

# Stability criteria for impulsive Kolmogorov-type systems of nonautonomous differential equations

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*Dedicated to Fabio Zanolin on the occasion of his sixtieth birthday*

**ABSTRACT.** *In this paper we consider a class of impulsive Kolmogorov-type systems. The problems of uniform stability and uniform asymptotic stability of the solutions are studied. We establish stability criteria by employing piecewise continuous Lyapunov functions. Examples are given to demonstrate the effectiveness of the obtained results. We show, also, that the role of impulses in changing the behavior of impulsive models is very important.*

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## 1. Introduction

The studies for Kolmogorov systems has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its theoretical and practical significance. Many authors established a series of criteria on the boundedness, persistence, permanence, global asymptotic stability and the existence of positive periodic solutions [8, 9, 12, 14, 16, 18]. Some interesting work on this topic of interest has been done by Zanolin and his co-authors [6, 19, 20].

On the other hand, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics, telecommunications, etc. Since time perturbations occur so often in nature, a number of models in ecology can be formulated as systems of impulsive differential equations [2, 3, 4, 5, 13, 15, 21]. One of the most important problems for these types of systems is to analyze the effect of impulsive time perturbations on the dynamic activity patterns in the systems. Impulses can make unstable systems stable; so they have been widely used as a control [17].

Recently, some qualitative properties of populations, which undergo impulsive effects at fixed times between interval of continuous evolutions, have been investigated for impulsive classes of Kolmogorov systems [5, 15, 21]. However, in all of these papers so far, authors mostly focused on the existence of periodic solutions and permanence.

In our previous papers [2] and [3] we studied stability properties of some special cases of impulsive Kolmogorov systems with or without delays.

In the present paper, we consider the uniform stability and uniform asymptotic stability of the solutions for a class of impulsive Kolmogorov-type systems of nonautonomous differential equations. For this purpose piecewise continuous auxiliary functions are used which are an analogue of Lyapunov functions. Examples are given to demonstrate the effectiveness of the obtained results. We show, also, that the role of impulses in changing the behavior of impulsive models is very important.

## 2. Preliminaries

Let  $R^n$  be the  $n$ -dimensional Euclidean space with norm  $\|x\| = \sum_{i=1}^n |x_i|$ . Let  $R_+ = [0, \infty)$ ,  $t_0 \in R_+$  and  $t_0 < t_1 < t_2 < \dots$ ,  $\lim_{k \rightarrow \infty} t_k = \infty$ .

Consider the following  $n$ - dimensional impulsive Kolmogorov-type system

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)f_i(t, x(t)), \quad t \neq t_k, \\ \Delta x_i(t_k) &= P_{ik}(x_i(t_k)), \quad k = 1, 2, \dots, \end{aligned} \quad (1)$$

$i = 1, 2, \dots, n$ , where  $n$  corresponds to the number of units in the system,  $x_i(t)$  corresponds to the state of the  $i$ th unit at time  $t$ ,  $f_i : [t_0, \infty) \times R_+^n \rightarrow R$ ,  $f = \text{col}(f_1, f_2, \dots, f_n)$ ,  $f \in C[[t_0, \infty) \times R_+^n, R^n]$ ,  $\Delta x_i(t) = x_i(t+0) - x_i(t-0)$ ,  $t_k$ ,  $k = 1, 2, \dots$  are the moments of impulsive perturbations and  $P_{ik}(x_i(t_k))$  represents the abrupt change of the state  $x_i(t)$  at the impulsive moment  $t_k$ ,  $P_k = \text{col}(P_{1k}, P_{2k}, \dots, P_{nk})$ ,  $P_k \in C[R_+^n, R^n]$ .

Let  $x_0 = \text{col}(x_{10}, x_{20}, \dots, x_{n0})$  and  $x_{i0} \geq 0$ ,  $i = 1, 2, \dots, n$ . Denote by  $x(t) = x(t; t_0, x_0) = \text{col}(x_1(t), x_2(t), \dots, x_n(t))$  the solution of system (1), satisfying the initial condition

$$x(t_0 + 0; t_0, x_0) = x_0. \quad (2)$$

We suppose that the existence, uniqueness, and continuous dependence of solutions of system (1) hold. For the efficient sufficient conditions which guarantee the existence, uniqueness, and continuous dependence of solutions of system (1) (see [11]).

The solutions  $x(t)$  of system (1) are piecewise continuous functions with points of discontinuity of the first kind  $t_k$  at which they are left continuous; i.e.

the following relations are satisfied:

$$x_i(t_k - 0) = x_i(t_k), \quad x_i(t_k + 0) = x_i(t_k) + P_{ik}(x_i(t_k)),$$

$$i = 1, 2, \dots, n, \quad k = 1, 2, \dots$$

We also assume that solutions of (1) with initial conditions (2) are nonnegative, and if  $x_{i0} > 0$  for some  $i$ , then  $x_i(t) > 0$  for all  $t \geq t_0$ . If, moreover,  $(t_k, x_i) \in (t_0, \infty) \times (0, \infty)$ , then  $x_i(t_k) + P_{ik}(x_i(t_k)) > 0$  for all  $i = 1, 2, \dots, n$  and  $k = 1, 2, \dots$ . Note that these assumptions are natural from the applicability point of view.

Let  $x(t) = x(t; t_0, x_0) = \text{col}(x_1(t), x_2(t), \dots, x_n(t))$  and  $x^*(t) = x^*(t; t_0, x_0^*) = \text{col}(x_1^*(t), x_2^*(t), \dots, x_n^*(t))$  be any two solutions of (1) with initial conditions

$$x(t_0 + 0; t_0, x_0) = x_0,$$

$$x^*(t_0 + 0; t_0, x_0^*) = x_0^*,$$

where  $x_0^* = \text{col}(x_{10}^*, x_{20}^*, \dots, x_{n0}^*)$  and  $x_{i0}^* \geq 0$ ,  $i = 1, 2, \dots, n$ .

We will use the following definitions of some stability properties of the solutions of (1).

**DEFINITION 2.1.** *The solution  $x^*(t)$  of system (1) is said to be:*

- (a) *stable, if for all  $t_0 \in R_+$  and for all  $\varepsilon > 0$  there exists  $\delta = \delta(t_0, \varepsilon) > 0$  such that if  $x_0, x_0^* \in R_+^n$ , with  $\|x_0 - x_0^*\| < \delta$ , then for all  $t \geq t_0$ :*

$$\|x(t; t_0, x_0) - x^*(t; t_0, x_0^*)\| < \varepsilon;$$

- (b) *uniformly stable, if the number  $\delta$  in (a) is independent of  $t_0 \in R_+$ ;*

- (c) *uniformly attractive, if there exists  $\lambda > 0$  such that for all  $\varepsilon > 0$  there exists  $\gamma = \gamma(\varepsilon) > 0$  such that if  $t_0 \in R_+$  and  $x_0, x_0^* \in R_+^n$ , with  $\|x_0 - x_0^*\| < \lambda$ , then for all  $t \geq t_0 + \gamma$ :*

$$\|x(t; t_0, x_0) - x^*(t; t_0, x_0^*)\| < \varepsilon;$$

- (d) *uniformly asymptotically stable, if it is uniformly stable and uniformly attractive.*

Introduce the sets

$$G_k = \left\{ (t, x, x^*) \in [t_0, \infty) \times R_+^n \times R_+^n : t_{k-1} < t < t_k \right\}, \quad k = 1, 2, \dots,$$

$$G = \bigcup_{k=1}^{\infty} G_k.$$

DEFINITION 2.2. A function  $V : [t_0, \infty) \times R_+^n \times R_+^n \rightarrow R_+$  belongs to class  $V_0$ , if:

1.  $V$  is continuous in  $G$  and locally Lipschitz continuous with respect to its second and third arguments on each of the sets  $G_k$ ,  $k = 1, 2, \dots$  and

$$V(t, x^*, x^*) = 0, \quad t \in [t_0, \infty).$$

2. For each  $k = 1, 2, \dots$  there exist the finite limits

$$V(t_k - 0, x, x^*) = \lim_{\substack{t \rightarrow t_k \\ t < t_k}} V(t, x, x^*), \quad V(t_k + 0, x, x^*) = \lim_{\substack{t \rightarrow t_k \\ t > t_k}} V(t, x, x^*)$$

and the equality  $V(t_k - 0, x, x^*) = V(t_k, x, x^*)$  holds.

3. For each  $k = 1, 2, \dots$  and  $x, x^* \in R_+^n$  the following inequality holds:

$$V(t_k + 0, x + P_k(x), x^* + P_k(x^*)) \leq V(t, x, x^*). \quad (3)$$

Let  $V \in V_0$ . For  $(t, x, x^*) \in G$  we set

$$\dot{V}_{(1)}(t, x, x^*) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [V(t+h, x+hx f(t, x), x^*+hx^* f(t, x^*)) - V(t, x, x^*)].$$

Note that if  $x = x(t)$  and  $x^* = x^*(t)$  are solutions of system (1), then  $D_{(1)}^+ V(t, x(t), x^*(t)) = \dot{V}_{(1)}(t, x, x^*)$ ,  $t \geq t_0$ ,  $t \neq t_k$ , where

$$D_{(1)}^+ V(t, x(t), x^*(t)) = \lim_{h \rightarrow 0^+} \sup \frac{1}{h} [V(t+h, x(t+h), x^*(t+h)) - V(t, x(t), x^*(t))]$$

is the upper right Dini derivative of the function  $V(t, x(t), x^*(t))$  (with respect to the system (1)).

We shall use the following class of functions:

$$K = \{a \in C[R_+, R_+] : a(r) \text{ is strictly increasing and } a(0) = 0\}.$$

### 3. Main results

In the proofs of our main theorems in this section we shall use piecewise continuous Lyapunov functions  $V \in V_0$ . Similar results for systems with delays are discussed in [13].

**THEOREM 3.1.** *Assume that there exist functions  $V \in V_0$  and  $a, b \in K$  such that*

$$a(\|x - x^*\|) \leq V(t, x, x^*) \leq b(\|x - x^*\|), \quad t \in [t_0, \infty), \quad x, x^* \in R_+^n, \quad (4)$$

$$\dot{V}_{(1)}(t, x, x^*) \leq 0, \quad (t, x, x^*) \in G. \quad (5)$$

*Then the solution  $x^*(t)$  of system (1) is uniformly stable.*

*Proof.* Let  $\varepsilon > 0$  be chosen. Choose  $\delta = \delta(\varepsilon) > 0$  so that  $b(\delta) < a(\varepsilon)$ . Let  $t_0 \in R_+$ ,  $x_0, x_0^* \in R_+^n$ , with  $\|x_0 - x_0^*\| < \delta$ , and  $x(t) = x(t; t_0, x_0) = \text{col}(x_1(t), x_2(t), \dots, x_n(t))$ ,  $x^*(t) = x^*(t; t_0, x_0^*) = \text{col}(x_1^*(t), x_2^*(t), \dots, x_n^*(t))$  be the solutions of (1).

From the properties of the function  $V$  and conditions (4), (5), we get to the inequalities

$$\begin{aligned} a(\|x(t; t_0, x_0) - x^*(t; t_0, x_0^*)\|) &\leq V(t, x(t; t_0, x_0), x^*(t; t_0, x_0^*)) \\ &\leq V(t_0 + 0, x_0, x_0^*) \\ &\leq b(\|x_0 - x_0^*\|) < b(\delta) < a(\varepsilon), \end{aligned}$$

from which it follows that  $\|x(t; t_0, x_0) - x^*(t; t_0, x_0^*)\| < \varepsilon$  for  $t \geq t_0$ . This proves the uniform stability of the solution  $x^*(t)$  of system (1).  $\square$

**THEOREM 3.2.** *Let the condition (4) of Theorem 3.1 be fulfilled and let a function  $c \in K$  exist such that for  $x, x^* \in R_+^n$  the inequality*

$$\dot{V}_{(1)}(t, x, x^*) \leq -c(\|x - x^*\|), \quad t \in [t_0, \infty), t \neq t_k, \quad k = 1, 2, \dots \quad (6)$$

*holds.*

*Then the solution  $x^*(t)$  of system (1) is uniformly asymptotically stable.*

*Proof.* From Theorem 3.1 it follows that the solution  $x^*(t)$  of system (1) is uniformly stable. Hence, for any  $\varepsilon$ ,  $\varepsilon > 0$ , there exists  $\delta > 0$ , such that if  $t_0 \in R_+$ ,  $x_0, x_0^* \in R_+^n$ , with  $\|x_0 - x_0^*\| < \delta$ , then

$$\|x(t; t_0, x_0) - x^*(t; t_0, x_0^*)\| < \varepsilon$$

for  $t \geq t_0$ .

Now, we shall prove that the solution  $x^*(t)$  of system (1) is uniformly attractive.

1. Let  $\alpha = \text{const} > 0$  be so small, that  $\{x \in R^n : \|x - x^*(t)\| \leq \alpha\} \subset R_+^n$ . For any  $t \geq t_0$  denote

$$V_{t, \alpha}^{-1} = \{x \in R_+^n : V(t + 0, x, x^*) \leq a(\alpha)\}.$$

From (4) we deduce

$$V_{t,\alpha}^{-1} \subset \{x \in R^n : \|x - x^*\| \leq \alpha\}.$$

From conditions of Theorem 3.2 it follows that for any  $t_0 \in R_+$  and any  $x_0 \in R_+^n : x_0 \in V_{t_0,\alpha}^{-1}$  we have  $x(t; t_0, x_0) \in V_{t,\alpha}^{-1}$ ,  $t \geq t_0$ . Choose  $\eta = \eta(\varepsilon)$  so that  $b(\eta) < a(\varepsilon)$  and let  $\gamma = \gamma(\varepsilon) > \frac{b(\alpha)}{c(\eta)}$ . If we assume that for each  $t \in [t_0, t_0 + \gamma]$  the inequality  $\|x(t; t_0, x_0) - x^*(t; t_0, x_0^*)\| \geq \eta$  is valid, then from (3) and (6) we deduce the inequalities

$$\begin{aligned} & V(t_0 + \gamma, x(t_0 + \gamma; t_0, x_0), x^*(t_0 + \gamma; t_0, x_0^*)) \\ & \leq V(t_0 + 0, x_0, x_0^*) - \int_{t_0}^{t_0 + \gamma} c(\|x(s; t_0, x_0) - x^*(s; t_0, x_0^*)\|) ds \\ & \leq b(\alpha) - c(\eta)\gamma < 0, \end{aligned}$$

which contradicts (4). The contradiction obtained shows that there exists  $t^* \in [t_0, t_0 + \gamma]$  such that  $\|x(t^*; t_0, x_0) - x^*(t^*; t_0, x_0^*)\| < \eta$ . Then for  $t \geq t^*$  (hence for any  $t \geq t_0 + \gamma$ ) the following inequalities hold:

$$\begin{aligned} a(\|x(t) - x^*(t)\|) & \leq V(t; x(t), x^*(t)) \\ & \leq V(t^*, x(t^*), x^*(t^*)) \\ & \leq b(\|x(t^*; t_0, x_0) - x^*(t^*; t_0, x_0^*)\|) \\ & < b(\eta) < a(\varepsilon). \end{aligned}$$

Therefore  $\|x(t; t_0, x_0) - x^*(t; t_0, x_0^*)\| < \varepsilon$  for  $t \geq t_0 + \gamma$ .

2. Let  $\lambda = \text{const} > 0$  be such that  $b(\lambda) \leq a(\alpha)$ . Then if  $x_0 \in R_+^n : \|x_0 - x_0^*\| < \lambda$ , (4) implies

$$V(t_0 + 0, x_0, x_0^*) \leq b(\|x_0 - x_0^*\|) < b(\lambda) \leq a(\alpha),$$

which shows that for  $x_0 \in V_{t_0,\alpha}^{-1}$ . From what we proved in item 1 it follows that the solution  $x^*(t)$  of system (1) is uniformly attractive.

Therefore, the solution  $x^*(t)$  of system (1) is uniformly asymptotically stable.  $\square$

**COROLLARY 3.3.** *If in Theorem 3.2 condition (6) is replaced by the condition*

$$\dot{V}_{(1)}(t, x, x^*) \leq -cV(t, x, x^*), \quad t \neq t_k, \quad k = 1, 2, \dots, \quad x, x^* \in R_+^n, \quad (7)$$

where  $c = \text{const} > 0$ , then the solution  $x^*(t)$  of system (1) is uniformly asymptotically stable.

*Proof.* The proof of Corollary 3.3 is analogous to the proof of Theorem 3.2. It uses the fact that

$$V(t, x(t; t_0, x_0), x^*(t; t_0, x_0^*)) \leq V(t_0 + 0, x_0, x_0^*) \exp[-c(t - t_0)]$$

for  $t \geq t_0$ , which is obtained from (7) and (3).

In fact, let  $\alpha = \text{const} > 0 : \{x \in R^n : \|x - x^*(t)\| \leq \alpha\} \subset R_+^n$ . Choose  $\lambda > 0$  so that  $b(\lambda) < a(\alpha)$ . Let  $\varepsilon > 0$  and  $\gamma \geq \frac{1}{c} \ln \frac{a(\alpha)}{a(\varepsilon)}$ . Then for  $t_0 \in R_+$ ,  $x_0, x_0^* \in R_+^n$ , with  $\|x_0 - x_0^*\| < \lambda$  and  $t \geq t_0 + \gamma$  the following inequalities hold

$$V(t, x(t; t_0, x_0), x^*(t; t_0, x_0^*)) \leq V(t_0 + 0, x_0, x_0^*) \exp[-c(t - t_0)] < a(\varepsilon),$$

whence, in view of (4), we deduce that the solution  $x^*(t)$  of system (1) is uniformly attractive.  $\square$

#### 4. Applications

The results obtained can be applied in the investigation of the stability of any solution which is of interest. One of the solutions which is an object of investigations for the systems of type (1) is the *equilibrium* state, i.e. the constant solution  $x^* = \text{col}(x_1^*, x_2^*, \dots, x_n^*)$  such that

$$\begin{aligned} \dot{x}_i^*(t) &= 0, \quad t \neq t_k, \\ \Delta x_i^*(t_k) &= 0, \quad k = 1, 2, \dots, i = 1, 2, \dots, n. \end{aligned}$$

In the applications, uniform stability and uniform asymptotic stability of the equilibria will be discussed for a special case of impulsive Kolmogorov-type models.

Consider the following  $n$ -species Lotka-Volterra type impulsive system

$$\begin{cases} \dot{x}_i(t) = x_i(t) \left[ b_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t)x_j(t) \right], & t \neq t_k, \\ x_i(t_k + 0) = x_i(t_k) + P_{ik}(x_i(t_k)), & i = 1, \dots, n, \quad k = 1, 2, \dots, \end{cases} \quad (8)$$

where  $n \geq 2$ ,  $t \geq 0$ ,  $a_{ij} \in C[R_+, R_+]$ ,  $b_i \in C[R_+, R]$ ,  $P_{ik} : R_+ \rightarrow R$ ,  $i, j = 1, \dots, n$ ,  $k = 1, 2, \dots$ ,  $0 < t_1 < t_2 < \dots < t_k < \dots$  are fixed impulsive points and  $\lim_{k \rightarrow \infty} t_k = \infty$ . In mathematical ecology, the system (8) denotes a model of the dynamics of an  $n$ -species system in which each individual competes with all others of the system for a common resource and at the fixed moments of time  $t_k$ ,  $k = 1, 2, \dots$ , the system is subject to short-term perturbations. The

numbers  $x_i(t_k)$  and  $x_i(t_k + 0)$  are, respectively, the population densities of species  $i$  before and after impulse perturbation at the moment  $t_k$  and  $P_{ik}$  are functions which characterize the magnitude of the impulse effect on the species  $i$  at the moments  $t_k$ .

Let  $x_0 = \text{col}(x_{10}, x_{20}, \dots, x_{n0})$  and  $x_{i0} \geq 0$ ,  $i = 1, 2, \dots, n$ . Denote by  $x(t) = x(t; t_0, x_0) = \text{col}(x_1(t), x_2(t), \dots, x_n(t))$  the solution of system (8), satisfying the initial condition

$$x(t_0 + 0; t_0, x_0) = x_0. \quad (9)$$

Given a continuous function  $g(t)$  which is defined on  $J$ ,  $J \subseteq R$ , we set

$$g^L = \inf_{t \in J} g(t), \quad g^M = \sup_{t \in J} g(t).$$

For  $0 \leq \tau_1 < \tau_2$ , we define the following notation:

$$A[g, \tau_1, \tau_2] = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} g(s) ds.$$

The lower and upper averages of  $g(t)$ , denoted by  $m[g]$  and  $M[g]$  are defined by

$$m[g] = \lim_{s \rightarrow \infty} \inf \{A[g, \tau_1, \tau_2] \mid \tau_2 - \tau_1 \geq s\},$$

$$M[g] = \lim_{s \rightarrow \infty} \sup \{A[g, \tau_1, \tau_2] \mid \tau_2 - \tau_1 \geq s\}.$$

In our subsequent analysis, we shall assume that the functions  $b_i$  and  $a_{ij}$ ,  $i, j = 1, 2, \dots, n$ , are continuous on  $R_+$ ,  $a_{ij} \geq 0$ ,  $a_{ij}^M < \infty$ ,  $b_i^M < \infty$ ,  $b_i^L > 0$ , and  $a_{ii}^L > 0$  for  $i = 1, 2, \dots, n$ .

Furthermore, in order to restrict our attention only to those solutions which evolve in the phase space  $\{x \in R_+^n : x_i > 0, i = 1, 2, \dots, n\}$ , we also shall assume that the functions  $P_{ik}$  are continuous on  $R_+$ , and  $x_i + P_{ik}(x_i) > 0$  for  $x_i > 0$ ,  $i = 1, 2, \dots, n$ ,  $k = 1, 2, \dots$ . This restriction prevents the instantaneous extinction of any population  $x_i$  at an impulse time  $t_k$ . We point out that efficient sufficient conditions which guarantee the positivity of the solutions of such systems are given in [2].

Ahmad and Lazer [1] proved that, if for  $i = 1, \dots, n$ ,

$$m[b_i] > \sum_{\substack{i=1 \\ i \neq j}}^n \frac{a_{ij}^M}{a_{jj}^L} M[b_j], \quad (A)$$

then for any solution  $x(t) = \text{col}(x_1(t), \dots, x_n(t))$  of the corresponding system to system (8) without impulses (i.e. with  $x_i(t_k + 0) = x_i(t_k)$ ,  $i = 1, \dots, n$ ,  $k = 1, 2, \dots$ ) if  $x_i(0) > 0$ ,  $i = 1, \dots, n$ , then:

$$0 < \inf_{t \geq 0} x_i(t) < \sup_{t \geq 0} x_i(t) < \infty.$$



LEMMA 4.1. Assume that the condition (A) is satisfied and the functions  $P_{ik}$  are such that

$$-x_i \leq P_{ik}(x_i) \leq 0 \quad \text{for } x_i \in R_+, \quad i = 1, 2, \dots, n, \quad k = 1, 2, \dots$$

Then there exist positive constants  $r$  and  $R$  such that

$$r \leq x_i(t) \leq R, \quad t \in [0, \infty). \quad (10)$$

*Proof.* From corresponding theorem for the continuous case ([1]), it follows that for all  $t \in [0, t_1] \cup (t_k, t_{k+1}]$ ,  $k = 1, 2, \dots$  and  $1 \leq i \leq n$  there exist positive constants  $r_i^*$  and  $R_i^*$  such that the following inequalities hold:

$$r_i^* \leq x_i(t) \leq R_i^*.$$

Using the positivity of the solutions and the condition of Lemma 4.1, we obtain

$$0 < x_i(t_k + 0) = x_i(t_k) + P_{ik}(x_i(t_k)) \leq x_i(t_k) \leq R_i^*.$$

Therefore, there exist positive constants  $r$  and  $R$  such that the inequalities (10) are valid.  $\square$

Next, we will give sufficient conditions for the uniform stability and uniform asymptotic stability of the equilibrium states of (8). The problems of existence and uniqueness of equilibria of Lotka-Volterra systems with or without impulses have been investigated by many authors. Some sufficient conditions for impulsive models are given in [2, 3, 13].

THEOREM 4.2. Assume that:

1. The assumptions of Lemma 4.1 holds.
2.  $r \leq x_i + P_{ik}(x_i) \leq x_i \leq R$  for  $r \leq x_i \leq R$ ,  $i = 1, 2, \dots, n$ ,  $k = 1, 2, \dots$
3. The following inequalities are valid

$$a_{jj}(t) \geq \sum_{\substack{i=1 \\ i \neq j}}^n a_{ij}(t), \quad t \neq t_k, \quad k = 1, 2, \dots$$

Then the equilibrium  $x^*$  of system (8) is uniformly stable.

*Proof.* Define a Lyapunov function

$$V(t, x, x^*) = \sum_{i=1}^n \left| \ln \frac{x_i}{x_i^*} \right|. \quad (11)$$

By Mean Value Theorem and by (10), it follows that for any closed interval contained in  $[0, t_1] \cup (t_k, t_{k+1}]$ ,  $k = 1, 2, \dots$  and for all  $i = 1, 2, \dots$

$$\frac{1}{R}|x_i(t) - x_i^*| \leq |\ln x_i(t) - \ln x_i^*| \leq \frac{1}{r}|x_i(t) - x_i^*|. \quad (12)$$

For  $t > 0$  and  $t = t_k$ ,  $k = 1, 2, \dots$ , we have

$$\begin{aligned} V(t_k + 0, x(t_k + 0), x^*(t_k + 0)) &= \sum_{i=1}^n \left| \ln \frac{x_i(t_k + 0)}{x_i^*(t_k + 0)} \right| \\ &= \sum_{i=1}^n \left| \ln \frac{x_i(t_k) + P_{ik}(x_i(t_k))}{x_i^*(t_k)} \right| \\ &\leq \sum_{i=1}^n \left| \ln \frac{x_i(t_k)}{x_i^*(t_k)} \right| = V(t_k, x(t_k), x^*(t_k)). \end{aligned} \quad (13)$$

Consider the upper right-hand derivative  $D_{(8)}^+ V(t, x(t), x^*)$  of the function  $V(t, x(t), x^*)$  with respect to system (8). For  $t \geq 0$  and  $t \neq t_k$ ,  $k = 1, 2, \dots$ , we derive the estimate

$$D_{(8)}^+ V(t, x(t), x^*) = \sum_{i=1}^n \frac{\dot{x}_i(t)}{x_i(t)} \operatorname{sgn}(x_i(t) - x_i^*).$$

Since  $x^*$  is the equilibrium of (8) and  $b_i(t) = a_{ii}(t)x_i^* + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(t)x_j^*$ , then

$$D_{(8)}^+ V(t, x(t), x^*) \leq \sum_{j=1}^n \left[ -a_{jj}(t)|x_j(t) - x_j^*| + \sum_{\substack{i=1 \\ i \neq j}}^n a_{ij}(t)|x_j(t) - x_j^*| \right].$$

Thus in view of condition 3 of Theorem 4.2, we obtain

$$D_{(8)}^+ V(t, x(t), x^*) \leq 0,$$

$t \geq 0$  and  $t \neq t_k$ ,  $k = 1, 2, \dots$

Since all conditions of Theorem 3.1 hold, then the equilibrium  $x^*$  of system (8) is uniformly stable.  $\square$

**THEOREM 4.3.** *In addition to the assumptions of Theorem 4.2, suppose there exists a nonnegative constant  $\mu$  such that*

$$a_{jj}(t) \geq \mu + \sum_{\substack{i=1 \\ i \neq j}}^n a_{ij}(t), \quad t \neq t_k, \quad k = 1, 2, \dots \quad (14)$$

*Then the equilibrium  $x^*$  of system (8) is uniformly asymptotically stable.*

*Proof.* We consider again the Lyapunov function (11). From (13) and (14), we obtain

$$D_{(8)}^+ V(t, x(t), x^*) \leq -\mu \sum_{i=1}^n |x_i(t) - x_i^*(t)|,$$

$t \geq 0$  and  $t \neq t_k, k = 1, 2, \dots$

Since all conditions of Theorem 3.2 are satisfied, the solution  $x^*$  of system (8) is uniformly asymptotically stable.  $\square$

In order to illustrate some features of our results, in the following we will apply Theorem 4.3 to two-dimensional systems, which have been studied extensively in the literature.

EXAMPLE 4.4. *For the system*

$$\begin{cases} \dot{x}(t) = x(t) [8 - 14x(t) - y(t)], \\ \dot{y}(t) = y(t) [15 - 4x(t) - 13y(t)], \end{cases} \quad (15)$$

one can show that the point  $(x^*, y^*) = (\frac{1}{2}, 1)$  is an equilibrium which is uniformly asymptotically stable [1].

Now, we consider the impulsive Lotka-Volterra system

$$\begin{cases} \dot{x}(t) = x(t) [8 - 14x(t) - y(t)], & t \neq t_k, \\ \dot{y}(t) = y(t) [15 - 4x(t) - 13y(t)], & t \neq t_k, \\ \Delta x(t_k) = -\frac{1}{3} \left( x(t_k) - \frac{1}{2} \right), & k = 1, 2, \dots, \\ \Delta y(t_k) = -\frac{3}{5} \left( y(t_k) - 1 \right), & k = 1, 2, \dots, \end{cases} \quad (16)$$

where  $0 < t_1 < t_2 < \dots$  and  $\lim_{k \rightarrow \infty} t_k = \infty$ .

For the system (16), the point  $(x^*, y^*) = (\frac{1}{2}, 1)$  is an equilibrium and all conditions of Theorem 4.3 are satisfied. In fact, for  $\mu \leq 10$ ,  $r = \frac{1}{2}$  and  $R = 1$ , we have

$$\begin{aligned} \frac{1}{2} &\leq \frac{3x(t_k) + 1}{6} = x(t_k) + P_{1k}(x(t_k)) \\ &= x(t_k) - \frac{1}{3} \left( x(t_k) - \frac{1}{2} \right) = \frac{2}{3} \left( x(t_k) - \frac{1}{2} \right) + \frac{1}{2} \leq x(t_k) \leq 1, \\ \frac{1}{2} &\leq \frac{2y(t_k) + 3}{5} = y(t_k) + P_{2k}(y(t_k)) \\ &= y(t_k) - \frac{3}{5} \left( y(t_k) - 1 \right) = \frac{2}{5} \left( y(t_k) - 1 \right) + 1 \leq y(t_k) \leq 1, \end{aligned}$$

for  $\frac{1}{2} \leq x(t_k) \leq 1$ ,  $\frac{1}{2} \leq y(t_k) \leq 1$ ,  $k = 1, 2, \dots$

Therefore, the equilibrium  $(x^*, y^*) = (\frac{1}{2}, 1)$  is uniformly asymptotically stable.

If, in the system (16), we consider the impulsive perturbations of the form:

$$\begin{cases} \Delta x(t_k) = -3\left(x(t_k) - \frac{1}{2}\right), & k = 1, 2, \dots, \\ \Delta y(t_k) = -\frac{3}{5}\left(y(t_k) - 1\right), & k = 1, 2, \dots, \end{cases}$$

then the point  $(x^*, y^*) = (\frac{1}{2}, 1)$  is again an equilibrium, but there is nothing we can say about its uniform asymptotic stability, because for  $\frac{1}{2} \leq x(t_k) \leq 1$ , we have  $-\frac{1}{2} \leq x(t_k) + P_{1k}(x(t_k)) \leq \frac{1}{2}$ ,  $k = 1, 2, \dots$

The example shows that by means of appropriate impulsive perturbations we can control the system's population dynamics. We can see that impulses are used to keep the stability properties of the system.

EXAMPLE 4.5. *The system*

$$\begin{cases} \dot{x}(t) = x(t) [2 - 6x(t) - y(t)], \\ \dot{y}(t) = y(t) [3 - 2x(t) - 5y(t)]. \end{cases} \quad (17)$$

has a boundary equilibrium point  $(x^*, y^*) = (\frac{1}{3}, 0)$ . We point out that efficient sufficient conditions which guarantee the stability of such solutions of predator-prey systems are given in [7, 10].

However, for the impulsive Lotka-Volterra system

$$\begin{cases} \dot{x}(t) = x(t) [2 - 6x(t) - y(t)], & t \neq t_k, \\ \dot{y}(t) = y(t) [3 - 2x(t) - 5y(t)], & t \neq t_k, \\ \Delta x(t_k) = -\frac{1}{2}\left(x(t_k) - \frac{1}{4}\right), & k = 1, 2, \dots, \\ \Delta y(t_k) = -\frac{1}{3}\left(y(t_k) - \frac{1}{2}\right), & k = 1, 2, \dots, \end{cases}$$

where  $0 < t_1 < t_2 < \dots$  and  $\lim_{k \rightarrow \infty} t_k = \infty$ , the point  $(x^*, y^*) = (\frac{1}{4}, \frac{1}{2})$  is an equilibrium which is uniformly asymptotically stable. In fact, all conditions of

Theorem 4.3 are satisfied for  $\mu \leq 3$ ,  $r = \frac{1}{4}$ ,  $R = \frac{1}{2}$  and

$$\begin{aligned} \frac{1}{4} &\leq \frac{4x(t_k) + 1}{8} = x(t_k) + P_{1k}(x(t_k)) \\ &= x(t_k) - \frac{1}{2} \left( x(t_k) - \frac{1}{4} \right) = \frac{1}{2} \left( x(t_k) - \frac{1}{4} \right) + \frac{1}{4} \leq x(t_k) \leq \frac{1}{2}, \\ \frac{1}{4} &\leq \frac{4y(t_k) + 1}{6} = y(t_k) + P_{2k}(y(t_k)) \\ &= y(t_k) - \frac{1}{3} \left( y(t_k) - \frac{1}{2} \right) = \frac{2}{3} \left( y(t_k) - \frac{1}{2} \right) + \frac{1}{2} \leq y(t_k) \leq \frac{1}{2}, \end{aligned}$$

for  $\frac{1}{4} \leq x(t_k) \leq \frac{1}{2}$ ,  $\frac{1}{4} \leq y(t_k) \leq \frac{1}{2}$ ,  $k = 1, 2, \dots$

This shows that the impulsive perturbations can prevent the population from going extinct.

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