concerning the family of the continuous functions (from $\mathbb{R}$ to $\mathbb{R}$) holds, with the same conditions (1) through (9), for the family of the differentiable functions (from $\mathbb{R}$ to $\mathbb{R}$), as well as for the ones of class C^∞ (from $\mathbb{R}$ to $\mathbb{R}$); this tells us, as a consequence, that if a Σ_1 sentence holds for all functions of class C^∞ from $\mathbb{R}$ to $\mathbb{R}$ then it holds, more generally, for all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

A similar characterization theorem holds for the analytic functions from $\mathbb{R}$ to $\mathbb{R}$; but in this case the claim involves only conditions (1) through (7).

Yet an analogous theorem holds for the functions (continuous, differentiable, of class C^∞, or analytic) from $\mathbb{I}$ to $\mathbb{I}$. In this case the characterization is exactly the same for all of the four collections of functions; consequently, if a Σ_1 sentence holds for all analytic functions from $\mathbb{I}$ to $\mathbb{I}$ then it holds, more generally, for all continuous functions from $\mathbb{I}$ to $\mathbb{I}$.

What said so far enables us to state the following decidability result:

Proposition 1.5 (Decidability of the Σ_1 sentences of FS, cf. [10, Section 1, Theorems 1.3 through 1.6]). *The validity problem for Σ_1 sentences is solvable, relative to each one of the following families of functions from $\mathbb{R}$ to $\mathbb{R}$: continuous, differentiable, C^∞, and analytic. The same holds for the corresponding families of functions from $\mathbb{I}$ to $\mathbb{I}$.*

Otherwise stated: let $\mathscr{F}$ be the family of all continuous functions (or the one of the differentiable functions, or of the functions of class C^∞, or of the analytic functions) from $\mathbb{R}$ to $\mathbb{R}$. Then an algorithm exists which, given any sentence $(\forall f \in \mathscr{F})\varphi$, where φ is Σ_1, establishes whether it is true or false. The same holds about $\mathbb{I}$.

Let us now address the decidability problem for the $(\forall f \in \mathscr{F})\varphi$ sentences of FS where φ is a Π_1 sentence (namely, φ is of the form $\forall x_1 \cdots \forall x_n \varphi_0$, with φ_0 quantifier-free). Focusing, for the time being, on the case when $\mathscr{F}$ is the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, we have:

$$(\forall f \in \mathscr{F})\forall x_1 \cdots \forall x_n \, \varphi_0$$

is true if and only if its negation

$$(\exists f \in \mathscr{F})\exists x_1 \cdots \exists x_n \, \chi_0 \,,$$

where $\chi_0 = \neg\varphi_0$, is false. This happens if and only if the sentence, to be referred below as γ,

$$(\exists f \in \mathscr{F})\exists x_1 \cdots \exists x_n \left(\bigvee_{i=1}^{m} \left[\left(\bigwedge_{j=1}^{k_i - 1} (x_j < x_{j+1}) \right) \wedge \psi_i \right] \right) \,,$$

which results from application of Lemma 1.1 to χ_0, is false. This happens if and only if $m = 0$. In fact, if $m = 0$ then, as already said in the claim of Lemma 1.1, the variables of χ_0 do not admit a coherent ordering, and therefore γ is false. If, on the opposite, $m \geqslant 1$ holds, then it is possible (by assigning suitable values to the variables and by choosing a suitable interpolation polynomial $\boldsymbol{f}$ as f) to determine $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ and $\boldsymbol{f}$ so that they satisfy $(\bigwedge_{j=1}^{k_1-1}(x_j < x_{j+1})) \wedge \psi_1$; in particular, it suffices to assign values $\boldsymbol{x}_i = i$ $(i = 1, \ldots, n)$ to the variables and to choose as f a polynomial $\boldsymbol{f}$ such that $\boldsymbol{f}(a_j) = b_j$ whenever $f(x_{a_j}) = x_{b_j}$ occurs in ψ_1. Therefore, if $m \geqslant 1$, then γ is true.

What said so far entails a decision procedure for the case of the Π_1 sentences. Analogous considerations can be made if $\mathscr{F}$, instead of being the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, is either the family of all differentiable functions (from $\mathbb{R}$ to $\mathbb{R}$), the one of all functions of class C^∞ (from $\mathbb{R}$ to $\mathbb{R}$), or the one of all analytic functions (from $\mathbb{R}$ to $\mathbb{R}$). The same considerations can be made again for the corresponding families of functions from $\mathbb{I}$ to $\mathbb{I}$.

We hence get the following decidability result:

Proposition 1.6 (Decidability of the Π_1 sentences of FS, cf. [10, Section 1, Theorem 1.7]). *The validity problem for Π_1 sentences is solvable, relative to each one of the following families of functions: continuous, differentiable, C^∞, and analytic. The same holds for the corresponding families of functions from $\mathbb{I}$ to $\mathbb{I}$.*

Otherwise stated: let $\mathscr{F}$ be the family of all continuous functions (or the one of the differentiable functions, or of the functions of class C^∞, or of the analytic functions) from $\mathbb{R}$ to $\mathbb{R}$. Then an algorithm exists which, given any sentence $(\forall f \in \mathscr{F})\varphi$, where φ is Π_1, establishes whether it is true or false. The same holds for $\mathbb{I}$.

Notice also that, since the characterization for all of them is the same ($m = 0$ in the sentence obtained from $\neg\varphi_0$ through application of Lemma 1.1), it turns out that these families of functions are indistinguishable relative to the Π_1 sentences; among others, a Π_1 sentence is true for all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ if and only if it is true for all analytic functions from $\mathbb{I}$ to $\mathbb{I}$.

The following example formalizes a lemma (good definition of a function) expressible by means of a Π_1 sentence.

Example 1.7. *Consider the theorem "let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $x, y, z \in \mathbb{R}$; if $f(x) = y$ and $f(x) = z$, then $y = z$". This can be formalized as*

$$(\forall f \in \mathscr{F})\forall x \forall y \forall z((f(x) = y \wedge f(x) = z) \rightarrow y = z)$$

and therefore it can be proved automatically, thanks to the preceding results (recall that the derived connective $\rightarrow$, exploited in the formalization of this sentence, can be eliminated, e.g., through the rewriting $a \rightarrow b \equiv \neg(a \wedge \neg b)$).

Let us now introduce the notion of *separated formula*. Intuitively speaking, we are talking about formulae in which the elements of the domain of f are not compared with those of its range. To state this more accurately:

DEFINITION 1.8. *Let φ_0 be a quantifier-free formula involving a monadic function f along with variables ranging over $\mathbb{R}$ (resp., over $\mathbb{I}$), the comparators $>$, $<$, $=$, and the usual connectives $\wedge, \vee, \neg$.*

We will say that φ_0 is a SEPARATED FORMULA *if it meets the following conditions:*

(a) The terms of φ_0 are of either the form x or the form $f(x)$, where x is a variable (i.e., no composition of f with itself occurs in φ_0).

(b) There are two sets, formed by variables of φ_0 and to be called set of the DOMAIN VARIABLES *and of the* RANGE VARIABLES*, respectively, such that:*

- *(b1) every variable of φ_0 belongs to exactly one of the two sets;*
- *(b2) if the term $f(x)$ occurs in φ_0, then x is a domain variable;*
- *(b3) when $f(x) > y$, $f(x) < y$, or $f(x) = y$ occurs as a subformula in φ_0, then y is a range variable;*
- *(b4) when $x > y$, $x < y$, or $x = y$ occurs as a subformula in φ_0, then x and y are either both domain variables or both range variables (that is, a domain variable is never compared with a range variable).*

To end, we will say that a sentence φ in prenex form is SEPARATED *when its unquantified part is a separated formula.*

For instance, the sentence $\exists x(f(x) = x)$ is not separated (if it were such then, due to the conditions (b2) and (b3), x would be both domain variable and range variable, which would conflict with condition (b1)).

The sentence $\exists x \exists y(f(x) = y)$ is, instead, separated (with x domain variable and y range variable).

For the $(\forall f \in \mathscr{F})\varphi$ sentences of the theory FS, when φ is a Π_2 separated sentence (i.e., a sentence of the form $\forall x_1 \cdots \forall x_n \exists x_{n+1} \cdots \exists x_m \varphi_0$, with φ_0 devoid of quantifiers and separated), then the following decidability result holds:

PROPOSITION 1.9 (Decidability of the separated Π_2 sentences of FS, cf. [10, Section 2]). *The validity problem for separated Π_2 sentences is solvable, relative to the following families of functions from $\mathbb{R}$ to $\mathbb{R}$: continuous, differentiable, C^∞, and analytic. The same holds for the corresponding families of functions from $\mathbb{I}$ to $\mathbb{I}$.*

Otherwise stated: let $\mathscr{F}$ be the family of all continuous functions (or the one of all differentiable functions, or the one of all functions of class C^∞, or the one of all analytic functions) from $\mathbb{R}$ to $\mathbb{R}$. Then there is an algorithm *which, given any sentence* $(\forall f \in \mathscr{F})\varphi$*, where* φ *be a separated* Π_2 *sentence, establishes whether it is true or false. The same holds for* $\mathbb{I}$.

Also in this case, the decidability of sentences is obtained through a normalization lemma with the aid of characterization theorems.

The following example shows how the *intermediate value theorem* can be formalized by means of a separated Π_2 sentence.

Example 1.10. *Consider the (intermediate value) theorem:*
"Let f *be a continuous function from* $\mathbb{R}$ *to* $\mathbb{R}$ *and let* $x_1, x_2, y_1, y_2, t \in \mathbb{R}$ *be such that* $f(x_1) = y_1$*,* $f(x_2) = y_2$ *and* $y_1 \leqslant t \leqslant y_2$*. Then there is a* $z \in \mathbb{R}$ *such that* $x_1 \leqslant z \leqslant x_2$ *and* $f(z) = t$*".*
This claim can be formalized as

$$(\forall f \in \mathscr{F})\forall x_1 \forall x_2 \forall y_1 \forall y_2 \forall t \exists z \big((f(x_1) = y_1 \wedge f(x_2) = y_2 \wedge y_1 \leqslant t \leqslant y_2) \\ \rightarrow (x_1 \leqslant z \leqslant x_2 \wedge f(z) = t)\big)$$

and hence it can be proved automatically.

1.2. Undecidability of Σ_4 sentences

Indicate, as usual, by $\omega = \{0, 1, \ldots, n, n+1, \ldots\}$ the set of all finite ordinal numbers (where $0 = \emptyset$ and $n + 1 = \{0, \ldots, n\}$); also let $n \in \omega$. A dyadic antireflexive and symmetrical relation (on n) is a subset R of $n \times n$ which meets the following conditions (where aRb stands for $(a, b) \in R$):

antireflexivity if aRb then $a \neq b$;

symmetry if aRb then bRa.

The first-order theory of antireflexive and symmetrical relations with finite models (finite graph theory, to be indicated as GSF) is the set of all sentences φ_R, constructed from the variables (now ranging over natural numbers), by means of the dyadic predicate symbol R (to be interpreted as an antireflexive and symmetric relation), the identity relator $=$, the propositional connectives $\wedge, \vee, \neg$, and the $\exists/\forall$–quantifiers.

The validity problem for the Σ_2 sentences of this theory is undecidable [13]. Specifically, there cannot be any algorithm which, given a generic sentence of type $(\forall R)\varphi_R$ (where φ_R is a Σ_2 sentence of the GSF theory), establishes whether it is true or false.

As a matter of fact, there is an algorithm which associates with every Σ_2 sentence φ_R of the GSF theory a separated Σ_4 sentence φ of the FS theory so that $(\forall R)\varphi_R$ is true if and only if $(\forall f \in \mathscr{F})\varphi$ is true about the family $\mathscr{F}$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Consequently, if the truth problem for $(\forall f \in \mathscr{F})\varphi$ sentences (where φ is a separated Σ_4 sentence in FS and $\mathscr{F}$ is the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$) were decidable, then the analogous problem for $(\forall R)\varphi_R$ sentences (where φ_R is a Σ_2 sentence of GSF) would also be decidable, which is not the case as just recalled above.

Therefore the truth problem for $(\forall f \in \mathscr{F})\varphi$ sentences, where φ is a separated Σ_4 sentence of FS and $\mathscr{F}$ is the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, turns out to be undecidable. This result can be generalized, much by the same method, into the following theorem:

Proposition 1.11 (Undecidability of separated Π_4 sentences of FS, cf. [10, Section 4, Theorem 4.2] and [11, Section 4, Theorem 4.2]). *The set $\{\varphi | (\forall f \in \mathscr{F})\varphi$ is true$\}$ of sentences turns out to be undecidable in the following cases (where we say that a separated sentence of* FS *is* WEAK *if it has no subformulae of type $f(x) < y$, $y < f(x)$, $f(x) < f(t)$, or $y < z$, with y, z range variables; that is, if the ordering relation is not used, in it, to compare elements of the range of f).*

(a) $\mathscr{F}$ is the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and φ ranges over all separated Σ_4 sentences of FS*;*

(b) more generally, $\mathscr{F}$ is a family of functions from $\mathbb{R}$ to $\mathbb{R}$ comprising all analytic functions and φ ranges over the separated, weak Σ_4 sentences of FS*;*

(c) $\mathscr{F}$ is the family of all continuous functions from $\mathbb{I}$ to $\mathbb{I}$ and φ ranges over all separated Σ_4 sentences of FS*;*

(d) more generally, $\mathscr{F}$ is a family of functions from $\mathbb{I}$ to $\mathbb{I}$ comprising all polynomials and φ ranges over all separated, weak Σ_4 sentences of FS.

On the other hand, the said set $\{\varphi | (\forall f \in \mathscr{F})\varphi \text{ is true}\}$ of sentences, where $\mathscr{F}$ is the family of all polynomials from $\mathbb{R}$ to $\mathbb{R}$ (resp., from $\mathbb{I}$ to $\mathbb{I}$) and φ ranges over all sentences of FS, turns out to be *co-recursively enumerable* (cf. [11, Section 4, Theorem 4.7]). Otherwise stated, there exists a computing procedure which eventually halts if and only if a sentence of the said type is submitted to it which happens to be false.

1.3. Decidability and undecidability of sentences about families of monotone functions

Let us now consider the sentences $(\forall f \in \mathscr{F})\varphi$ of the FS theory, where $\mathscr{F}$ is the family of all functions (from $\mathbb{R}$ to $\mathbb{R}$) which are continuous, monotone strictly increasing, and unlimited below as well as above. The following lemma reduces the decidability issue for sentences of this type to the analogous issue regarding sentences of type $(\forall A_1, A_2, A_3, A_4, A_5)\varphi^+$, where $A_1, A_2, A_3, A_4, A_5 \subseteq \mathbb{R}$ and φ^+ is a sentence involving real-valued variables, the comparators $<, =$, the usual connectives $\wedge, \vee, \neg$, $\exists/\forall$–quantifiers, and predicates of type $x \in A_i$. The latter was solved in the positive, cf. [1].

LEMMA 1.12 ([10, Section 3, Lemma 3.5]). *To each sentence φ there corresponds a sentence φ^+ for which the following sentences are logically equivalent.*

(a) $(\forall f \in \mathscr{F})\varphi$, where $\mathscr{F}$ is the family of all functions (from $\mathbb{R}$ to $\mathbb{R}$) which are continuous, monotone strictly increasing and unlimited below as well as above.

(b) $(\forall A_1, A_2, A_3, A_4, A_5)\varphi^+$, where $A_1, A_2, A_3, A_4, A_5 \subseteq \mathbb{R}$ and φ^+ is a sentence that involves variables ranging over $\mathbb{R}$, the comparators $<, =$, the propositional connectives $\wedge, \vee, \neg$, $\exists/\forall$–quantifiers, and predicates of type $x \in A_i$.

Such a φ^+ can be obtained from φ through a suitable algorithm.

Here we will content ourselves with providing the intuitive idea, lying behind this lemma, that the first-order properties of a function f (which is continuous, monotone strictly increasing, and unlimited below as well as above) can be expressed as properties of sets, which are defined starting from the function (for instance, the set $\alpha(f)$ of all fixpoints of f and the set $\beta(f)$ of all left endpoints of the intervals of $\mathbb{R}\backslash\alpha(f)$).

This lemma yields, in view of the decidability of $(\forall A_1, A_2, A_3, A_4, A_5)\varphi^+$ sentences, decidability of the $(\forall f \in \mathscr{F})\varphi$ sentences of the FS theory (where $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous, monotone strictly increasing and unlimited below as well as above). This decidability result can be enhanced, much by the same method, into the following proposition:

PROPOSITION 1.13 ([10, Section 3]; [11, Sections 2 and 3]). *The set $\{\varphi | (\forall f \in \mathscr{F})\varphi$ is true$\}$ of sentences turns out to be decidable in the following cases.*

(a) $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous, monotone strictly increasing and unlimited below as well as above.

(b) $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous and monotone strictly increasing.

(c) $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous and monotone strictly decreasing.

(d) $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous and strictly monotone.

(e) $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are monotone nondecreasing, such that there are at most n intervals on which each of them is constant, and each of them has at most n discontinuity points (where n is a fixed number in $\mathbb{N}$).

(f) $\mathscr{F}$ is the family of all functions from $\mathbb{I}$ to $\mathbb{I}$ which are monotone nondecreasing, such that there are at most n intervals on which each of them is constant, and each of them has at most n discontinuity points (where n is a fixed number in $\mathbb{N}$).

(g) $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are monotone and φ is a separated sentence (as by the definition seen earlier).

The following example formalizes the property of a function from $\mathbb{R}$ to $\mathbb{R}$, continuous and monotone strictly decreasing, of having exactly one fixpoint.

EXAMPLE 1.14. *Consider the claim:*
"Let f be a function from $\mathbb{R}$ to $\mathbb{R}$, continuous and monotone strictly decreasing. Then there exists exactly one $x \in \mathbb{R}$ such that $f(x) = x$."
This claim can be formalized as

$$(\forall f \in \mathscr{F})\exists x \forall y[f(x) = x \wedge (f(y) = y \rightarrow y = x)],$$

where $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous and monotone strictly decreasing. Therefore this theorem can be proved automatically.

On the opposite, decidability gets lost if one takes, as $\mathscr{F}$, the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are continuous and monotone and have an arbitrarily large finite number of intervals on which they are constant.

As a matter of fact, given a Turing machine T endowed with symbols $\{a_0, \ldots, a_h\}$ (where a_0 stands for the *blank*) and states $\{q_0, q_1, \ldots, q_k\}$ (where q_0 is the initial state and q_1 is the final state), it is possible to construct a sentence $\varphi(T)$ such that $(\exists f \in \mathscr{F})\varphi(T)$ is true if and only if the machine T, starting with an empty tape, halts after a finite number of steps. Since $(\exists f \in \mathscr{F})\varphi(T)$ is true if and only if $(\forall f \in \mathscr{F})\neg\varphi(T)$ is false, if the truth of the $(\forall f \in \mathscr{F})\varphi$ sentences were decidable, then the truth of the $(\exists f \in \mathscr{F})\varphi(T)$ sentences would also be decidable, and therefore the problem "T will halt" would turn out to be such; however, as is well-known, the halting problem is undecidable [20].

This argument can be adjusted to all families of functions $\mathscr{F}$ (either from $\mathbb{R}$ to $\mathbb{R}$ or from $\mathbb{I}$ to $\mathbb{I}$) which include all nondecreasing monotone functions of class C^∞ and have any finite number of intervals where they are constant. The same holds for the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are monotone, continuous on the left, and have an arbitrary finite number of discontinuity points. Hence we have the following undecidability result:

PROPOSITION 1.15 ([11, Section 1]). *The set of all* $\{\varphi|(\forall f \in \mathscr{F})\varphi$ *is true*$\}$ *sentences has, in each of the following cases, an unsolvable decision problem (case (b) generalizes case (a)).*

(a) $\mathscr{F}$ *is the family of all functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *(resp., from* $\mathbb{I}$ *to* $\mathbb{I}$*) which are continuous and monotone and have an arbitrary, though finite, number of intervals over which they are constant.*

(b) $\mathscr{F}$ *is a family of functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *(resp., from* $\mathbb{I}$ *to* $\mathbb{I}$*) containing all nondecreasing monotone functions of class* C^∞ *which have an arbitrary finite number of intervals over which they are constant.*

(c) $\mathscr{F}$ *is the family of all functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *which are strictly monotone, continuous on the left, and have an arbitrary finite number of discontinuity points.*[2]

Nevertheless, the set $\{\varphi|(\forall f \in \mathscr{F})\varphi \text{ is true}\}$ of sentences, where $\mathscr{F}$ is the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ (resp., from $\mathbb{I}$ to $\mathbb{I}$) which are monotone nondecreasing and have an arbitrary finite number of intervals over which they are constant and an arbitrary finite number of discontinuity points, turns out to be *co-recursively enumerable* (cf. [11, Section 3, Corollary 3.6]). In other words, there exists a computing procedure which eventually halts if and only if a sentence of the said type is initially submitted to it which happens to be false.

2. The theories RMCF, RMCF$^+$, and RDF

As said in the introduction, Tarski's elementary algebra is decidable; i.e., there is an algorithm telling one, of any given closed formula Φ of this theory, whether Φ is true or false. As recalled there, Tarski's elementary algebra is the first-order theory supplying a denumerable infinity of real-valued variables, the numerical constants 0, 1, -1 (interpreted as the corresponding real numbers), the operations $+$, $-$, and $\cdot$ (designating the familiar arithmetic operations over $\mathbb{R}$), the standard comparators $>$, $<$, and $=$, the propositional connectives $\wedge$, $\vee$, and $\neg$, and the quantifiers $\exists$ and $\forall$.

The decidability of Tarski's elementary algebra readily entails the decidability of its own existential sub-theory, consisting of all statements of the form

$$\exists x_1 \exists x_2 \cdots \exists x_n \vartheta \,,$$

where ϑ is quantifier-free and involves only variables from among $x_1, x_2, \ldots, x_n$.

[2]With regard to item (c), [11] does not discuss the case of functions from $\mathbb{I}$ to $\mathbb{I}$.

The existential theory of reals can be thought of as a quantifier-free language. For, a prenex sentence $\exists x_1 \exists x_2 \cdots \exists x_n \vartheta$ is true if and only if its unquantified matrix ϑ is satisfiable, and hence any truth-decision algorithm for the existential theory of reals can be used also to solve the satisfiability problem for the corresponding theory devoid of quantifiers.

The fragments of real analysis RMCF, RMCF^+, and RDF, which will be reviewed in this section, are in quantifier-free form. They extend the quantifier-free theory of reals with various predicates over real functions of a real variable. More specifically, the theories RMCF and RMCF^+ deal with continuous functions, whereas the theory RDF refers to differentiable functions with a continuous derivative.

We begin with a brief description of the theory RMCF. Later we will review in some detail RMCF^+, and will also give a brief outline of the theory RDF.

The theory RMCF (of Reals with Monotone and Convex Functions) [3] involves predicates for function comparison, and predicates about monotonicity of functions (strict and non-strict), and about concavity and convexity of functions (only non-strict). The atomic formulae of RMCF are of these forms:

$$\begin{array}{ll}
t_1 = t_2\,, & t_1 > t_2\,, \\
F_1 = F_2\,, & F_1 > F_2\,, \\
\mathsf{Up}(F)_{[t_1,t_2]}\,, & \mathsf{Strict_Up}(F)_{[t_1,t_2]}\,, \\
\mathsf{Down}(F)_{[t_1,t_2]}\,, & \mathsf{Strict_Down}(F)_{[t_1,t_2]}\,, \\
\mathsf{Convex}(F)_{[t_1,t_2]}\,, & \mathsf{Concave}(F)_{[t_1,t_2]}\,.
\end{array}$$

Here t_1, t_2 are numerical expression (involving real variables, the real constants 0, 1, function images of numerical expressions, and the arithmetic operations) and F_1, F_2 are functional expressions (involving function variables and constants and the operations of sum and difference between functional expressions). The functional constants are $\mathbf{0}$, $\mathbf{1}$, interpreted as the functions with fixed values 0 and 1, respectively. Function symbols are interpreted as continuous real functions of a real variable having as their domain the whole real axis $\mathbb{R}$. The predicate $F_1 = F_2$ (resp., $F_1 > F_2$) states that the real functions $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ interpreting the expressions F_1 and F_2 coincide over the whole real axis (resp., $\boldsymbol{f}_1(x) > \boldsymbol{f}_2(x)$ holds for all $x \in \mathbb{R}$). The predicate symbols express monotonicity (strict or non-strict), non-strict convexity, and non-strict concavity of functions; each of them refers to a closed bounded interval $[t_1, t_2]$. The formulae of RMCF result from propositional combinations of atomic formulae by means of the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. As said, explicit quantification is not allowed in RMCF formulae.

The above considerations could easily be formalized in a definition of the (RMCF) interpretations of formulae of RMCF. We say that an RMCF formula ϑ is *satisfiable* if there exists an RMCF interpretation (*real model*) of the symbols of ϑ which makes ϑ true. We say that an RMCF formula ϑ is *valid* (or is a *theorem*) if ϑ is true in all RMCF interpretations.

As shown in [3], there is a decision procedure which determines, for any given RMCF formula, whether it is satisfiable or not. Such a procedure is achieved through satisfiability-preserving transformations which reduce the satisfiability problem for RMCF to the satisfiability problem for Tarski's theory of reals.[3] To prove the correctness of these formula transformations, function variables are interpreted as piecewise linear functions. In addition, since a formula is valid if and only if its negation is unsatisfiable, the same algorithm tells one whether a given RMCF formula is valid or not; hence one can fully mechanize recognition of any theorem expressible in RMCF.

In [3], a variant of the theory RMCF in which function variables are interpreted as multivariate continuous real functions is also studied and a decision procedure is provided for it.

As an ending remark, note that Proposition 1.6 about the Π_1-decidability of FS, to the extent to which it refers to continuous real functions of one real variable defined all over $\mathbb{R}$, readily follows from the decidability of RMCF.

2.1. The theory RMCF$^+$

The theory RMCF$^+$ [2] (cf. also [16, pp. 165–177]) is an extension of RMCF with predicates on strict convexity and concavity of real continuous functions of a real variable. In addition, most of the predicates on functions apply both to bounded and unbounded intervals.

2.1.1. Syntax of RMCF$^+$

The language of RMCF$^+$ contains

- a denumerable infinity of individual variables, called *numerical variables*, which are denoted by $x, y, z, \ldots$;
- two *numerical constants* $0, 1$;
- a denumerable infinity of *function variables*, denoted by $f, g, h, \ldots$;
- two *functional constants* $\mathbf{0}$, $\mathbf{1}$.

The language of RMCF$^+$ also includes two distinguished symbols, $-\infty, +\infty$, which are restricted to occur only within range defining parameters, as stated in the definition of atomic RMCF$^+$-formulae below.

Numerical terms are recursively defined as follows:

(a) numerical variables and the constants $0, 1$ are numerical terms;

[3] We will be a bit more specific on this, and also about syntax and semantics matters, in the next section, in the context of the extension RMCF$^+$ of RMCF.

(b) if t_1, t_2 are numerical terms, so are $(t_1 + t_2), (t_1 - t_2)$, and $(t_1 \cdot t_2)$;

(c) if t is a numerical term and f is a function variable, then $f(t)$ is a numerical term.

Functional terms are recursively defined as follows:

(a) function variables and the functional constants **0**, **1** are functional terms;

(b) if F_1, F_2 are functional terms, so are $(F_1 + F_2)$ and $(F_1 - F_2)$.

In the following, the expression *numerical variable* will be used also to denote the constants $0, 1$. Likewise, the expression *function variable* will be used also to denote the functional constants **0**, **1**

By *extended numerical variable* we mean a numerical variable or one of the symbols $-\infty, +\infty$. Likewise, by *extended numerical term* we mean a numerical term or one of the symbols $-\infty, +\infty$.

An atomic RMCF^+-formula is an expression having one of the following forms:

$$\begin{array}{ll} t_1 = t_2\,, & t_1 > t_2\,, \\ (F_1 = F_2)_A\,, & (F_1 > F_2)_{[t_1,t_2]}\,, \\ \mathsf{Up}(F)_A\,, & \mathsf{Strict\_Up}(F)_A\,, \\ \mathsf{Down}(F)_A\,, & \mathsf{Strict\_Down}(F)_A\,, \\ \mathsf{Convex}(F)_A\,, & \mathsf{Strict\_Convex}(F)_A\,, \\ \mathsf{Concave}(F)_A\,, & \mathsf{Strict\_Concave}(F)_A\,, \end{array}$$

where A stands for any of the following interval terms

$$[t_1, t_2], \quad [t_1, +\infty[, \quad]-\infty, t_2], \quad]-\infty, +\infty[\,,$$

t_1, t_2 are numerical terms, and F, F_1, F_2 are functional terms.[4]

The formulae of RMCF^+ are propositional combinations of atomic formulae by means of the usual connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. Let us stress again that explicit quantification is not admitted.

To ease readability, occasionally we will use abbreviations. For instance, if t_1, t_2, t_3 are numerical terms, then $t_1 = t_2/t_3$ is a shorthand for the conjunction

$$(t_2 = t_1 \cdot t_3) \wedge (\neg(t_3 = 0))\,.$$

[4]Notice that literals of type $F_1 > F_2$ are admitted in RMCF^+-formulae only if restricted to finite closed intervals, rather than to possibly infinite closed intervals, as is the case for all remaining literals involving functional terms. This is due to the facts that (a) the satisfiability test for RMCF^+-formulae is based on the property that any satisfiable RMCF^+-formula admits a *canonical* model M sending function variables to piecewise linear functions with *small* quadratic perturbations on finite internal intervals and *small* exponential perturbations on the two external infinite intervals; (b) there are problems in satisfying literals of type $F_1 > F_2$ on the two external infinite intervals using linear functions with exponential perturbations in the presence of literals of the remaining types, involving functional terms.

Likewise, $t_1 > t_2/t_3$ is a shorthand for the formula

$$\Big((t_1 \cdot t_3 > t_2) \wedge (t_3 > 0)\Big) \vee \Big((t_2 > t_1 \cdot t_3) \wedge (0 > t_3)\Big).$$

And so on.

2.1.2. Semantics of RMCF^+

An RMCF *interpretation* for the language RMCF^+ is a map M defined over terms and formulae of RMCF^+ as follows:

(a) for every numerical variable x distinct from $0, 1$, Mx is a real number;

(b) the numerical constants $0, 1$ are interpreted as the real numbers $0, 1$, respectively;

(c) the functional constants $\mathbf{0}$, $\mathbf{1}$ are interpreted as the constant functions with values 0 and 1, respectively, defined over the whole real axis $\mathbb{R}$;

(d) for each function variable f distinct from $\mathbf{0}$, $\mathbf{1}$, Mf is a continuous real function of a real variable over the whole axis $\mathbb{R}$;

(e) for each numerical term $t_1 \otimes t_2$, with $\otimes \in \{+, -, \cdot\}$, $M(t_1 \otimes t_2)$ is the real number $Mt_1 \otimes Mt_2$;

(f) for each numerical term $f(t)$, $M(f(t))$ is the real number $(Mf)(Mt)$;

(g) for each functional term $F_1 \oplus F_2$, with $\oplus \in \{+, -\}$, $M(F_1 \otimes F_2)$ is the function $MF_1 \oplus MF_2$;

(h) let t_1, t_2 be numerical terms, F, G functional terms, and A an interval term of the form

$$[t_1, t_2], \quad [t_1, +\infty[, \quad]-\infty, t_2], \quad]-\infty, +\infty[\,.$$

Let MA be the interpretation of the interval term A, namely

$$MA = \begin{cases} [Mt_1, Mt_2] & \text{if } A = [t_1, t_2]\,, \\ [Mt_1, +\infty[& \text{if } A = [t_1, +\infty]\,, \\ ]-\infty, Mt_2] & \text{if } A =]-\infty, t_2]\,, \\ ]-\infty, +\infty[& \text{if } A =]-\infty, +\infty]\,. \end{cases}$$

(h.1) $M(t_1 = t_2)$ (resp., $M(t_1 > t_2)$) is true if and only if $Mt_1 = Mt_2$ (resp., $Mt_1 > Mt_2$);

(h.2) $M\big((F > G)_{[t_1,t_2]}\big)$ is true if and only if $(MF)(x) > (MG)(x)$ for all $x \in [Mt_1, Mt_2]$ (thus $M\left((F > G)_{[t_1,t_2]}\right)$ is vacuously true whenever $Mt_1 > Mt_2$; a similar observation applies to the cases below);

(h.3) $M((F = G)_A)$ is true if and only if $(MF)(x) = (MG)(x)$ for all $x \in MA$;

(h.4) $M(\mathsf{Up}(F)_A)$ (resp., $M(\mathsf{Strict\_Up}(F)_A)$) is true if and only if the function MF is monotonically nondecreasing (resp., strictly increasing) in the interval MA;

(h.5) $M(\mathsf{Down}(F)_A)$ (resp., $M(\mathsf{Strict\_Down}(F)_A)$) is true if and only if the function MF is monotonically nonincreasing (resp., strictly decreasing) in the interval MA;

(h.6) $M(\mathsf{Convex}(F)_A)$ (resp., $M(\mathsf{Strict\_Convex}(F)_A)$) is true if and only if the function MF is convex (resp., strictly convex) in the interval MA;

(h.7) $M(\mathsf{Concave}(F)_A)$ (resp., $M(\mathsf{Strict\_Concave}(F)_A)$) is true if and only if the function MF is concave (resp., strictly concave) in the interval MA.

2.1.3. A decision procedure for RMCF^+ formulae: an overview

We briefly review below a decision procedure for the satisfiability problem for RMCF^+ formulae, namely an algorithm which given any RMCF^+ formula φ tells one whether or not φ is satisfiable by a real model.

Phase 1: The first phase of the algorithm consists in transforming the input formula φ into an equisatisfiable formula of the form $\bigvee_{i=1}^{n} \varphi_i$, where each φ_i, for $i = 1, \ldots, n$, is in *standard ordered form*, i.e.,

(a) φ_i is a conjunction of literals of the following simple types

$$\begin{array}{ll} x = y + w\,, & x = y \cdot w\,, \\ x > y\,, & y = f(x)\,, \\ (f = g + h)_A\,, & (f > g)_{[x_1,x_2]}\,, \\ \mathsf{Up}(f)_A\,, & \mathsf{Strict\_Up}(f)_A\,, \\ \mathsf{Convex}(f)_A\,, & \mathsf{Strict\_Convex}(f)_A\,, \end{array} \tag{1}$$

where A is an interval term of any of the following types

$$[x_1, x_2]\,, \quad [x_1, +\infty[\,, \quad]-\infty, x_2]\,, \quad]-\infty, +\infty[\,,$$

x, y, w, x_1, x_2 are numerical variables, and f, g, h are function variables.

(b) Let $x_1, \dots, x_n$ be the *domain variables* of φ_i, namely the numerical variables x which appear in φ_i either within a functional term of the form $f(x)$ or as one of the two extremes w_b (other than $\pm\infty$) in an interval term of the form $[w_1, w_2]$. Then there exists a permutation π of $\langle 1, \dots, n \rangle$ such that φ_i contains the literals $x_{\pi(j+1)} > x_{\pi(j)}$, for $j = 1, \dots, n-1$ (the conjunction of such literals yields a strict ordering of the domain variables).

For instance, the formula

$$\mathsf{Down}(f)_{[x,y]} \wedge y = f(x)$$

is transformed into the equisatisfiable formula

$$\begin{aligned}\Big((\mathbf{0} = f + g)_{[x,y]} \wedge \mathsf{Up}(g)_{[x,y]} \wedge y = f(x) \wedge x > y\Big) \\ \vee \Big((\mathbf{0} = f + g)_{[x,y]} \wedge \mathsf{Up}(g)_{[x,y]} \wedge y = f(x) \wedge (x = y + 0)\Big) \\ \vee \Big((\mathbf{0} = f + g)_{[x,y]} \wedge \mathsf{Up}(g)_{[x,y]} \wedge y = f(x) \wedge y > x\Big) .\end{aligned}$$

Since φ is satisfiable if and only if at least one of the φ_i is satisfiable, Phase 1 allows one to reduce the satisfiability problem for general RMCF^+ formulae to the satisfiability problem for RMCF^+ conjunctions of simple atomic formulae of the types (1) in standard ordered form.

As we have noted for Lemma 1.1, in this phase a combinatorial explosion can take place, which should be counteracted by suitable measures in the implementation of the algorithm (cf. [2, p. 775]).

The subsequent phases of the algorithm will therefore address the satisfiability problem for RMCF^+ conjunctions in standard ordered form.

Thus, let φ_i be a RMCF^+ conjunction in standard ordered form (for instance, one of the conjuncts resulting from Phase 1).

Phase 2: In this phase all function variables present in φ_i are evaluated over the domain variables of φ_i. In other words, for each domain variable v_j of φ_i and each function variable f occurring in φ_i, the conjunct

$$y_j^f = f(v_j) ,$$

where y_j^f is a freshly introduced numerical variable, is added to φ_i.

In addition, for each literal $x = f(v_j)$ initially present in φ_i, the literal

$$x = y_j^f$$

is added to φ_i.

Let ψ be the resulting formula. Plainly, ψ and φ_i are equisatisfiable.

For instance, the formula

$$\mathsf{Convex}(f)_{[x,y]} \wedge y > x$$

is transformed into the equisatisfiable formula

$$\mathsf{Convex}(f)_{[x,y]} \wedge y > x \wedge z = f(x) \wedge t = f(y) .$$

Phase 3: During this phase, all literals involving function variables, namely those of the form

$$\begin{array}{lll} y = f(x) , & (f = g + h)_A , & (f > g)_{[x_1,x_2]} , \\ \mathsf{Up}(f)_A , & \mathsf{Convex}(f)_A , & \\ \mathsf{Strict\_Up}(f)_A , & \mathsf{Strict\_Convex}(f)_A , & \end{array}$$

are removed from the formula ψ resulting from Phase 2 and are replaced by suitable RMCF^+ conjuncts not involving function variables. Thus, the resulting conjunction is a quantifier-free formula, which can be readily tested for satisfiability by any decider for Tarski's theory of reals.

This is the most critical phase of the algorithm, from the correctness point of view. Indeed, while it is not difficult to eliminate function symbols from ψ in such a way that the resulting RMCF^+ formula ψ_1 is satisfiable whenever so is the input formula ψ, particular care must be taken in order that the reverse implication holds too, namely that ψ is satisfiable whenever so is ψ_1.

Let us see in detail the steps of Phase 3. Let $V = \{v_1, \dots, v_r\}$ be the collection of the domain variables of ψ and assume that ψ contains the literals $v_{i+1} > v_i$, for $i = 1, \dots, r-1$ (see (b) in Phase 1). Let $ind : V \cup \{-\infty, +\infty\} \longrightarrow \{1, 2, \dots, r\}$ be the *index function* of V, where

- $ind(v_i) = i$, for $i = 1, \dots, r$,
- $ind(-\infty) = 1$ and $ind(+\infty) = r$.

Also, for each function variable f in ψ, let us introduce the new numerical variables γ_0^f, γ_r^f, and α_j^f, for $j = 0, 1, \dots, r$.

We perform the following six transformation steps (five addition steps and one, the last, elimination step).

1. For each literal of the type $(f = g + h)_{[w_1,w_2]}$ in ψ, where f, g, h are function variables and w_1, w_2 are extended numerical variables, we add the following literals:

$$y_i^f = y_i^g + y_i^h , \qquad \alpha_j^f = \alpha_j^g + \alpha_j^h ,$$

for every i such that $ind(w_1) \leqslant i \leqslant ind(w_2)$ and for every j such that $ind(w_1) \leqslant j \leqslant ind(w_2) - 1$.

In addition, if $w_1 = -\infty$, we add also the following two literals:

$$\alpha_0^f = \alpha_0^g + \alpha_0^h\,, \qquad \gamma_0^f = \gamma_0^g + \gamma_0^h\,.$$

Likewise, if $w_2 = +\infty$, we add also the following two literals:

$$\alpha_r^f = \alpha_r^g + \alpha_r^h\,, \qquad \gamma_r^f = \gamma_r^g + \gamma_r^h\,.$$

2. For each literal of the type $(f > g)_{[w_1,w_2]}$ present in ψ, where f, g are function variables and w_1, w_2 are numerical variables, we add the following literals:

$$y_j^f - y_j^g > |\alpha_j^f| + |\alpha_j^g|\,, \qquad y_{j+1}^f - y_{j+1}^g > |\alpha_j^f| + |\alpha_j^g|\,,$$

for every j such that $ind(w_1) \leqslant j \leqslant ind(w_2)$ (here and in the following it is to be understood that literals containing the absolute value function are to be considered as shorthands for equivalent RMCF^+ formulae with no occurrence of the absolute value).

3. For each literal of the form $\mathsf{Up}(f)_{[w_1,w_2]}$ in ψ, where f is a function variable and w_1, w_2 are extended numerical variables, we add the following literals:

$$y_{j+1}^f - y_j^f \geqslant 4|\alpha_j^f|\,,$$

for every j such that $ind(w_1) \leqslant j \leqslant ind(w_2) - 1$.

In addition, if $w_1 = -\infty$, we add also the following two literals:

$$\gamma_0^f \geqslant 0\,, \qquad \gamma_0^f \geqslant \alpha_0^f\,.$$

Likewise, if $w_2 = +\infty$, we add also the following two literals:

$$\gamma_r^f \geqslant 0\,, \qquad \alpha_r^f + \gamma_r^f \geqslant 0\,.$$

For literals of the form $\mathsf{Strict\_Up}(f)$, we proceed much in the same way, but using the strict inequality $>$ in place of $\geqslant$.

4. For each literal of the type $\mathsf{Convex}(f)_{[w_1,w_2]}$ in ψ, where f is a function variable and w_1, w_2 are extended numerical variables, we add the following literals:

$$0 \geqslant \alpha_i^f\,, \qquad \alpha_j^f \geqslant \frac{1}{4}\Big[y_j^f - y_{j+1}^f + (y_j^f - y_{j-1}^f - 4\alpha_{j-1}^f)\frac{v_{j+1} - v_j}{v_j - v_{j-1}}\Big]\,,$$

for every i such that $ind(w_1) \leqslant i \leqslant ind(w_2) - 1$ and every j such that $ind(w_1) < j < ind(w_2)$.

In addition, if $w_1 = -\infty$, we add also the following literal

$$0 \geqslant \alpha_0^f$$

and, provided that $w_2 \neq v_1$, also the literal

$$\frac{y_2^f - y_1^f + 4\alpha_1^f}{v_2 - v_1} \geqslant \gamma_0^f - \alpha_0^f \,.$$

Likewise, if $w_2 = +\infty$, we add also the following literal

$$0 \geqslant \alpha_r^f$$

and, provided that $w_1 \neq v_r$, also the literal

$$\alpha_r^f + \gamma_r^f \geqslant \frac{y_r^f - y_{r-1}^f - 4\alpha_{r-1}^f}{v_r - v_{r-1}} \,.$$

5. For each literal of the type $\mathsf{Strict\_Convex}(f)_{[w_1,w_2]}$ in ψ, where f is a function variable and w_1, w_2 are extended numerical variables, we add the following literals:

$$0 > \alpha_i^f \,, \qquad \alpha_j^f \geqslant \frac{1}{4}\Big[y_j^f - y_{j+1}^f + (y_j^f - y_{j-1}^f - 4\alpha_{j-1}^f)\frac{v_{j+1} - v_j}{v_j - v_{j-1}}\Big] \,,$$

for every i such that $ind(w_1) \leqslant i \leqslant ind(w_2) - 1$ and every j such that $ind(w_1) < j < ind(w_2)$.

In addition, if $w_1 = -\infty$, we add also the following literal

$$0 > \alpha_0^f$$

and, provided that $w_2 \neq v_1$, also the literal

$$\frac{y_2^f - y_1^f + 4\alpha_1^f}{v_2 - v_1} \geqslant \gamma_0^f - \alpha_0^f \,.$$

Likewise, if $w_2 = +\infty$, we add also the following literal

$$0 > \alpha_r^f$$

and, provided that $w_1 \neq v_r$, also the literal

$$\alpha_r^f + \gamma_r^f \geqslant \frac{y_r^f - y_{r-1}^f - 4\alpha_{r-1}^f}{v_r - v_{r-1}} \,.$$

6. Finally, we drop from ψ all literals involving function variables.

For instance, the formula

$$(f = g + h)_{[x,y]} \wedge y > x \wedge z_1 = f(x) \wedge z_2 = f(y)$$
$$\wedge\, t_1 = g(x) \wedge t_2 = g(y) \wedge s_1 = h(x) \wedge s_2 = h(y)$$

is transformed into the equisatisfiable formula

$$y > x \wedge z_1 = f(x) \wedge z_2 = f(y)$$
$$\wedge\, t_1 = g(x) \wedge t_2 = g(y) \wedge s_1 = h(x) \wedge s_2 = h(y) \wedge$$
$$(z_1 = t_1 + s_1) \wedge (z_2 = t_2 + s_2) \wedge (\alpha_f = \alpha_g + \alpha_h)\,.$$

Let ψ_1 be the resulting formula, after the execution of the steps 1–6 above. As already remarked, it can easily be shown that if ψ is satisfiable, so is ψ_1. On the other hand, if ψ_1 is satisfied by a real model M, then for each function variable f thanks to the constraints introduced during the first five addition steps above, it can be shown that there exists a function Mf which can be obtained by perturbing quadratically and exponentially a piecewise linear function through the points (Mv_j, My_j^f), for $j = 1, \ldots, r$. It turns out that the real assignment M so extended over the function variables of ψ is a model for all literals of ψ. Since ψ_1 is a quantifier-free formula of Tarski's theory of reals, its satisfiability can be tested algorithmically.

As a universally closed RMCF^+ statement is valid if and only if its negation is unsatisfiable, the satisfiability test for RMCF^+ outlined above can also be used to test the validity (i.e., theoremhood) of the universal closure of formulae of RMCF^+. Thus we have the following result.

Proposition 2.1 ([2, Section 3, Theorem 1]). *The validity problem for universally closed* RMCF^+ *statements is decidable. In other words, one can test algorithmically whether any universally closed* RMCF^+ *statement is a theorem or not.*

2.1.4. Formalization in RMCF^+ of elementary lemmas in real analysis

We show by way of some examples that the theory RMCF^+ is expressive enough to allow the formulation of some elementary lemmas in real analysis, which can be proved automatically by the decision procedure outlined above.

Example 2.2. *Consider the claim:*

"Let f and g be two real functions defined over a closed bounded interval $[a,b]$, such that $f(a) = g(a)$ and $f(b) = g(b)$. If f is strictly convex and g is concave, then $f(x) < g(x)$ for each $x \in]a,b[$."

This can be formalized by the universal closure of the RMCF^+ *formula*

$$\Big(\mathsf{Strict\_Convex}(f)_{[a,b]} \wedge \mathsf{Concave}(g)_{[a,b]} \wedge f(a) = g(a) \\ \wedge f(b) = g(b) \wedge b > x \wedge x > a\Big) \rightarrow (g(x) > f(x)) . \quad (2)$$

To show that (2) is valid, it is sufficient to prove that its negation

$$\mathsf{Strict\_Convex}(f)_{[a,b]} \wedge \mathsf{Concave}(g)_{[a,b]} \wedge f(a) = g(a) \\ \wedge f(b) = g(b) \wedge b > x \wedge x > a \wedge \neg(g(x) > f(x))$$

is unsatisfiable. After the normalization phase (Phase 1), we obtain

$$\Big[\mathsf{Strict\_Convex}(f)_{[a,b]} \wedge \mathsf{Convex}(h)_{[a,b]} \wedge (\mathbf{0} = g + h)_{[a,b]} \\ \wedge f(a) = g(a) \wedge f(b) = g(b) \wedge b > x \wedge x > a \wedge (f(x) > g(x))\Big] \\ \vee \Big[\mathsf{Strict\_Convex}(f)_{[a,b]} \wedge \mathsf{Convex}(h)_{[a,b]} \wedge (\mathbf{0} = g + h)_{[a,b]} \\ \wedge f(a) = g(a) \wedge f(b) = g(b) \wedge b > x \wedge x > a \wedge (f(x) = g(x))\Big] .$$

Then, after executing the subsequent phases of the decision algorithm, we obtain the inequalities

$$(f(b) - f(x)) \cdot (x - x_1) > (f(x) - f(a)) \cdot (x_2 - x_1) \\ (-g(b) + g(x)) \cdot (x - x_1) \geqslant (-g(x) + g(a)) \cdot (x_2 - x_1)$$

which, together with $f(a) = g(a)$ *e* $f(b) = g(b)$*, imply* $f(x) < g(x)$*, contradicting both* $f(x) > g(x)$ *and* $f(x) = g(x)$*.*

Having proved that the negation of (2) is unsatisfiable, it follows that (2) is valid, thus proving that our claim expresses a theorem.

A second example is the following.

EXAMPLE 2.3. *Consider the claim:*

"Let f *and* g *be two real functions defined over a closed bounded interval* $[a, b]$*, such that* f *is strictly convex and* g *is concave in* $[a, b]$*. Then there exist at most two distinct points* $x, y \in [a, b]$ *such that* $f(x) = g(x)$ *and* $f(y) = g(y)$ *(i.e., the graphs of* f *and* g *meet in at most two points in* $[a, b]$*)."*

Observe that it can be formalized as the universal closure of the following

RMCF^+ *formula*

$$\Big[\mathsf{Strict\_Convex}(f)_{[a,b]} \wedge \mathsf{Concave}(g)_{[a,b]} \\ \wedge (a \leqslant x_1 \leqslant b) \wedge (a \leqslant x_2 \leqslant b) \wedge (a \leqslant x_3 \leqslant b) \\ \wedge f(x_1) = g(x_1) \wedge f(x_2) = g(x_2) \wedge f(x_3) = g(x_3)\Big] \\ \rightarrow \Big[(x_1 = x_2) \vee (x_1 = x_3) \vee (x_2 = x_3)\Big]$$

and therefore it can be proved automatically.

2.2. An overview of the theory RDF

The theory RDF (of Reals with Differentiable Functions) is an unquantified first-order theory involving various predicates on real functions of class C^1 of one real variable, namely functions with continuous first derivative. Predicates of RDF concern comparison of functions, strict and non-strict monotonicity, strict and non-strict convexity (and concavity), and comparison of first derivatives with real constants. Specifically, the atomic formulae of RDF are:

$$\begin{array}{ll} t_1 = t_2\,, & t_1 > t_2\,, \\ (f = g)_A\,, & (f > g)_{[t_1,t_2]}\,, \\ \mathsf{Up}(f)_A\,, & \mathsf{Strict\_Up}(f)_A\,, \\ \mathsf{Down}(f)_A\,, & \mathsf{Strict\_Down}(f)_A\,, \\ \mathsf{Convex}(f)_A\,, & \mathsf{Strict\_Convex}(f)_A\,, \\ \mathsf{Concave}(f)_A\,, & \mathsf{Strict\_Concave}(f)_A\,, \\ (D[f] \geqslant t)_A\,, & (D[f] > t)_A\,, \\ (D[f] \leqslant t)_A\,, & (D[f] < t)_A\,, \\ (D[f] = t)_A\,, & \end{array}$$

where A stands for any of the following interval terms

$$[t_1, t_2], \quad [t_1, +\infty[, \quad]-\infty, t_2], \quad]-\infty, +\infty[\,,$$

t_1, t_2 are numerical terms, and f, g stand for function variables or the functional constants $\mathbf{0}$ and $\mathbf{1}$. Numerical terms are arithmetic expressions involving real variables, the real constants 0, 1, functional expressions of the form $f(t)$, and the arithmetic operators.

Formulae of RDF are propositional combinations of atomic RDF-formulae with the usual logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. Again, explicit quantification is not allowed.

Function variables are interpreted by real functions of a real variable, defined on the whole real axis $\mathbb{R}$, differentiable over $\mathbb{R}$ and with continuous derivative. The functional constants $\mathbf{0}$ and $\mathbf{1}$ are interpreted as the constant functions with values 0 and 1, respectively. Predicates of type $(f > g)_{[t_1,t_2]}$ assert that the function f strictly dominates g in the closed bounded interval $[t_1, t_2]$. The remaining atomic formulae on functions can refer also to closed half-bounded intervals $[t_1, +\infty[$ and $]-\infty, t_2]$ and to the whole real axis $]-\infty, +\infty[$.

Based on the above indications and in analogy with what has been done in the preceding section, one can give a precise definition of RDF-interpretations. Then, satisfiable RDF-formulae are those which admit at least one satisfying interpretation (real model), and valid RDF-formulae (RDF-theorems) are those which are satisfied by all interpretations.

Domenico Cantone and Gianluca Cincotti have proved in recent years that:

- An RDF-formula φ is satisfiable if and only if it admits a *canonical* real model M which interprets the function variables of φ as piecewise linear real functions with *small* quadratic and exponential perturbations.
- Canonical models can be encoded by finitely many parameters satisfying suitable arithmetical conditions. These can be tested for satisfiability by any decision procedure for the existential Tarski's theory of reals.
- Thereby one gets the solvability of the satisfiability problem for RDF-formulae; consequently, solvability of the validity problem for RDF-formulae, because a formula is valid if and only if its negation is unsatisfiable.

The results on which we are reporting can be summarized as follows:

PROPOSITION 2.4. RDF *has solvable satisfiability and validity problems.*[5]

Before outlining the decision algorithm for RDF, we illustrate the expressiveness of this theory by formalizing in it some simple lemmas of elementary real analysis.

EXAMPLE 2.5. *Consider the claim:*
"Let f be a real function of class C^1 on the closed interval $[a, b]$, with constant first derivative. Then f is linear in $[a, b]$."
Plainly, this claim can be formalized by the RDF*-formula*

$$(D[f] = t)_{[a,b]} \rightarrow \Big(\mathsf{Convex}(f)_{[a,b]} \wedge \mathsf{Concave}(f)_{[a,b]}\Big)$$

and therefore it can be verified automatically by a decision procedure for RDF.

---

[5]A communication—as yet unpublished—of these results, "*Decision algorithms for fragments of real analysis. II. A theory of differentiable functions with convexity and concavity predicates*" was offered by D. Cantone and G. Cincotti at the Italian conference "Convegno italiano di Logica Computazionale" (CILC'07), 21–22 June 2007, Messina.

A continuation, due to D. Cantone and G.T. Spartà, of that study is in progress: "*Decision algorithms for fragments of real analysis. III. A theory of differentiable functions with (semi-) open intervals*". Motivations for extending RDF so as to overcome some of its expressive limitations will emerge from the discussion of Examples 2.6 and 2.7 below.

Another example is the following.

EXAMPLE 2.6 (Weak form of Rolle's theorem). *Consider the claim:*
"Let f be a real function of class C^1 on the closed interval $[a, b]$ such that $f(a) = f(b)$, $f'(a) \neq 0$, and $f'(b) \neq 0$. Then there exists $c \in]a, b[$ such that $f'(c) = 0$."
In view of the continuity of the first derivative f', this claim can be formalized by the following RDF*-formula*

$$\Big(a < b \wedge f(a) = f(b) \wedge D[f](a) \neq 0 \wedge D[f](b) \neq 0\Big) \\ \rightarrow \neg\Big((D[f] > 0)_{[a,b]} \vee (D[f] < 0)_{[a,b]}\Big)$$

and therefore it can be verified automatically by a decision procedure for RDF.

A final example is the following.

EXAMPLE 2.7 (Weak form of the mean-value theorem). *Consider the claim:*
"Let f be a real function of class C^1 on the closed interval $[a, b]$ such that $f'(a) \neq \dfrac{f(b) - f(a)}{b - a}$, and $f'(b) \neq \dfrac{f(b) - f(a)}{b - a}$. Then there exists $c \in]a, b[$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$."
Note that this claim generalizes that of the preceding example. Thus, again by the continuity of the first derivative f', it can be formalized in RDF *as follows:*

$$\Big(a < b \wedge x = \frac{f(b) - f(a)}{b - a} \wedge D[f](a) \neq x \wedge D[f](b) \neq x\Big) \\ \rightarrow \neg\Big((D[f] > x)_{[a,b]} \vee (D[f] < x)_{[a,b]}\Big) .$$

In Example 2.6 we had to exclude the cases in which either $f'(a) = 0$ or $f'(b) = 0$, because $(D[f] > 0)_{[a,b]} \vee (D[f] < 0)_{[a,b]}$ expresses that $D[f]$ is nonzero in the closed interval $[a, b]$, rather than in the open interval $]a, b[$. A similar remark applies to Example 2.7, where we had to assume the extra assumptions $f'(a) \neq \frac{f(b)-f(a)}{b-a}$, and $f'(b) \neq \frac{f(b)-f(a)}{b-a}$. If we could express literals of the forms $(D[f] < t)_{]a,b[}$ and $(D[f] > t)_{]a,b[}$, relative to open intervals, in both cases we could get rid of those extra assumptions.

Such remarks have motivated the study—just mentioned in a footnote—of the extension RDF^+ of RDF with literals of any of the forms

$$(f > g)_A\,, \quad (D[f] > t)_B\,, \quad (D[f] < t)_B\,, \quad (D[f] \neq t)_B\,,$$

where A stands for an open or semi-open bounded interval and B stands for an open or semi-open interval which is not necessarily bounded.

2.2.1. The decision algorithm for RDF, in outline

Much like the decision algorithm for RMCF^+, the one for RDF begins with a *normalization phase* which transforms the input formula φ into an equisatisfiable disjunction $\bigvee_{i=1}^{n} \varphi_i$, where each φ_i is a conjunction in *standard ordered form*. While the ordering condition concerning the *domain variables* of each φ_i is as before (but here we include among the domain variables also every x appearing in a term $D[f](x)$ within φ_i), the forms of the literals constituting φ_i are, for the theory at hand:

$$\begin{array}{ll}
x = y + w\,, & x = y \cdot w\,, \\
x > y\,, & y = f(x)\,, \\
(f = g)_A\,, & (f > g)_{[x_1,x_2]}\,, \\
y = D[f](x)\,, & (D[f] \bowtie y)_A\,, \\
\mathsf{Strict_Up}(f)_A\,, & \mathsf{Strict_Down}(f)_A\,, \\
\mathsf{Convex}(f)_A\,, & \mathsf{Strict_Convex}(f)_A\,, \\
\mathsf{Concave}(f)_A\,, & \mathsf{Strict_Concave}(f)_A\,,
\end{array} \tag{3}$$

where $\bowtie\ \in \{=, >, \geqslant, <, \leqslant\}$, A is an interval term of any of the following types

$$[x_1, x_2]\,, \qquad [x_1, +\infty[\,, \qquad]-\infty, x_2]\,, \qquad]-\infty, +\infty[\,,$$

x, y, w, x_1, x_2 are numerical variables, and f, g are function variables. Notice that all negative literals are eliminated by the transformation rules exploited in this phase (all of which are, conceptually, rather simple).

In order to determine whether or not φ is satisfiable, we must check one by one its disjuncts φ_i until either one of them turns out to be satisfiable, or all disjuncts have been examined without success. In preparation for this, we explicitly evaluate all function variables present in each φ_i over the domain variables of φ_i. The way to do this is closely analogous to the one discussed earlier for RMCF^+: we associate new variables y_j^f, t_j^f with each combination of a domain variable v_j of φ_i with a function variable f also appearing in φ_i, and conjoin the literals

$$y_j^f = f(v_j)\,, \qquad t_j^f = D[f](v_j)$$

with φ_i. For each literal $x = f(v_j)$ occurring in φ_i, we then insert the literal $x = y_j^f$ into φ_i; likewise, for each literal $x = D[f](v_j)$ in φ_i, we introduce the equality $x = t_j^f$. Each φ_i produced by the normalization phase is thereby transformed by the present phase into an equisatisfiable conjunction ψ_i.

We will now describe the main phase, which eliminates from each ψ_i all literals that involve function variables.

Let $V = \{v_1, v_2, \ldots, v_r\}$ be the collection of the domain variables of ψ_i with their implicit ordering, and let the index function $ind\, :\, V \cup \{-\infty, +\infty\} \longrightarrow$

$\{1, 2, \ldots, r\}$ be defined as follows:

$$ind(x) =_{\text{Def}} \begin{cases} 1 & \text{if } x = -\infty, \\ l & \text{if } x = v_l, \text{ for some } l \in \{1, 2, \ldots, r\}, \\ r & \text{if } x = +\infty. \end{cases}$$

For each function symbol f occurring in ψ_i, introduce new numerical variables γ_0^f, γ_r^f and proceed as follows:

1. For each literal of type $(f{=}g)_{[z_1,z_2]}$ occurring in ψ_i, add the literals:

$$y_i^f = y_i^g, \qquad t_i^f = t_i^g,$$

for $i \in \{ind(z_1), \ldots, ind(z_2)\}$; moreover, if $z_1 = -\infty$, add the literal:

$$\gamma_0^f = \gamma_0^g\,;$$

likewise, if $z_2 = +\infty$, add the literal:

$$\gamma_r^f = \gamma_r^g\,.$$

2. For each literal of type $(f{>}g)_{[w_1,w_2]}$ occurring in ψ_i, add the literal:

$$y_i^f > y_i^g\,,$$

for $i \in \{ind(w_1), \ldots, ind(w_2)\}$.

3. For each literal of type $(D[f]{\bowtie}y)_{[z_1,z_2]}$ occurring in ψ_i, where $\bowtie \in \{=, <, \leqslant, >, \geqslant\}$, add the formulae:

$$t_i^f \bowtie y\,,$$
$$\frac{y_{j+1}^f - y_j^f}{v_{j+1} - v_j} \bowtie y,$$

for $i, j \in \{ind(z_1), \ldots, ind(z_2)\}$, $j \neq ind(z_2)$. Moreover, if $\bowtie \in \{\leqslant, \geqslant\}$ also add the implication:

$$\left(\frac{y_{j+1}^f - y_j^f}{v_{j+1} - v_j} = y\right) \longrightarrow (t_j^f = y \wedge t_{j+1}^f = y);$$

moreover, if $z_1 = -\infty$, add the formula:

$$\gamma_0^f \bowtie y,$$

and if $z_2 = +\infty$, add the formula:

$$\gamma_r^f \bowtie y.$$

4. For each literal of type $\mathsf{Strict\_Up}(f)_{[z_1,z_2]}$ (resp. $\mathsf{Strict\_Down}(f)_{[z_1,z_2]}$) occurring in ψ_i, add the formulae:
$$t_i^f \geqslant 0 \qquad (\text{resp. } t_i^f \leqslant 0),$$
$$y_{j+1}^f > y_j^f \qquad (\text{resp. } y_{j+1}^f < y_j^f),$$
for $i, j \in \{ind(z_1), \ldots, ind(z_2)\}$, $j \neq ind(z_2)$. Moreover, if $z_1 = -\infty$, add the formula:
$$\gamma_0^f > 0 \qquad (\text{resp. } \gamma_0^f < 0),$$
and if $z_2 = +\infty$, add the formula:
$$\gamma_r^f > 0 \qquad (\text{resp. } \gamma_r^f < 0).$$

5. For each literal of type $\mathsf{Convex}(f)_{[z_1,z_2]}$ (resp. $\mathsf{Concave}(f)_{[z_1,z_2]}$) occurring in ψ_i, add the following formulae:[6]
$$t_i^f \leqslant \frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} \leqslant t_{i+1}^f \qquad (\text{resp. } \geqslant),$$
$$\left(\frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} = t_i^f \ \vee \ \frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} = t_{i+1}^f \right) \longrightarrow (t_i^f = t_{i+1}^f),$$
for $i \in \{ind(z_1), \ldots, ind(z_2)-1\}$; moreover, if $z_1 = -\infty$, add the formula:
$$\gamma_0^f \leqslant t_1^f \qquad (\text{resp. } \gamma_0^f \geqslant t_1^f),$$
and if $z_2 = +\infty$, add the formula:
$$\gamma_r^f \geqslant t_r^f \qquad (\text{resp. } \gamma_r^f \leqslant t_r^f).$$

6. For each literal of type $\mathsf{Strict\_Convex}(f)_{[z_1,z_2]}$ (resp. $\mathsf{Strict\_Concave}(f)_{[z_1,z_2]}$) occurring in ψ_i, add the following formulae:
$$t_i^f < \frac{y_{i+1}^f - y_i^f}{v_{i+1} - v_i} < t_{i+1}^f \qquad (\text{resp. } >),$$
for $i \in \{ind(z_1), \ldots, ind(z_2)-1\}$; moreover, if $z_1 = -\infty$, add the formula:
$$\gamma_0^f < t_1^f \qquad (\text{resp. } \gamma_0^f > t_1^f),$$
and if $z_2 = +\infty$, add the formula:
$$\gamma_r^f > t_r^f \qquad (\text{resp. } \gamma_r^f < t_r^f).$$

[6] Observe that this group of formulae implicitly forces the relations $\frac{y_j^f - y_{j-1}^f}{v_j - v_{j-1}} \leqslant \frac{y_{j+1}^f - y_j^f}{v_{j+1} - v_j}$ for each $j \in \{ind(z_1)+1, \ldots, ind(z_2)-1\}$. Geometrically, the point of coordinates (v_j, y_j^f) does not lie above (resp. lies below) the straight line joining the two points (v_{j-1}, y_{j-1}^f) and (v_{j+1}, y_{j+1}^f).

7. Withdraw all literals where function variables appear.

In conclusion, the formula χ_i resulting from ψ_i through the function variable removal phase just described only involves literals of the following types:

$$t_1 \leqslant t_2\,, \qquad t_1 < t_2\,, \qquad t_1 = t_2\,,$$

where t_1 and t_2 are terms involving only real variables, the real constants 0 and 1, and the arithmetic operators $+$ and $\cdot$ (and their counterparts $-$ and $/$), so that the formula χ_i belongs to the decidable (existential) Tarski's theory of reals. Showing that our theory RDF has a solvable satisfiability problem simply amounts to showing that the main phase leading from ψ_i to χ_i preserves satisfiability. The proof of this fact, albeit not particularly deep, requires a somewhat technical and lengthy proof, which we omit here.

References

[1] J. P. Burgess and Y. Gurevich, *The decision problem for linear temporal logic*, Notre Dame J. Formal Logic **26** (1985), 115–128.

[2] D. Cantone, G. Cincotti and G. Gallo, *Decision algorithms for fragments of real analysis. I. Continuous functions with strict convexity and concavity predicates*, J. Symbolic Comput. **41** (2006), 763–789.

[3] D. Cantone, A. Ferro, E. G. Omodeo and J. T. Schwartz, *Decision algorithms for some fragments of analysis and related areas*, Comm. Pure Appl. Math. **40** (1987), 281–300.

[4] A. Church, *An unsolvable problem of elementary number theory*, Amer. J. Math. **58** (1936), 345–363.

[5] G. Collins, *Quantifier elimination for real closed fields by cylindrical algebraic decomposition*, in "Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern", May 20-23, 1975 (H. Brakhage, ed.), Lecture Notes in Computer Science, vol. 33, Springer Berlin (1975), pp. 134–183.

[6] M. Davis, H. Putnam and J. Robinson, *The decision problem for exponential Diophantine equations*, Ann. of Math. **74** (1961), 425–436.

[7] H. B. Enderton, *A Mathematical Introduction to Logic*, Academic Press (1972).

[8] M. J. Fisher and M. O. Rabin, *Super-exponential complexity of Presburger arithmetic*. Complexity and Computation, Vol. VII, SIAM-AMS, Philadelphia (1974), 27–41. (Cf. http://publications.csail.mit.edu/lcs/specpub.php?id=42)

[9] A. Formisano and E. Omodeo, *Theory-specific automated reasoning*, in "A 25-year perspective on Logic Programming: Achievements of the Italian Association for Logic Programming, GULP" (A. Dovier and E. Pontelli eds.), Lecture Notes in Computer Science, vol. 6125, Springer (2010), pp. 37–63.

[10] H. Friedman and À. Seress, *Decidability in elementary analysis. I*, Adv. Math. **76** (1989), no. 1, 94–115.

[11] H. Friedman and À. Seress, *Decidability in elementary analysis. II*, Adv. Math. **79** (1990), no. 1, 1–17.

[12] D. Grigoriev, *Complexity of deciding Tarski algebra*, J. Symbolic Comput. **5** (1988), 65–108.
[13] Y. Gurevich, *Existential interpretation. II*, Arch. Math. Logic **22** (1982), 103–120.
[14] J. Renegar, *A faster PSPACE algorithm for deciding the existential theory of the reals*, 29th Annual Symposium on Foundations of Computer Science (FOCS 1988, Los Angeles, Ca., USA), IEEE Computer Society Press, Los Alamitos (1988), pp. 291–295.
[15] D. Richardson, *Some undecidable problems involving elementary functions of a real variable*, J. Symbolic Logic **33** (1968), 514–520.
[16] J. T. Schwartz, D. Cantone, and E. G. Omodeo, *Computational Logic and Set Theory*, Springer (2011). Foreword by Martin Davis.
[17] A. Tarski, *A decision method for elementary algebra and geometry*, Tech. report, RAND Corporation, Santa Monica, CA (1951), Prepared for publication with the assistance of J.C.C. McKinsey. Available at http://www.rand.org/pubs/reports/R109.
[18] A. Tarski, *What is elementary geometry?, The Axiomatic Method with Special Reference to Geometry and Physics*, North-Holland, Amsterdam (1959), pp. 16–29.
[19] A. Tarski and S. Givant, *Tarski's system of geometry*, Bull. Symbolic Logic **5** (1999), no. 2, 175–214.
[20] A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proc. London Math. Soc. **42** (1936), 230–265, a correction appeared on Proc. London Math. Soc. **43** (1937), 544–546.

Authors' addresses:

Domenico Cantone
Dipartimento di Matematica e Informatica, University of Catania
Viale Andrea Doria 6, I-95125 Catania, Italy
E-mail: `cantone@dmi.unict.it`

Eugenio G. Omodeo
Dipartimento di Matematica e Geoscienze, DMI, University of Trieste
Via Alfonso Valerio 12/1, I-34127 Trieste, Italy
E-mail: `eomodeo@units.it`

Gaetano T. Spartà
Dipartimento di Metodi e Modelli per l'Economia, il Territorio e la Finanza (MEMOTEF), University of Roma "La Sapienza"
Via Del Castro Laurenziano 9, I-00161 Roma, Italy
E-mail: `gaetanosparta@virgilio.it`

Received June 4, 2012
Revised October 10, 2012