# From probability to sequences and back

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*Dedicated to Fabio Zanolin*

Abstract. *This is a survey covering sequential structures and their applications to the foundations of probability theory. Sequential convergence, convergence groups and the extension of sequentially continuous maps belong to general topology and Trieste for long has been a center of sequential topology. We begin with some personal reflections, continue with topological problems motivated by the extension of probability measures, and close with some recent results related to the categorical foundations of probability theory.*

## 1. Introduction

My PhD advisor Professor Josef Novák (1905 - 1999) and Professor Mario Dolcher (1920 - 1997), the PhD advisor of Fabio Zanolin, have had a common interest in sequential convergence and sequential topology (cf. [3]). Fabio has solved some problems posed by Novák related to sequential convergence spaces and groups ([37, 38]) and our personal meeting at the Prague Topological Symposium in 1982 resulted in friendship, fruitful cooperation, and a series of joint papers ([2, 12, 13, 14, 15, 16, 17, 18, 19]).

During my first visit of Italy in 1986, my homeland Slovakia (part of Czechoslovakia until 1993) and Italy have been separated by the Iron Curtain. That time, due to the Helsinky Agreement in 1975, scientific contacts and even joint research have been more easy and, thanks to a generous support by the Consiglio Nazionale delle Ricerche, I had both honor and pleasure to spend few fantastic weeks within the mathematical community in Trieste. Besides intensive joint research on convergence groups with Fabio, my plan was to present some results of Novák and members of his research team. The topic was "topological (sequential) aspects of the extension of measure". While working on my colloquium presentation, I have solved the "product problem for

sequential envelopes" (the product of sequential envelopes is equal to the sequential envelope of product, cf. [5]). The theory of sequential envelopes and its applications to probability has been a big theme for people around Novák ([5, 6, 9, 10, 25, 26, 29, 30, 31, 34]). Indeed, sequential envelopes are epireflections similar to the Čech-Stone $\beta$-compactification, the Hewitt $\upsilon$-realcompactification, and the $E$-compactifications of S. Mrówka, for which the product problems and their solutions are really "hard mathematics" (cf. [21]). I remember being so happy, that even the bad news about Chernobyl looked unimportant to me (that time the information was very limited).

At this point, let me provide some background information about Josef Novák and his interest in the relationship between (sequential) topology and probability. He was a student of Eduard Čech and hence a topologist by faith. During WWII, Czech universities have been closed by the Nazi authorities and Novák became involved in statistical applications. Continuity in applications usually means sequential continuity, while the "real topology" means ultrafilters, compactness, and the like... The idea of Novák was to utilize sequences in general topology as much as possible (remember his construction of a regular topological space every continuous function on which is constant). The extension of probability measures (in fact bounded sequentially continuous functions) from a field $\mathbb{A}$ of subsets to the generated $\sigma$-field $\sigma(\mathbb{A})$ served as a canonical example in three directions.

1. Operations in $\mathbb{A}$ are sequentially continuous, hence we can study $\mathbb{A}$ as a sequential convergence algebra (group) and $\sigma(\mathbb{A})$ can be considered as its sequential completion.

2. The sequential convergence in a field of sets is determined by probability measures (a sequence $\{A_n\}_{n=1}^{\infty}$ converges to $A$ iff the sequence $\{p(A_n)\}_{n=1}^{\infty}$ converges to $p(A)$ for all probability measures $p$) - a sequential version of complete regularity of a topological space. The problem is to find suitable sequential absolute properties of $\sigma(\mathbb{A})$ analogous to absolute properties like compactness or realcompactness.

3. Sequential convergence structures do not belong to the mainstream of general topology, hence there was a need to develop a suitable classification of such structures and to introduce characteristic properties guaranteeing relevant constructions in the realm of sequential structures. Observe that sequences are "short and meager", so that analogous topological and sequential constructions usually have different properties, for example, unlike $\beta X$ and $\upsilon X$, the extension of bounded sequentially continuous functions and unbounded sequentially continuous functions are equivalent constructions ([5]).

An interested reader can find more detailed information about sequential structures in [6] and references therein.

In the present paper I will concentrate on the outcome of research related

to the second of the three directions. Most of our joint research with Fabio Zanolin concerned the other two directions. Here I mention two main themes related to sequential convergence groups, also known as $\mathcal{L}$-groups.

1. Free convergence groups. Beside being a natural construction, the free group serves as a vehicle to transport properties of sequential convergence spaces to $\mathcal{L}$-groups (cf. [12, 14, 15, 16]).

2. Coarse convergence groups. To define a compatible sequential convergence (we assume unique limits) for a given group $G$, it is the same as to define a suitable subgroup of $G^N$ (the group of all sequences converging to the neutral element of $G$). This relates algebraic properties of $G$, resp. $G^N$, and certain properties of the convergence in question. Coarse convergence means that it cannot be enlarged without ruining the compatibility (e.g. the uniqueness of limits). The coarseness can be characterized by an algebraic condition, which results in an nice interplay between algebra and sequential topology. Coarse groups have interesting nontrivial properties (cf. [2, 13, 17, 19, 35]).

## 2. Measure extension theorem and more

In this section we outline the basic ideas of Josef Novák related to the extension of probability measures and leading to the notion of sequential envelope (cf. [8]).

THEOREM 2.1 (METHM – classical). *Let $\mathbb{A}$ be a field of sets, let $\sigma(\mathbb{A})$ be the generated $\sigma$-field, and let $p$ be a probability measure on $\mathbb{A}$. Then there exists a unique probability measure $\overline{p}$ on $\sigma(\mathbb{A})$ such that $\overline{p}(A) = p(A)$ for all $A \in \mathbb{A}$.*

The proof (usually based on the outer measure) can be found in any treatise on measure. However, additional properties of $\sigma(\mathbb{A})$ are usually not mentioned there. J. Novák pointed out that from the "topological viewpoint" $\sigma(\mathbb{A})$ can be viewed as a maximal object over which all probability measures on $\mathbb{A}$ can be extended.

In order to make the text more self-contained, we recall some facts about fields of sets. Let $X$ be a set. Then each subset $A \subseteq X$ can be viewed as the indicator function $\chi_A \in \{0,1\}^X$, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Moreover, a sequence $\{A_n\}_{n=1}^{\infty}$ converges to $A$ (i.e. $A = \limsup A_n = \liminf A_n$) iff the sequence $\{\chi_{A_n}\}_{n=1}^{\infty}$ converges pointwise to $\chi_A$. If $\mathbb{A}$ is a field of subsets of $X$, then the generated $\sigma$-field $\sigma(\mathbb{A})$ is the smallest sequentially closed subset of $\{0,1\}^X$ containing $\mathbb{A}$ and $\mathbb{A}$ is sequentially dense in $\sigma(\mathbb{A})$ (i.e. each $A \in \sigma(\mathbb{A})$ can be reached by iterations, up to $\omega_1$ times, of adding sequential limits, starting with sequences from $\mathbb{A}$). Observe that if two probability measures on $\sigma(\mathbb{A})$ coincide on $\mathbb{A}$, then a topological argument guarantees that they are identical. Let $\mathbb{A}, \mathbb{B}$ be fields of subsets of $X$ and let $\mathbb{A} \subseteq \mathbb{B}$. A sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathbb{A}$ is said to be *P-Cauchy* if for each probability measure $p$ on $\mathbb{A}$ the sequence $\{p(A_n)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers. If for

each probability measure $p$ on $\mathbb{A}$ there exists a probability measure $\bar{p}$ on $\mathbb{B}$ such that $\bar{p}(A) = p(A)$ for all $A \in \mathbb{A}$, then $\mathbb{A}$ is said to be *P-embedded* in $\mathbb{B}$.

THEOREM 2.2. *The following are equivalent*
  *(i)* $\mathbb{A} = \sigma(\mathbb{A})$;
  *(ii) Each P-Cauchy sequence converges in* $\mathbb{A}$;
  *(iii)* $\mathbb{A}$ *is sequentially closed in each field of sets* $\mathbb{B}$ *in which* $\mathbb{A}$ *is P-embedded.*

*Proof.* (i) implies (ii). Assume (i) and let $\{A_n\}_{n=1}^{\infty}$ be a $P$-Cauchy sequence in $\mathbb{A}$. Since each $x \in X$ represents a point-probability, the sequence $\{A_n\}_{n=1}^{\infty}$ (pointwise) converges in $\{0,1\}^X$. From $\mathbb{A} = \sigma(\mathbb{A})$ it follows that $\mathbb{A}$ is sequentially closed and hence $\{A_n\}_{n=1}^{\infty}$ converges in $\mathbb{A}$.

(ii) implies (iii). Let $\mathbb{A}$ be $P$-embedded in $\mathbb{B}$ and let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{A}$ which converges in $\mathbb{B}$. Since each $\bar{p} \in P(\mathbb{B})$ is sequentially continuous, $\{A_n\}_{n=1}^{\infty}$ is $P$-Cauchy and hence converges in $\mathbb{A}$.

(iii) implies (i). From the classical METHM it follows that $\mathbb{A}$ is $P$-embedded in $\sigma(\mathbb{A})$. Thus (iii) implies that $\mathbb{A}$ sequentially closed in $\sigma(\mathbb{A})$ and hence $\mathbb{A} = \sigma(\mathbb{A})$. This completes the proof. $\qquad\square$

THEOREM 2.3 (METHM – Novák). *Let* $\mathbb{A}$ *be a field of subsets of* $X$ *and let* $\sigma(\mathbb{A})$ *be the generated* $\sigma$*-field. Then* $\sigma(\mathbb{A})$ *is a maximal field of subsets of* $X$ *in which* $\mathbb{A}$ *is P-embedded and sequentially dense.*

*Proof.* The assertion follows from the preceding theorem. Let $\mathbb{A}$ be a field of subsets of $X$. Assume that $\mathbb{A}$ is $P$-embedded and sequentially dense in a field $\mathbb{B}$. Clearly, $\mathbb{A}$ is $P$-embedded and sequentially dense in $\sigma(\mathbb{B})$. Since the generated $\sigma$-field of a field of subsets of $X$ is the smallest sequentially closed system in $\{0,1\}^X$ containing the field in question, necessarily $\sigma(\mathbb{B}) = \sigma(\mathbb{A})$. Thus $\sigma(\mathbb{A})$ is maximal. This completes the proof. $\qquad\square$

Observe that $\sigma$-fields form a special class of fields of subsets. Indeed, $\mathbb{A} = \sigma(\mathbb{A})$ means that $\mathbb{A}$ has the following absolute property with respect to the extension of probability measures (cf. [7]): $\mathbb{A}$ is sequentially closed in each field of subsets in which it is $P$-embedded (in this respect, this absolute property is similar to the compactness).

J. Novák showed that each bounded $\sigma$-additive measure on a ring of sets $\mathbb{A}$ is sequentially continuous ([28]) and pointed out the topological aspects of the extension of such measures on $\mathbb{A}$ over the generated $\sigma$-ring $\sigma(\mathbb{A})$: it is of a similar nature as the extension of bounded continuous functions on a completely regular topological space $X$ over its Čech-Stone compactification $\beta X$ (or as the extension of continuous functions on $X$ over its Hewitt realcompactification $\upsilon X$). He developed a theory of sequential envelopes and (exploiting the Measure Extension Theorem) he proved that $\sigma(\mathbb{A})$ is the sequential envelope of $\mathbb{A}$ with respect to the probabilities. However, the sequential continuity

does not capture other properties (e.g. additivity) of probability measures. We show that in the category $ID$ of $D$-posets of fuzzy sets (such $D$-posets generalize both fields of subsets and their fuzzy counterparts called bold algebras) probabilities are morphisms and the extension of probabilities on $\mathbb{A}$ over $\sigma(\mathbb{A})$ is a completely categorical construction (an epireflection, see [1]).

OBSERVATION 2.4. *Novák's original construction of the sequential envelope of a space $X$ (a set carrying sequential convergence and the corresponding convergence closure) with respect to a given class $\mathcal{C}_0$ of sequentially continuous functions into $[0,1]$ follows the usual construction of $\beta$-compactification: embedding $X$ into the power $[0,1]^{\mathcal{C}_0}$ and taking the closure (instead of the product topology, $[0,1]^{\mathcal{C}_0}$ carries the pointwise convergence, i.e. the categorical product convergence, and instead of the topological closure we take the smallest sequentially closed set containing the embedded $X$). In fact, this is a categorical construction of an epireflection of $X$, belonging to the category of space embeddable into powers $[0,1]^S$, into the subcategory of spaces embeddable as sequentially closed subspaces of powers $[0,1]^S$ (cf. [5]).*

OBSERVATION 2.5. *In the realm of sequential convergence spaces, the sequentially closed subspaces of categorical convergence powers $[0,1]^S$ possess the quality of being absolutely sequentially closed with respect to the extension of sequentially continuous functions of a given class, i.e., sequentially closed in every larger space to which sequentially continuous functions of a given class can be extended.*

OBSERVATION 2.6. *The category $ID$ of $D$-posets of fuzzy sets is the result of a quest for a natural domain of generalized random events in which "all goes well":*

*1. Both the classical Kolmogorovian probability theory, or CPT, and the fuzzy probability theory, or FPT, initiated by A. L. Zadeh ([36]) "live as minimal models having simple characteristic properties".*

*2. Probability measures, observables (i.e. preimages of random variables) and their fuzzy counterparts are morphisms.*

*3. Basic probability notions and constructions are categorical.*

## 3. Notes on probability

In this section we present some notes about the foundations of probability. We will put into a perspective CPT and FPT and show why in the category $ID$ "all goes well".

A. N. Kolmogorov in his famous "Grundbegriffe" ([22]) has "mathematized" probability via set-theoretic and maesure-theoretic constructions. Roughly, random events are "measurable" subsets of the outcomes, and probability is a measure (normed and $\sigma$-additive) on the random events. Observe that

- Random events form a $\sigma$-complete lattice of sets;

- In fact, every random event, as a subset of $\Omega$, is a propositional function (Boolean logic).

In 1968 L. A. Zadeh ([36]) proposed to extend the classical probability to the realm of fuzzy mathematics. His idea was to extend classical random events, i.e. measurable $\{0, 1\}$-valued (propositional) functions, to fuzzy random events, i.e. measurable $[0, 1]$-valued (propositional) functions, and the probability measure to the integral with respect to a probability measure.

There are conceptual and theoretical differences and similarities between randomness and fuzziness (cf. [24]).

- Both systems describe uncertainty with numbers in the unit interval $[0, 1]$ and both systems combine sets and propositions associatively, commutatively, and distributively;

- The key distinction concerns how the systems deal with a thing $A$ and its opposite $A^c$;

- Classical logic and set theory assume that the law of noncontradiction (the law of excluded middle) is never violated. That is what makes the classical theory black or white;

- Fuzziness begins where Western logic ends. Fuzziness describes *event ambiguity*. It measures the degree to which an event occurs, not whether it occurs;

- Randomness describes the uncertainty of *event occurrence*. An event occurs or not;

- At issue is the nature of the occurring event: whether it is uncertain in any way, in particular whether it can be unambiguously distinguished from its opposite.

In order to represent a *classical object o*

- We choose a set $X$ of attributes;

- We identify $o$ and the set $A_o = \{x \in X; o \text{ does have } x\}$.

Observe that, in fact, $o$ can be viewed as a propositional function $o \in \{0, 1\}^X$ and $x \in A_o$ iff the proposition $o(x)$ is true. Clearly, $x$ cannot be at the same time in $A_o$ and in its complement.

In order to represent a *fuzzy object o*

- We choose a set $X$ of attributes;

- We identify $o$ and the fuzzy set $o \in [0,1]^X$, where $o(x)$ is the degree to which $o$ possesses the attribute $x$.

Observe that, in fact, $o$ can be viewed as a "fuzzy propositional function" $o \in \{0,1\}^X$ and $o(x)$ tells us how much $o$ is true at $x$. It can happen, that at some $x$ both $o$ and its complement $o^c = 1_X - o$ are "partially true", i.e., both $o(x)$ and $o^c(x) = 1 - o(x)$ are positive numbers.

QUESTION: Is it possible to build a generalized probability so that the CPT and FPT are special cases?
ANSWER: YES.

- We start with a set $X$ of attributes and the system of potential generalized random events $[0,1]^X$ carrying the natural pointwise partial order;

- Any minimal model of generalized random events $\mathcal{X} \subseteq [0,1]^X$ has to contain the maximal and minimal random events (constant functions $0_X$, $1_X$) and has to be closed with respect to the relative complementation: if $u, v \in \mathcal{X}$ and $v \leq u$, i.e. $v(x) \leq u(x)$ for all $x \in X$, then $u - v \in \mathcal{X}$;

- If we assume that it is a $\sigma$-complete lattice (defined pointwise), then there exists a $\sigma$-field $\mathbf{A}$ of subsets of $X$ such that $\mathbf{A} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbf{A})$, where $\mathcal{M}(\mathbf{A})$ is the family of all measurable functions ranging in $[0,1]$;

- If we assume that $\mathcal{X}$ is divisible, i.e., for each $u \in \mathcal{X}$ and each natural number $n$ there exists $v \in \mathcal{X}$ such that $nv = u$, and a $\sigma$-complete lattice, then $\mathcal{X} = \mathcal{M}(\mathbf{A})$.

The last two items are in fact deep results about the structure of "fuzzy random events" (cf. [27, Theorem 5.1]). To sum up, random events in CPT and random events in FPT are the minimal models of random events in a reasonable generalized probability; divisibility characterizes the transition from random events in CPT to random events in FPT.

## 4. From extension to epireflection

This section is devoted to bold algebras, distinguished domains of generalized probability (cf. [33]). First, we recall some notions used in the sequel.

*D-posets* have been introduced in [23] in order to model events in quantum probability. They generalize Boolean algebras, $MV$-algebras and other probability domains (cf. [4]) and provide a category in which generalized probability measures, called states, become morphisms. Recall that a $D$-poset is a partially ordered set $X$ with the greatest element $1_X$, the least element $0_X$, and a partial binary operation called *difference*, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1)  $a \ominus 0_X = a$ for each $a \in X$;

(D2)  If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

A map $h$ of a $D$-poset $X$ into a $D$-poset $Y$ which preserves the $D$-structure is said to be a *D-homomorphism*. Consider the unit interval $I = [0,1]$ carrying the natural order, algebraic operations and convergence. Define a partial operation "$\ominus$" as follows: for $a, b \in I, b \leq a$, put $a \ominus b = a - b$. Then $I$ carrying the natural (total) order, together with the partial operation is a $D$-poset. A sequentially continuous $D$- homomorphism of $\mathcal{X}$ into $I$ is said to be a *state*.

Fundamental to applications are *D-posets of fuzzy sets*, i.e. systems $\mathcal{X} \subseteq [0,1]^X$ carrying the coordinatewise partial order, coordinatewise convergence of sequences, containing the top and bottom elements of $I^X$, and closed with respect to the partial operation difference defined coordinatewise. We always assume that $\mathcal{X}$ is *reduced*, i.e., for $x, y \in X$, $x \neq y$, there exists $u \in \mathcal{X}$ such that $u(x) \neq u(y)$. Denote $ID$ the category having (reduced) $D$-posets of fuzzy sets as objects and having sequentially continuous $D$-homomorphisms as morphisms. Objects of $ID$ are subobjects of the powers $I^X$.

Recall ([4, 7]) that a *bold algebra* is a system $\mathcal{X} \subseteq [0,1]^X$ containing the constant functions $0_X$, $1_X$ and closed with respect to the usual Łukasiewicz operations: for $u, v \in \mathcal{X}$ put $(u \oplus v)(x) = u(x) \oplus v(x) = \min\{1, u(x) + v(x)\}$, $u^*(x) = 1 - u(x)$, $x \in X$. Bold algebras are $MV$-algebras representable as $[0,1]$-valued functions, $MV$-algebras generalize Boolean algebras and bold algebras generalize in a natural way fields of sets (viewed as indicator functions). More information concerning $MV$-algebras and probability on $MV$-algebras can be found in [33]. If a bold algebra $\mathcal{X} \subseteq [0,1]^X$ is sequentially closed in $[0,1]^X$ (with respect to the coordinatewise sequential convergence), then $\mathcal{X}$ is a *Łukasiewicz tribe* ($\mathcal{X}$ is closed not only with respect to finite, but also with respect to countable Łukasiewicz sums, cf. [7, Corollary 2.8]). Let $\mathcal{X} \subseteq [0,1]^X$ be a bold algebra. Then $[0,1]^X$ is a Łukasiewicz tribe containing $\mathcal{X}$ and the intersection of all Łukasiewicz tribes $\mathcal{Y} \subseteq [0,1]^X$ such that $\mathcal{X} \subseteq \mathcal{Y}$ is a Łukasiewicz tribe; it will be called the *induced* Łukasiewicz tribe and denoted by $\sigma(\mathcal{X})$. Each bold algebra can be considered as on object of $ID$. Finally, each bold algebra $\mathcal{X} \subseteq [0,1]^X$ is a lattice, where for $u, v \in \mathcal{X}$ we have $(u \vee v)(x) = u(x) \vee v(x)$ and $(u \wedge v)(x) = u(x) \wedge v(x)$, $x \in X$.

Denote $FSD$ the full subcategory of $ID$ the objects of which are fields of sets and $CFSD$ its full subcategory consisting of $\sigma$-fields. It is known (cf. [32]) that sequentially continuous $D$-homomorphisms of a field of sets ranging in $I$ are exactly $\sigma$-additive probability measures.

Denote $BID$ the full subcategory of $ID$ whose objects are bold algebras (the morphisms are exactly sequentially continuous $D$-morphisms). Let $CBID$ be

the subcategory of $BID$ consisting of Łukasiewicz tribes (remember, a bold algebra $\mathcal{X} \subseteq I^X$ is a tribe iff $\mathcal{X}$ is a sequentially closed in $I^X$).

THEOREM 4.1. *Let $\mathcal{X} \subseteq I^X$ be a bold algebra and let $\sigma(\mathcal{X}) \subseteq I^X$ be the induced Łukasiewicz tribe. Let $h$ be a sequentially continuous D-homomorphism of $\mathcal{X}$ into a Łukasiewicz tribe $\mathcal{Y}$. Then $h$ can be uniquely extended to a sequentially continuous D-homomorphism $h_\sigma$ of $\sigma(\mathcal{X})$ into $\mathcal{Y}$.*

*Proof.* Let $\mathcal{Y} = \sigma(\mathcal{Y}) \subseteq I^Y$. For each $y \in Y$, let $pr_y$ be the $y$-th projection of $I^Y$ to the factor space $I^{\{y\}}$. Then each composition $pr_y \circ h$ is a state on $\mathcal{X}$ and (cf. [7, Proposition 2.1]) it can be uniquely extended to a state $\overline{pr_y \circ h}$ on $\sigma(\mathcal{X})$. Since $I^Y$ is a categorical product, there is a unique $ID$-morphism $h_\sigma$ of $\sigma(\mathcal{X})$ into $I^Y$ such that $pr_y \circ h_\sigma = \overline{pr_y \circ h}$. Clearly, for each $u \in \mathcal{X}$ and each $y \in Y$ we have $\overline{pr_y \circ h}(u) = (pr_y \circ h)(u)$. Hence $h_\sigma(u) = h(u)$ for each $u \in \mathcal{X}$. A topological argument shows that $h_\sigma$ maps $\sigma(\mathcal{X})$ into $\mathcal{Y} = \sigma(\mathcal{Y})$ and that $h_\sigma$ is uniquely determined (indeed, the pointwise convergence has unique limits, $\mathcal{X}$ is sequentially dense in $\sigma(\mathcal{X})$, $h_\sigma$ is sequentially continuous and hence $h_\sigma(\sigma(\mathcal{X})) \subseteq \sigma((h(\mathcal{X})) \subseteq \sigma(\mathcal{Y}) = \mathcal{Y}$, (cf. [30]). $\qquad\square$

REMARK 4.2. *If $\mathcal{Y}$ is the unit interval $[0,1]$ carrying the canonical D-structure, then the previous theorem becomes the usual "State Extension Theorem" for bold algebras.*

REMARK 4.3. *Note that the embedding of a bold algebra $\mathcal{X}$ into $\sigma(\mathcal{X})$ is an epimorphism (two morphisms on $\sigma(\mathcal{X})$ agreeing on $\mathcal{X}$ are identical). This is a standard topological fact following from the uniqueness of limits, sequential continuity of morphisms, and the sequential density of $\mathcal{X}$ in $\sigma(\mathcal{X})$ (cf. [30]).*

COROLLARY 4.4. *The subcategory $CBID$ is an epireflective subcategory of $BID$.*

Observe ([1]) that an epireflector is (roughly) a nice functor sending each object having some fundamental properties to the unique object in the subcategory of objects having some extreme properties, its epireflection, and sending each morphism to the unique morphism of the epireflection of its domain into the epireflection of its range (e.g. the completion of a metric space is an epireflection into complete metric spaces).

COROLLARY 4.5. *The subcategory $CFSD$ is an epireflective subcategory of $FSD$.*

*Proof.* Let $\mathbb{A} \subseteq \{0,1\}^X$ be a field of subset of $X$ and let $\sigma(\mathbb{A})$ be the generated $\sigma$-field. Let $h$ be an $ID$-morphism of $\mathbb{A}$ into a $\sigma$-field $\mathbb{B} = \sigma(\mathbb{B})$. Clearly, it suffices to prove that $h$ can be uniquely extended to an $ID$-morphism $h_\sigma$ of $\sigma(\mathbb{A})$ into $\mathbb{B}$. But $\sigma(\mathbb{A})$ and $\mathbb{B}$ are the induced Łukasiewicz tribes and the assertion follows from Theorem 4.1. $\qquad\square$

As stated earlier, in the category $ID$ the extension of probability measures on a field of subsets over the generated $\sigma$-field becomes a purely categorical construction. Moreover, the categorical approach leads to a better understanding of the foundations of probability theory (cf. [11, 20, 27]). Finally, observe that the sequential continuity of morphisms plays an an important role.

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