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An Elementary Hilbert Space Approach to Evolutionary Partial Differential Equations

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Abstract. *The purpose of this paper is to provide a survey of an approach to evolutionary problems originally developed in [5], [4] for a special case. The ideas are extended to a much larger problem class and the utility of the approach is exemplified by a Robin type initial boundary value problem for acoustic waves. The paper concludes with an outlook to open directions of further research.*

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1. Introduction

In [5] a theoretical framework has been developed to discuss typical linear evolutionary problems as they arise in various fields of applications. The suitability of the problem class described for such applications has been demonstrated by numerous examples of varying complexity. To briefly introduce the ideas, we first recall that the time derivative ∂_0 can be realized as a normal operator in an exponentially weighted, Hilbert-space-valued L^2-type space $H_{\varrho,0}(\mathbb{R}, H)$, which may be described by completing the space $\mathring{C}_\infty(\mathbb{R}, H)$ of smooth $H-$valued functions, H a (complex) Hilbert space, with compact support in $\mathbb{R}$ with respect to the norm $|\cdot|_{\varrho,0,0}$ induced by the inner product $\langle \cdot | \cdot \rangle_{\varrho,0,0}$ given by

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_0 \exp(-2\varrho t)\ dt,$$

where $\langle \cdot | \cdot \rangle_0$ denotes the inner product of H, $\varrho \in \mathbb{R}_{>0}$. The real part of this normal operator realization of the time-derivative is simply multiplication by the parameter ϱ:

$$\mathfrak{Re}\,\partial_0 = \varrho.$$

Consequently,

$$\partial_{0,\varrho} := \partial_0 - \varrho = \mathrm{i}\ \mathfrak{Im}\,\partial_0$$

is skew-selfadjoint. The Fourier-Laplace transform given as the unitary extension $\mathcal{L}_\varrho$ of the mapping

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}, H) \subseteq H_{\varrho,0}(\mathbb{R}, H) &\to L^2(\mathbb{R}, H) \\ \varphi &\mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\mathrm{i} \cdot (t - \mathrm{i}\,\varrho))\, \varphi(t)\, dt \end{aligned}$$

is a spectral representation of the selfadjoint operator $\frac{1}{\mathrm{i}}\,\partial_{0,\varrho}$ and so we find the unitary equivalence

$$\partial_0 = \mathcal{L}_\varrho^* (\mathrm{i}\, m_0 + \varrho)\, \mathcal{L}_\varrho\,,$$

where m_0 denotes the multiplication-by-the-argument operator given as the closure of

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R}, H) \subseteq L^2(\mathbb{R}, H) &\to L^2(\mathbb{R}, H) \\ \varphi &\mapsto m_0\varphi \end{aligned}$$

with

$$(m_0\varphi)(\lambda) := \lambda\, \varphi(\lambda) \text{ in } H$$

for every $\lambda \in \mathbb{R}$. This observation allows us to consistently define an operator function calculus for operator-valued functions such as a uniformly bounded, analytic function

$$\begin{aligned} B_{\mathbb{C}}\left(\frac{1}{2\varrho}, \frac{1}{2\varrho}\right) \subseteq \mathbb{C} &\to L(H, H) \\ z &\mapsto M(z) \end{aligned}$$

mapping the ball $B_{\mathbb{C}}\left(\frac{1}{2\varrho}, \frac{1}{2\varrho}\right)$ in $\mathbb{C}$ of radius $\frac{1}{2\varrho}$ centered at $\frac{1}{2\varrho}$ into the Banach space $L(H, H)$ of bounded linear operators in H by letting

$$M\left(\partial_0^{-1}\right) := \mathcal{L}_\varrho^* M\left(\frac{1}{\mathrm{i}\, m_0 + \varrho}\right) \mathcal{L}_\varrho.$$

Here the linear operator $M\left(\frac{1}{\mathrm{i}\, m_0 + \varrho}\right) : L^2(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$ is determined uniquely by

$$\left(M\left(\frac{1}{\mathrm{i}\, m_0 + \varrho}\right) \varphi\right)(\lambda) := M\left(\frac{1}{\mathrm{i}\, \lambda + \varrho}\right) \varphi(\lambda) \text{ in } H$$

for every $\lambda \in \mathbb{R}$, $\varphi \in \mathring{C}_\infty(\mathbb{R}, H)$. Considering the canonical extension of a densely defined, closed linear operator $A : D(A) \subseteq H \to H$ to $H_{\varrho,0}(\mathbb{R}, H)$ denoted for simplicity again by A, we are led to the closure of the operator sum $M(\partial_0)\,\partial_0 + A$ (with natural domain $D(\partial_0) \cap D(A)$) for which again we utilize the same name (this notational simplification can be legitimized in the framework of extrapolation spaces, which we shall, however, avoid to invoke here). It has been shown in [5], [4], that the standard initial boundary value problems of mathematical physics have the common form

$$\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)U = F, \tag{1}$$

where A is in fact skew-selfadjoint and the so-called material law operator $M\left(\partial_0^{-1}\right)$ satisfies the following strict positivity constraint. There is a constant $c_0 \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} \mathfrak{Re}\left\langle M\left(\partial_0^{-1}\right)\partial_0 U|U\right\rangle_{\varrho,0,0} &\geq c_0\ \langle U|U\rangle_{\varrho,0,0}\ , \\ \mathfrak{Re}\left\langle M\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^* U|U\right\rangle_{\varrho,0,0} &\geq c_0\ \langle U|U\rangle_{\varrho,0,0} \end{aligned} \tag{2}$$

for every sufficiently large $\varrho \in \mathbb{R}_{>0}$ and every $U \in D\left(\partial_0\right) = D\left(\partial_0^*\right)$.

Then the solution operator

$$\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)^{-1} : H_{\varrho,0}\left(\mathbb{R}, H\right) \to H_{\varrho,0}\left(\mathbb{R}, H\right)$$

is well-defined for all sufficiently large $\varrho \in \mathbb{R}_{>0}$.

A superficial look at (1) could lead to the misconception that this class merely describes a class of evolution equations, which by simply solving for $\partial_0 U$ would yield a more familiar form. A somewhat artifical example may serve to avoid this misunderstanding (for physically relevant examples see [5]). Consider the (on $\mathbb{C} \setminus 0$) analytic operator-valued function

$$z \mapsto M\left(z\right) = P + \left(1 - P\right)\ z\ \exp\left(-h/z\right),$$

where P is an arbitrary orthogonal projector in H, $h \in \mathbb{R}_{>0}$. Then (observing (12) below) leads to a material law operator of the appropriate form

$$M\left(\partial_0^{-1}\right) = P + \left(1 - P\right)\ \partial_0^{-1}\tau_{-h},$$

where τ_{-h} denotes backwards time-translation by a time span $h \in \mathbb{R}_{>0}$. The case $P = 1$ leads to solving $\left(\partial_0 + A\right)U = F$. This is a simple evolution equation, but considered in our approach considered as an invertible equation involving a normal operator $\partial_0 + A$ (in $H_{\varrho,0}\left(\mathbb{R}, H\right)$ with real part given by multiplication by the parameter $\varrho \in \mathbb{R}_{>0}$. Note that, in particular, the concept of semi-groups is completely by-passed here. If P differs from the identity then the resulting equation $\left(P\partial_0 + \left(1 - P\right)\tau_{-h} + A\right)U = F$ is not at all an evolution equation in the usual sense. It involves a time delay term and there is no obvious way in writing this in the form of an explicit ODE anyway. Indeed, the case $P = 0$ comfortably covered by our theory does not even contain a time-derivative!

This observation indicates why we are speaking of *evolutionary* equations and not of evolution equation in describing our problem class.

The central aim of this paper is in particular to extend the described solution theory to encompass an even larger class of problems allowing for certain non-skew-selfadjoint operators A and to survey some specific aspects of this generalization. The presentation will rest on a conceptually more elementary version of this theory as presented in [4].

After briefly describing the corner stones of a general solution theory in section 2 we shall discuss in section 3 the particular issues of causality and memory, which are characteristic features of problems we may rightfully call *evolutionary.* We will then conclude the presentation with a particular application involving an implementation of a boundary condition of third kind (Robin boundary condition) for acoustic wave propagation.

2. General Solution Theory

The basic solution theory of a problem of the form

$$\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)U = F \tag{3}$$

in $H_{\varrho,0}(\mathbb{R},H)$, where $A : D(A) \subseteq H_{\varrho,0}(\mathbb{R},H) \to H_{\varrho,0}(\mathbb{R},H)$ is a closed, densely defined, linear operator, rests on strict positive definite conditions of the form

$$\mathfrak{Re}\left\langle U\right|\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)U\big\rangle_{\varrho,0,0} \geq \beta_0 \left\langle U|U\right\rangle_{\varrho,0,0}\ , \tag{4}$$

$$\mathfrak{Re}\left\langle V\right|\left(M^*\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^* + A^*\right)V\Big\rangle_{\varrho,0,0} \geq \beta_0 \left\langle V|V\right\rangle_{\varrho,0,0} \tag{5}$$

for some $\beta_0 \in \mathbb{R}_{>0}$ and every $U \in D(\partial_0) \cap D(A)$ and $V \in D(\partial_0) \cap D(A^*)$. Here

$$M\left(\partial_0^{-1}\right)^* = M^*\left(\left(\partial_0^*\right)^{-1}\right)$$

is the adjoint of the material law operator $M\left(\partial_0^{-1}\right)$. We recall from the introduction the concept of a material law operator and for sake of reference give a formal definition by paraphrasing our introductory description.

Definition 2.1. *A material law operator $M\left(\partial_0^{-1}\right)$ in $H_{\varrho,0}(\mathbb{R},H)$, $\varrho \in \mathbb{R}_{>0}$, is given via the Fourier-Laplace transform as*

$$M\left(\partial_0^{-1}\right) := \mathcal{L}_\varrho^* M\left(\frac{1}{\mathrm{i}\,m_0 + \varrho}\right)\mathcal{L}_\varrho.$$

Here $M\left(\frac{1}{\mathrm{i}\,m_0+\varrho}\right) : L^2(\mathbb{R},H) \to L^2(\mathbb{R},H)$ is a multiplication operator (operator-valued) uniquely determined by

$$\left(M\left(\frac{1}{\mathrm{i}\,m_0 + \varrho}\right)\varphi\right)(\lambda) := M\left(\frac{1}{\mathrm{i}\,\lambda + \varrho}\right)\varphi(\lambda) \quad \text{in } H$$

for every $\lambda \in \mathbb{R}$, $\varphi \in \mathring{C}_\infty(\mathbb{R}, H)$, *where*

$$\begin{aligned} B_{\mathbb{C}}\left(\frac{1}{2\varrho}, \frac{1}{2\varrho}\right) \subseteq \mathbb{C} &\to L(H,H) \\ z &\mapsto M(z) \end{aligned}$$

is a uniformly bounded, analytic function mapping the ball $B_{\mathbb{C}}\left(\frac{1}{2\varrho}, \frac{1}{2\varrho}\right)$ *in* $\mathbb{C}$ *of radius* $\frac{1}{2\varrho}$ *centered at* $\frac{1}{2\varrho}$ *into the Banach space* $L(H,H)$ *of continuous linear operators in* H.

Remark 2.2. *Note that* $M\left(\partial_0^{-1}\right)$ *is strictly speaking depending on* $\varrho \in \mathbb{R}_{>0}$. *Since, however,* $B_{\mathbb{C}}(r,r) \subseteq B_{\mathbb{C}}(s,s)$ *for* $s \geq r$, $s, r \in \mathbb{R}$, *we see that by this definition the material law operators are continuous extensions of the same operator family* M *restricted to* $\mathring{C}_\infty(\mathbb{R}, H)$, *which is dense in every* $H_{\widetilde{\varrho}}(\mathbb{R}, H)$ *for* $\widetilde{\varrho} \in \mathbb{R}_{\geq \varrho}$. *Thus, if* $M\left(\partial_0^{-1}\right)$ *is defined for one* $\varrho \in \mathbb{R}_{>0}$ *it is essentially the same for every larger* $\widetilde{\varrho}$.

Since in the general case the central issue of causality is still an open issue, so far only the case of A being the canonical extension to $H_{\varrho,0}(\mathbb{R}, H)$ of a skew-selfadjoint operator in H has been considered more closely, [5, 4]. In the following we want to go a step further and extend the theory to a class of non-skew-selfadjoint operators A in H.

The essential requirement for A is that A and its adjoint A^* are both bounded below in the sense that there is a $\gamma_0 \in \mathbb{R}$ such that

$$\mathfrak{Re} \langle U | AU \rangle_0 \geq \gamma_0 \langle U | U \rangle_0 \,, \tag{6}$$

$$\mathfrak{Re} \langle V | A^* V \rangle_0 \geq \gamma_0 \langle V | V \rangle_0 \tag{7}$$

for all $U \in D(A)$, $V \in D(A^*)$. Moreover, we must have

$$c_0 > -\gamma_0 \,,$$

where $c_0 \in \mathbb{R}_{>0}$ is a constant such that

$$\begin{aligned} \textstyle\int_{]-\infty,a]} \mathfrak{Re} \left\langle M\left(\partial_0^{-1}\right) \partial_0 U | U \right\rangle_0 (t) \, \exp(-2\varrho t) \; dt &\geq \\ \geq c_0 \textstyle\int_{]-\infty,a]} \langle U | U \rangle_0 (t) \, \exp(-2\varrho t) \; dt \,, \\ \textstyle\int_{]-\infty,a]} \mathfrak{Re} \left\langle M\left((\partial_0^*)^{-1}\right) \partial_0^* U | U \right\rangle_0 (t) \, \exp(-2\varrho t) \; dt &\geq \\ \geq c_0 \textstyle\int_{]-\infty,a]} \langle U | U \rangle_0 (t) \, \exp(-2\varrho t) \; dt \end{aligned} \tag{8}$$

for every sufficiently large $\varrho \in \mathbb{R}_{>0}$ and every $a \in \mathbb{R}$, $U \in D(\partial_0) = D(\partial_0^*)$ holds, in order to maintain (4), (5), (with $\beta_0 = c_0 + \gamma_0$). This assumption is slightly more restrictive than (2) to simplify later arguments. It should be noted, however, that a completely analogous reasoning to the presentation in [4] would also lead to a proof of the following theorem 2.3.

Clearly, the skew-selfadjoint case, i.e. $A = -A^*$, is a special case with $\gamma_0 = 0$. We are now ready to formulate the basic solution theory of (3).

THEOREM 2.3. *Let $M\left(\partial_0^{-1}\right)$ be a material law operator in $H_{\varrho,0}(\mathbb{R}, H)$ and $A : D(A) \subseteq H \to H$ a closed, densely defined, linear operator in a Hilbert space H satisfying assumptions (4), (5). Then the solution operator of (3)*

$$\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)^{-1} : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)$$

exists for all sufficiently large $\varrho \in \mathbb{R}_{>0}$.

Proof. We have for sufficiently large $\varrho \in \mathbb{R}_{>0}$ and all $U \in D(\partial_0) \cap D(A)$ by letting $a \to \infty$ in (8)

$$\begin{aligned}
|U|_{\varrho,0,0}\left|\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)U\right|_{\varrho,0,0} &\geq \mathfrak{Re}\left\langle U\right|\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)U\left.\right\rangle_{\varrho,0,0} \\
&= \mathfrak{Re}\left\langle U|M\left(\partial_0^{-1}\right)\partial_0 U\right\rangle_{\varrho,0,0} + \\
&\quad + \int_{\mathbb{R}} \mathfrak{Re}\left\langle U(t)\,|AU(t)\right\rangle_0 \exp(-2\varrho t)\;dt \\
&\geq c_0\left\langle U|U\right\rangle_{\varrho,0,0} + \\
&\quad + \gamma_0 \int_{\mathbb{R}} \left\langle U(t)\,|U(t)\right\rangle_0 \exp(-2\varrho t)\;dt \\
&\geq (c_0 + \gamma_0)\left\langle U|U\right\rangle_{\varrho,0,0}
\end{aligned}$$

and so in particular that $\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)$ is invertible and its inverse, defined on the range $\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)[H_{\varrho,0}(\mathbb{R}, H)]$ of $\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)$

$$\begin{aligned}
&\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)^{-1} : \\
&\quad : \left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)[H_{\varrho,0}(\mathbb{R}, H)] \subseteq H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)
\end{aligned}$$

is continuous. Indeed,

$$\left\|\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)^{-1}\right\| \leq \frac{1}{c_0 + \gamma_0}.$$

By the spectral cut-off technique with

$$\chi_{]-N+1,N+1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right) := \chi_{]-\infty,-N-1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right) - \chi_{]-\infty,N+1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right),\; N \in \mathbb{N},$$

where $\left(\chi_{]-\infty,\lambda]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right)\right)_{\lambda\in\mathbb{R}}$ is the spectral family associated with the selfadjoint $\mathfrak{Im}\,\partial_0 = \frac{1}{\mathrm{i}}\partial_{0,\varrho}$, as utilized in [4], it follows that $\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)^*$ is given by the closure of $\left(M\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^*+A^*\right)$. Recalling that we decided to drop the closure bar in such a case, this is

$$\left(M\left(\partial_0^{-1}\right)\partial_0+A\right)^*=\left(M\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^*+A^*\right).$$

By analogy we find that also

$$\begin{aligned}&\left(M\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^*+A^*\right)^{-1}:\\&\quad:\left(M\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^*+A^*\right)\left[H_{\varrho,0}\left(\mathbb{R},H\right)\right]\subseteq H_{\varrho,0}\left(\mathbb{R},H\right)\to H_{\varrho,0}\left(\mathbb{R},H\right)\end{aligned}$$

is continuous with

$$\left\|\left(M\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^*+A^*\right)^{-1}\right\|\leq\frac{1}{c_0+\gamma_0}.$$

Since in particular the null space of $\left(M\left(\left(\partial_0^*\right)^{-1}\right)\partial_0^*+A^*\right)$ is trivially equal to $\{0\}$, we have

$$\overline{\left(M\left(\partial_0^{-1}\right)\partial_0+A\right)\left[H_{\varrho,0}\left(\mathbb{R},H\right)\right]}=H_{\varrho,0}\left(\mathbb{R},H\right).$$

Thus, $\left(M\left(\partial_0^{-1}\right)\partial_0+A\right)^{-1}$ is a densely defined, (by construction and convention) closed, continuous linear operator. Therefore, $\left(M\left(\partial_0^{-1}\right)\partial_0+A\right)^{-1}$ must already be defined on $H_{\varrho,0}\left(\mathbb{R},H\right)$. □

Now that we have a well-established solution operator, we address the question of the dependence of the solution on the parameter $\varrho\in\mathbb{R}_{>0}$.

We want to show that for suitable right-hand sides F in (3) the solution is largely independent of $\varrho\in\mathbb{R}_{>0}$. For this let

$$F\in\bigcap_{\varrho\in\mathbb{R}_{>\varrho_0}}H_{\varrho,0}\left(\mathbb{R},H\right)\tag{9}$$

for some $\varrho_0\in\mathbb{R}_{>0}$.

THEOREM 2.4. *Let $\varrho_0\in\mathbb{R}_{>0}$ be sufficiently large and let F satisfy condition (9). Then, for $\varrho,\widetilde{\varrho}\in\mathbb{R}_{>\varrho_0}$ the solutions $U_\varrho\in H_{\varrho,0}\left(\mathbb{R},H\right)$ and $U_{\widetilde{\varrho}}\in H_{\widetilde{\varrho},0}\left(\mathbb{R},H\right)$ of (3) must coincide.*

Proof. Let ϱ_0 be sufficiently large so that (8) is also satisfied for all $\varrho\in\mathbb{R}_{>\varrho_0}$. Without loss of generality let $\widetilde{\varrho}\leq\varrho$. Then, we have

$$\varphi\left(m_0\right)\left[H_{\varrho,0}\left(\mathbb{R},H\right)\right]\subseteq H_{\widetilde{\varrho},0}\left(\mathbb{R},H\right)$$

for any smooth cut-off function φ with $\varphi(t) = 1$ on $]-\infty, a]$ and $\varphi(t) = 0$ on $]\, a+1, \infty[$. Moreover, with the same cut-off technique as in the proof of theorem 2.3 and noting that $\chi_{]-N+1,N+1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right) U_\varrho \in D(A)$, see [4], we get

$$\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)\varphi(m_0)\,\chi_{]-N+1,N+1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right) U_\varrho =$$
$$= \varphi(m_0)\,\chi_{]-N+1,N+1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right) F + \varphi(m_0)\,A\chi_{]-N+1,N+1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right) U_\varrho +$$
$$+M\left(\partial_0^{-1}\right)\varphi'(m_0)\,\chi_{]-N+1,N+1]}\left(\frac{1}{\mathrm{i}}\partial_{0,\varrho}\right) U_\varrho.$$

From this together with (8) we obtain letting $N \to \infty$

$$\mathfrak{Re} \int_{]-\infty,a]} \langle U_\varrho(t)\,|F(t)\rangle_0 \exp\left(-2\widetilde{\varrho}t\right)\,dt \geq$$
$$\geq \beta_0 \int_{]-\infty,a]} \langle U_\varrho(t)\,|U_\varrho(t)\rangle_0 \exp\left(-2\widetilde{\varrho}t\right)\,dt,$$

which implies similar as in the proof of theorem 2.3 that

$$\beta_0^2 \int_{]-\infty,a]} |U_\varrho(t)|_0^2 \exp\left(-2\widetilde{\varrho}t\right)\,dt \leq \int_{]-\infty,a]} |F(t)|_0^2 \exp\left(-2\widetilde{\varrho}t\right)\,dt. \tag{10}$$

Letting $a \to \infty$ in this yields

$$U_\varrho \in H_{\widetilde{\varrho},0}(\mathbb{R}, H).$$

By uniqueness the solution $U_{\widetilde{\varrho}} \in H_{\widetilde{\varrho},0}(\mathbb{R}, H)$ must now coincide with U_ϱ. □

3. Causality and Memory

Clearly, the basic solution theory can be generalized to more general situations. Keeping in mind that the condition is actually a special case of monotonicity, even the case of material law operators for which $\partial_0 M\left(\partial_0^{-1}\right)$ is strictly monotone appears accessible. Note that the monotonicity in the weighted space-time spaces discussed here is not the monotonicity in H, which frequently occurs in other frameworks. An extension in this direction is, however, not the topic we want to persue here. In the solution theory described so far there is a more subtle point to be considered, going beyond the issue of mere well-posedness. For an “evolution” – in the intuitive meaning – to take place we also expect an additional qualitative property securing “causality” for the solution operator.

To discuss this aspect rigorously, we first need to properly define, what we should mean by “causality” of the solution operator. We utilize here again the multiplier notation for the cut-off by a characteristic function defining $\chi_M(m_0)\,u$ by

$$(\chi_M(m_0)\,u)(t) = \chi_M(t)\;u(t)$$

for (almost) every $t \in \mathbb{R}$ and every $u \in H_{\varrho,0}(\mathbb{R}, H)$, $M \subseteq \mathbb{R}$. With this notation we propose the following (in comparison with [5] somewhat simplified) version of a mapping being causal.

Definition 3.1. *A mapping* $W : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)$, $\varrho \in \mathbb{R}_{>0}$, *is called causal if*

$$\bigwedge_{a\in\mathbb{R}} \bigwedge_{u,v\in H_{\varrho,0}(\mathbb{R},H)} \big(\chi_{]-\infty,a]}(m_0)(u-v) = 0 \implies \chi_{]-\infty,a]}(m_0)(W(u) - W(v)) = 0\big).$$

This can be stated in somewhat heuristical terms as "the values of $W(u)$ and $W(v)$ are equal as long as the data u, v are equal". The assumptions on the material law operator $M\left(\partial_0^{-1}\right)$ have been specified to ensure that it is a causal operator, as one would expect it to be, see [5, 4]. In the case which is our current focus we have an additional feature that simplifies matters. We have time-translation invariance and so it suffices to consider just one particular $a \in \mathbb{R}$, say $a = 0$. Time-translation τ_h is given by

$$(\tau_h\varphi)(t) = \varphi(t+h)$$

for $t, h \in \mathbb{R}$ and $\varphi \in H_{\varrho,0}(\mathbb{R}, H)$. We summarize this observation in the following lemma.

Lemma 3.2. *Let* $W : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)$, $\varrho \in \mathbb{R}_{>0}$, *commute with time-translation* τ_h *for every* $h \in \mathbb{R}$, *then* W *is causal if and only if*

$$\bigwedge_{u,v\in H_{\varrho,0}(\mathbb{R},H)} \big(\chi_{]-\infty,0]}(m_0)\;(u-v) = 0 \implies \chi_{]-\infty,0]}(m_0)\;(W(u) - W(v)) = 0\big).$$

Proof. The result is immediate if one observes the commutator relations

$$\tau_a\chi_{]-\infty,a]} = \chi_{]-\infty,0]}\tau_a,\;\; \tau_a W = W\tau_a\;,\;\; a \in \mathbb{R}.$$

□

Note that in case W is linear and translationinvariant causality simplifies further by specializing to $v = 0$

$$\bigwedge_{u\in H_{\varrho,0}(\mathbb{R},H)} \big(\chi_{]-\infty,0]}(m_0)\;u = 0 \implies \chi_{]-\infty,0]}(m_0)\;Wu = 0\big)\,.$$

Causality of such a W clearly implies

$$\chi_{]-\infty,0]}(m_0)\;W\,\chi_{]-\infty,0]}(m_0)\; = \chi_{]-\infty,0]}(m_0)\;W. \tag{11}$$

3.1. Causality.

Noting that

$$\tau_h = \exp(h\partial_0) \tag{12}$$

in the sense of the function calculus of normal operators, time-translation invariance of the solution operator is clear, see [4]. Moreover, causality follows now very elementarily from (2.4) for $\varrho = \widetilde{\varrho}$. We record this as the following proposition:

PROPOSITION 3.3. *The solution operator* $\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)^{-1} : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)$ *is causal for all sufficiently large* $\varrho \in \mathbb{R}_{>0}$.

3.2. Memory.

Since the type of material law operators we consider may show "memory effects", we also need to make precise, what this should mean. In a possibly somewhat surprising analogy to causality we give the following definition.

DEFINITION 3.4. *We say* $W : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)$, $\varrho \in \mathbb{R}_{>0}$, *has no memory if*

$$\bigwedge_{a\in\mathbb{R},\, u,v\in H_{\varrho,0}(\mathbb{R},H)} \chi_{[a,\infty[}(m_0)(u-v) = 0 \implies \chi_{[a,\infty[}(m_0)(W(u) - W(v)) = 0. \tag{13}$$

If this is not *the case we say that* W *has memory or* W *is a memory term.*

REMARK 3.5. *The case that* $W : H_{\varrho,0}(\mathbb{R}, H) \to H_{\varrho,0}(\mathbb{R}, H)$, $\varrho \in \mathbb{R}_{>0}$, *is a memory term is obviously characterized by a simple negation of (13) as*

$$\bigvee_{a\in\mathbb{R},\, u,v\in H_{\varrho,0}(\mathbb{R},H)} \chi_{[a,\infty[}(m_0)\ (u-v) = 0 \wedge \chi_{[a,\infty[}(m_0)\ (W(u) - W(v)) \neq 0. \tag{14}$$

Clearly ∂_0^{-1} is a linear and time-translation invariant memory term. So we may expect that any non-constant family M generates in the above sense a material law operator with memory for all sufficiently large $\varrho \in \mathbb{R}_{>0}$. In the case that M has a continuous extension to $B_{\mathbb{C}}\left(\frac{1}{2\varrho}, \frac{1}{2\varrho}\right) \cup \{0\}$ and denoting the continuous extension at zero as $M(0)$, however, one usually says that $\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)$ has memory if the closure $M_1\left(\partial_0^{-1}\right)$ of $\partial_0\left(M\left(\partial_0^{-1}\right) - M(0)\right)$ is a memory term in $H_{\varrho,0}(\mathbb{R}, H)$ for all sufficiently large $\varrho \in \mathbb{R}_{>0}$. If M_1 is constant, then the material law operator is simply

$$M\left(\partial_0^{-1}\right) = M(0) + \partial_0^{-1} M_1$$

and $M\left(\partial_0^{-1}\right)\partial_0 + A = M(0)\partial_0 + M_1 + A$.

4. Some Observations and Special Cases

To illuminate the above abstract theory we shall inspect some special aspects. Typical for initial boundary value problems of mathematical physics is that A and $-A^*$ are extensions of the same skew-symmetric operator. In section 5 we shall have occasion to inspect such a case more closely. As stated in the introduction the standard cases are indeed linked to the case of A being skew-selfadjoint. It is in itself a remarkable fact that the skew-selfadjointness of A is predominantly due to a specific block operator matrix structure of A namely that

$$A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},$$

where $C : D(C) \subseteq H_0 \to H_1$ is a closed, densely defined, linear operator acting between Hilbert spaces H_0 and H_1. It is a straight-forward calculation to confirm that an operator A of this structure is always skew-selfadjoint in the Hilbert space $\mathcal{H}$ given by the direct sum $H_0 \oplus H_1$ of the Hilbert spaces H_0, H_1. In the simple case $M\left(\partial_0^{-1}\right) = 1$ we are led to the evolution operator $\partial_0 + A$. Keeping in mind that the null space of A reduces A we may assume without loss of generality that A is injective (the null space of A corresponds to the static behavior $\partial_0 + A$). The polar decomposition $C = U|C|$ of C, where now $U : H_0 \to H_1$ is unitary, implies $C^* = |C|\,U^*$ and shows that $A = \mathcal{U}\begin{pmatrix} 0 & -|C| \\ |C| & 0 \end{pmatrix}\mathcal{U}^*$ with $\mathcal{U} = \begin{pmatrix} -U & 0 \\ 0 & 1 \end{pmatrix} : H_0 \oplus H_0 \to H_0 \oplus H_1$ unitary. In other words, A is unitarily equivalent to $\begin{pmatrix} 0 & |C| \\ -|C| & 0 \end{pmatrix}$, which is skew-selfadjoint in the real Hilbert space $H_0 \oplus H_0$. Thus, $\partial_0 + A$ is also unitarily equivalent to $\partial_0 - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}|C|$. The latter clearly shows the structure of a Hamiltonian system[1]. Indeed, the functional $\mathbb{H}$ given by $(p,q) \to \langle p|\,|C|\,p\rangle_{H_0} + \langle q|\,|C|\,q\rangle_{H_0}$ on $D(|C|) \oplus D(|C|)$ is the corresponding Hamiltonian. It is $d_p\mathbb{H} = d_q\mathbb{H} = |C|$ in the sense of Frechét derivatives.

Because of this remarkable connection we shall say that an operator A of the above block operator matrix form has *Hamiltonian structure.* It is already an interesting unifying feature that the typical linear evolutionary equations of mathematical physics (acoustics, electrodynamics, visco-elastics etc.) show this Hamiltonian structure. The more complex coupling mechanisms becoming more recently of interest maintain – after suitable symmetric permutation of

[1]In this direction it may also be noteworthy that assuming without loss of generality H_0 to be a real Hilbert space and going over to its complexification $\mathcal{H}_0$ multiplication by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ turns into multiplication by the imaginary unit i. Thus, $\partial_0 + A$ is transformed to the Schrdinger type operator $\partial_0 - \mathrm{i}\,|C|$. Indeed, with $C = |C| = -\Delta$ we get the usual free Schrdinger operator $\partial_0 + \mathrm{i}\Delta$.

rows and colums – this Hamiltonian structure. The complexity resides solely in the material law operator. This structural observation has been exemplified by numerous examples in [5].

Although, the Hamiltonian structure is at the root of the derivation of many evolution equations associated with classical physical phenomena it has been customary to deal with initial boundary value problems as second order equations. Historically, this appears to be motivated by the desire to get the Laplacian as a well-studied operator as soon as possible into play. Consequently, model modifications started from the second order case. For moderately more complex materials it is, however, easy to see, that they do *not* correspond to a second order partial differential equation.

Apart from the complexity of material properties being modelled into the material law operator, it may also be surprising that the required positivity conditions (8) also allow for the case that some equations in the system may not contain any time-derivative. This situation leads to so-called *partial differential algebraic equation* (PDAE). In the first order framework presented here, the heat diffusion provides already a natural example of a PDAE.

The positive definiteness condition (8) is also general enough to include fractional integrals in the material law operator. Here only the case of a material law of the form

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1/2} M_{1/2} + \partial_0^{-1} M_1\left(\partial_0^{-1}\right)$$

has been considered so far and will be published in a forthcoming paper.

In all this we have ignored the issue of initial data. Restricting our attention to the case $M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1} M_1\left(\partial_0^{-1}\right)$ with M_1 analytic and uniformly bounded in $B_{\mathbb{C}}\left(\frac{1}{2\varrho}, \frac{1}{2\varrho}\right)$, we recall that initial data are implemented in this setting by cancelling the prescribed jump M_0U_0 of the term M_0U at time zero, compare [5]. Clearly, at this point we need to invoke the theory of Sobolev lattices from [5] in order to argue rigorously. However, to avoid lengthy theoretical considerations for the purpose of this survey we shall rely on [5] for the details. Based on the concepts developed in [5], we find that a solution $U \in H_{\varrho,0}(\mathbb{R}, H)$ of

$$\partial_0\left(M_0U - \chi_{\mathbb{R}_{>0}} \otimes M_0U_0\right) + M_1\left(\partial_0^{-1}\right)U + AU = F,$$

where $F = 0$ on $\mathbb{R}_{<0}$ and $U_0 \in D(A)$, would be such that

$$\partial_0\left(M_0U - \chi_{\mathbb{R}_{>0}} \otimes M_0U_0\right) \in H_{\varrho,0}(\mathbb{R}, \mathbb{C}) \otimes H_{-1}(|A| + \mathrm{i}),$$

where $H_{-1}(|A| + \mathrm{i})$ is the completion of $H_{\varrho,0}(\mathbb{R}, \mathbb{C}) = H_{\varrho,0}(\mathbb{R}, \mathbb{C}) \otimes H$ with respect to the norm $\left|(|A| + \mathrm{i})^{-1} \cdot\right|_0$. We may conclude that

$$M_0U - \chi_{\mathbb{R}_{>0}} \otimes M_0U_0 \in H_{\varrho,1}(\mathbb{R}, \mathbb{C}) \otimes H_{-1}(|A| + \mathrm{i}).$$

By a 1-dimensional ($H_{-1}(|A|+\mathrm{i})$-valued) Sobolev imbedding result we obtain

$$(M_0U)(0+) = M_0U_0\,.$$

compare [5] for the details of the reasoning. This observation leads to $U \in H_{\varrho,0}(\mathbb{R}, H)$ being found as given by

$$U = V + \chi_{\mathbb{R}_{>0}} \otimes U_0\,,$$

where $V \in H_{\varrho,0}(\mathbb{R}, H)$ is the solution of

$$\left(M_0\partial_0 + M_1\left(\partial_0^{-1}\right) + A\right)V = F - M_1\left(\partial_0^{-1}\right)\chi_{\mathbb{R}_{>0}} \otimes U_0 + \chi_{\mathbb{R}_{>0}} \otimes AU_0.$$

A related reformulation difficulty occurs for general memory problems. Although here it is common to assume $U_0 = 0$ leading to continuity at time zero, which is exactly the case dealt with in the above, one usually assumes a different perspective. Instead of looking for the solution U it is said that U is supposed to be known prior to time zero and the solution we seek is the future development of this known past. To consolidate this perspective with our framework we can argue similarly as for the initial data case relying again on [5] for the details of the arguments. Splitting up the past and the future in

$$\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)U = F$$

yields in a first cut-off step

$$\chi_{\mathbb{R}_{>0}}(m_0)\,M\left(\partial_0^{-1}\right)\partial_0\left(\left(\chi_{\mathbb{R}_{<0}}(m_0)\,U + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right) + \left(\chi_{\mathbb{R}_{>0}}(m_0)\,U - \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right)\right) + \\ + A\chi_{\mathbb{R}_{>0}}(m_0)\,U = \chi_{\mathbb{R}_{>0}}(m_0)\,F.$$

Restoring the full material law operator $M\left(\partial_0^{-1}\right)$ on the left leads to

$$M\left(\partial_0^{-1}\right)\partial_0\left(\left(\chi_{\mathbb{R}_{<0}}(m_0)\,U + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right) + \left(\chi_{\mathbb{R}_{>0}}(m_0)\,U - \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right)\right) + \\ + A\chi_{\mathbb{R}_{>0}}(m_0)\,U = \\ = \chi_{\mathbb{R}_{>0}}(m_0)\,F + \\ + \chi_{\mathbb{R}_{<0}}(m_0)\,M\left(\partial_0^{-1}\right)\partial_0\left(\left(\chi_{\mathbb{R}_{<0}}(m_0)\,U + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right) + \left(\chi_{\mathbb{R}_{>0}}(m_0)\,U - \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right)\right) = \\ = \chi_{\mathbb{R}_{>0}}(m_0)\,F + \chi_{\mathbb{R}_{<0}}(m_0)\,M\left(\partial_0^{-1}\right)\partial_0\left(\left(\chi_{\mathbb{R}_{<0}}(m_0)\,U + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right)\right).$$

The last simplification follows from the causality of the material law operator, which is implied by its general assumptions. Separating future and past further finally yields

$$\left(M\left(\partial_0^{-1}\right)\partial_0 + A\right)\left(\chi_{\mathbb{R}_{>0}}(m_0)\,U - \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right) = \\ = \chi_{\mathbb{R}_{>0}}(m_0)\,F - \chi_{\mathbb{R}_{>0}}(m_0)\;M\left(\partial_0^{-1}\right)\partial_0\left(\left(\chi_{\mathbb{R}_{<0}}(m_0)\,U + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right)\right) + \\ -\chi_{\mathbb{R}_{>0}} \otimes AU(0-)\,.$$

So assuming that $\left(\chi_{\mathbb{R}_{<0}}(m_0)\,U + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right)$ is in $D(\partial_0)$ for all sufficiently large $\varrho \in \mathbb{R}_{>0}$ and $U(0-) \in D(A)$ makes this a problem of the above type. Thus, the future development $\chi_{\mathbb{R}_{>0}}(m_0)\,U$ can be found as

$$\chi_{\mathbb{R}_{>0}}(m_0)\,U = V + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\,,$$

where $V \in H_{\varrho,0}(\mathbb{R}, H)$ is the solution of

$$\begin{aligned}\left(M\left(\partial_0^{-1}\right)\partial_0 \;+\; A\right)V &= \\ &= \chi_{\mathbb{R}_{>0}}(m_0)\,F + \\ &\quad -\chi_{\mathbb{R}_{>0}}(m_0)\;M\left(\partial_0^{-1}\right)\partial_0\left(\left(\chi_{\mathbb{R}_{<0}}(m_0)\,U + \chi_{\mathbb{R}_{>0}} \otimes U(0-)\right)\right) + \\ &\quad -\chi_{\mathbb{R}_{>0}} \otimes AU(0-)\,.\end{aligned}$$

In conclusion of this chapter address a modelling aspect resulting from the theory.

5. An Application: Acoustic Waves with Robin Type Boundary Condition

We want to conclude our discussion with a more substantial utilization of the theory presented. Let us consider a (time-translation invariant) linear system operator of the form

$$\partial_0 M\left(\partial_0^{-1}\right) + A$$

with

$$\mathfrak{Re}\,\langle U|AU\rangle_0 \geq 0$$
$$\mathfrak{Re}\,\langle V|A^*V\rangle_0 \geq 0$$

for all $U \in D(A)$, $V \in D(A^*)$. This implies that (6) (7) are satisfied if the material law is of the form

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1} M_1\left(\partial_0^{-1}\right)$$

with M_0 selfadjoint and strictly positive definite.

We have more specifically

$$A = \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}$$

with

$$D(A) := \left\{ \begin{pmatrix} p \\ v \end{pmatrix} \in D\left(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right) \,\middle|\, ap - v \in H\left(\mathring{\operatorname{div}}, \Omega\right) \right\}.$$

Here we assume

$$a : L^2(\Omega) \to \left(L^2(\Omega)\right)^n$$

and

$$\operatorname{div} a : L^2(\Omega) \to L^2(\Omega)$$

and a is multiplicative in the sense that the product rule holds

$$\operatorname{div}(a\,p) = (\operatorname{div} a)\,p + a \cdot \operatorname{grad} p\,.$$

Such an operator combined with a suitable material law yields an evolutionary problem of the stated type. Particular material law operators are in the simplest case of the block diagonal form

$$M\left(\partial_0^{-1}\right) = \begin{pmatrix} \kappa & 0 \\ 0 & q + \varrho^{-1}\,\partial_0^{-1} \end{pmatrix},$$

where $\kappa : L^2(\Omega) \to L^2(\Omega)$ and $q : L^2(\Omega)^3 \to L^2(\Omega)^3$ are suitable continuous, selfadjoint, non-negative mappings and $\varrho^{-1} : L^2(\Omega)^3 \to L^2(\Omega)^3$ is continuous and linear. Such material laws are suggested by models of linear acoustics, see e.g. [3], or by the so-called Maxwell-Cattaneo-Vernotte law [1, 2] describing heat propagation. For our purposes we may allow for more general material laws merely constrained by the above conditions, in particular (8).

In order to confirm conditions (6), (7), we calculate

$$\begin{aligned}
&\mathfrak{Re}\,\langle U|AU\rangle_0 = \\
&= \mathfrak{Re}\left(\langle p|\operatorname{div} v\rangle_0 + \langle \operatorname{grad} p|v\rangle_0\right) \\
&= \mathfrak{Re}\left(\left\langle p|\mathring{\operatorname{div}}\,(v - ap)\right\rangle_0 + \langle p|\operatorname{div} ap\rangle_0 + \langle \operatorname{grad} p|v\rangle_0\right) \\
&= \mathfrak{Re}\left(-\langle \operatorname{grad} p|\,(v - ap)\rangle_0 + \langle p|\operatorname{div} ap\rangle_0 + \langle \operatorname{grad} p|v\rangle_0\right) \\
&= \mathfrak{Re}\left(\langle \operatorname{grad} p|ap\rangle_0 + \langle p|\operatorname{div} ap\rangle_0\right) \\
&= \mathfrak{Re}\left(\langle \operatorname{grad} p|ap\rangle_0 + \langle p|\,(\operatorname{div} a)\,p\rangle_0 + \langle p|a \cdot \operatorname{grad} p\rangle_0\right) + \\
&= 2\,\mathfrak{Re}\int_{\mathbb{R}}\int_{\Omega} \exp\left(-2\varrho m_0\right)\,\operatorname{div}\left(a\,|p|^2\right) \\
&= 2\int_{\mathbb{R}}\int_{\Omega} \exp\left(-2\varrho m_0\right)\,\operatorname{div}\left(\mathfrak{Re}\,a\,|p|^2\right).
\end{aligned}$$

For this to be non-negative we assume that a is such that

$$\int_{\Omega} \operatorname{div}\left(\mathfrak{Re}\,a\,|\varphi|^2\right) \geq 0 \tag{15}$$

for every $\varphi \in H(\operatorname{grad}, \Omega)$. It should be noted that in case of a smooth boundary this amounts to

$$\int_{\partial\Omega} \left(n \cdot \mathfrak{Re}\,a\,|\varphi|^2\right) \geq 0,$$

which holds if

$$n \cdot \mathfrak{Re}\, a \geq 0,$$

where n denotes the exterior normal field to $\partial\Omega$. So, condition (15) generalizes the latter to the case of non-smooth boundaries.

A re-formulating of the constraint (15) is to require that the quadratic functional $Q_{\Omega,a}$ given by

$$p \mapsto \langle \operatorname{grad} p | \left(\mathfrak{Re}\, a\right) p\rangle_0 + \langle \left(\mathfrak{Re}\, a\right) p | \operatorname{grad} p\rangle_0 + \langle \left(\operatorname{div} \mathfrak{Re}\, a\right) p | p\rangle_0$$

is non-negative on $H(\operatorname{grad}, \Omega)$. Note that this functional vanishes on $H\left(\mathring{\operatorname{grad}}, \Omega\right)$ and therefore the non-negativity condition constitutes a boundary constraint[2] on a and on the underlying domain Ω. The implicit constraint on Ω is that the requirement

$$Q_{\Omega,a}\left[H(\operatorname{grad}, \Omega)\right] \subseteq \mathbb{R}_{\geq 0}$$

must be non-trivial, i.e. there must be an a for which this does not hold. For this surely we must have $H\left(\mathring{\operatorname{grad}}, \Omega\right) \neq H(\operatorname{grad}, \Omega)$.

We need to find the adjoint of A, which must be a restriction of

$$-\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}$$

and an extension of

$$-\begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}.$$

[2]A boundary constraint is a non-trivial proposition P on a mathematical object expressed in terms of its interaction with a function space over a domain $\Omega \subseteq \mathbb{R}^{n+1}$, $n \in \mathbb{N}$, which is required to be true and is true for all subspaces of elements with compact support. A boundary condition is a non-trivial proposition imposed on elements u of a function space over a domain $\Omega \subseteq \mathbb{R}^{n+1}$, $n \in \mathbb{N}$, which is also satisfied for $u + \varphi$ for all φ in the function space having compact support in Ω. An example for a boundary constraint for an open set $\Omega \subseteq \mathbb{R}^{n+1}$, $n \in \mathbb{N}$, is: $H\left(\mathring{\operatorname{grad}}, \Omega\right)$ is compactly embedded into $L^2(\Omega)$. This constraint is non-trivial, since there are cases in which this is not true. On the other hand, the property is true for every subspace $\left\{\varphi \in H\left(\mathring{\operatorname{grad}}, \Omega\right) \mid \operatorname{supp}\varphi \subseteq K \subset\subset \Omega\right\}$.

Given $f \in H(\operatorname{grad}, \Omega)$, imposing on $u \in H(\operatorname{grad}, \Omega)$ the requirement $u - f \in H\left(\mathring{\operatorname{grad}}, \Omega\right)$ is a boundary condition if $H\left(\mathring{\operatorname{grad}}, \Omega\right) \neq H(\operatorname{grad}, \Omega)$. Indeed, in this case the proposition $u - f \in H\left(\mathring{\operatorname{grad}}, \Omega\right)$ is non-trivial, since there are elements $u \in H(\operatorname{grad}, \Omega)$ not satisfying the proposition, and obviously $u + \varphi - f \in H\left(\mathring{\operatorname{grad}}, \Omega\right)$ for every $\varphi \in H(\operatorname{grad}, \Omega)$ with compact support in Ω, since such φ is in $H\left(\mathring{\operatorname{grad}}, \Omega\right)$.

We suspect it is

$$D\left(A^{*}\right) := \left\{ \begin{pmatrix} p \\ v \end{pmatrix} \in D\left(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \right) \,\Big|\, \overline{a}p + v \in H\left(\mathring{\operatorname{div}}, \Omega\right) \right\}.$$

Indeed,

$$U \in D\left(A\right)$$

implies

$$\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} U \in H\left(\operatorname{grad}, \Omega\right) \oplus H\left(\mathring{\operatorname{div}}, \Omega\right).$$

$$\begin{aligned} \begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a\left(m\right) & 1 \end{pmatrix} &= \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a\left(m\right) & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -a\left(m\right) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} - \begin{pmatrix} \left(\operatorname{div} a\right)\left(m\right) & 0 \\ 0 & 0 \end{pmatrix} + \\ &\quad - \begin{pmatrix} a\left(m\right) \cdot \operatorname{grad} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -a\left(m\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} - \begin{pmatrix} \left(\operatorname{div} a\right)\left(m\right) & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} U &= \begin{pmatrix} 1 & a\left(m\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a\left(m\right) & 1 \end{pmatrix} U + \\ &\quad + \begin{pmatrix} 1 & a\left(m\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \left(\operatorname{div} a\right)\left(m\right) & 0 \\ 0 & 0 \end{pmatrix} U \\ &= \begin{pmatrix} 1 & a\left(m\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a\left(m\right) & 1 \end{pmatrix} U + \\ &\quad + \begin{pmatrix} \left(\operatorname{div} a\right)\left(m\right) & 0 \\ 0 & 0 \end{pmatrix} U. \end{aligned}$$

Letting $\begin{pmatrix} 1 & 0 \\ -a\left(m\right) & 1 \end{pmatrix} U = W$ we have $V \in D\left(A^{*}\right)$ and for every

$$W \in D\left(\begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix}\right),$$

$$\begin{aligned} 0 &= \left\langle \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a\,(m) & 1 \end{pmatrix} W | V \right\rangle_0 + \left\langle \begin{pmatrix} 1 & 0 \\ a\,(m) & 1 \end{pmatrix} W | \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} V \right\rangle_0, \\ &= \left\langle \begin{pmatrix} 1 & a\,(m) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix} W | V \right\rangle_0 + \\ &\quad + \left\langle \begin{pmatrix} (\operatorname{div} a)\,(m) & 0 \\ 0 & 0 \end{pmatrix} U | V \right\rangle_0 + \left\langle \begin{pmatrix} 1 & 0 \\ a\,(m) & 1 \end{pmatrix} W | \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} V \right\rangle_0, \\ &= \left\langle \begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix} W | \begin{pmatrix} 1 & 0 \\ \overline{a}\,(m) & 1 \end{pmatrix} V \right\rangle_0 + \\ &\quad + \left\langle W | \begin{pmatrix} (\operatorname{div} \overline{a})\,(m) & 0 \\ 0 & 0 \end{pmatrix} V \right\rangle_0 + \left\langle W | \begin{pmatrix} 1 & \overline{a}\,(m) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} V \right\rangle_0. \end{aligned}$$

This implies that

$$\begin{pmatrix} 1 & 0 \\ \overline{a}\,(m) & 1 \end{pmatrix} V \in D\left(\begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix}\right),$$

which is the above characterization and also

$$\begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{a}\,(m) & 1 \end{pmatrix} V = \begin{pmatrix} (\operatorname{div} \overline{a})\,(m) & 0 \\ 0 & 0 \end{pmatrix} V + \begin{pmatrix} 1 & \overline{a}\,(m) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} V.$$

As a consequence of the similarity between A and A^* we find by analogous reasoning that not only

$$\mathfrak{Re}\,\langle U|AU\rangle_0 \geq 0$$

for all $U \in D\,(A)$ but also

$$\mathfrak{Re}\,\langle U|A^*U\rangle_0 \geq 0$$

for all $V \in D\,(A^*)$. Thus we have indeed that

$$\partial_0 M\left(\partial_0^{-1}\right) + A$$

is continuously invertible with inverse $\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)^{-1} : H_{\varrho,0,0}\,(\mathbb{R}, H) \to H_{\varrho,0,0}\,(\mathbb{R}, H)$ for any $\varrho \in \mathbb{R}_{>0}$.

6. Conclusion and Outlook

We have discussed a general class of evolutionary differential equations and discussed a particular application to demonstrate the utility of the framework.

Although, the solution idea appears to be quite powerful as far as providing a unified set-up for various evolutionary problems is concerned, there are many aspects begging for further investigation.

We have noted that the whole class shows time-translation invariance as a characteristic feature. It is a largely open field of investigation to consider systems lacking this property, i.e. so called time-dependent material law operators.

This is particularly important in order to consider non-linear problems. First steps are in preparation to utilize the suggestive extension of the positivity conditions (8) to include monotone operators in the material relation turning the evolutionary equation into a differential inclusion. But there is still ample to do to obtain a satisfactory understanding of a general solution theory.

It should be noted, that in particular with regards to non-linear lower order perturbations, however, there is a simple mechanism to obtain a satisfactory solution theory. In the simplest case, where M_0 is strictly positive definite the constant $c_0 \in \mathbb{R}_{>0}$ is actually a lower bound for the parameter $\varrho \in \mathbb{R}_{>0}$ and can be chosen as large as needed as long as ϱ is sufficiently large. In other words, by choice of the parameter ϱ the solution operator $\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)^{-1}$ has arbitrarily small operator norm. Thus any globally Lipschitz continuous right-hand side $U \mapsto f(U)$ leads to a fixed point problem for a contractive mapping. This basic mechanism is frequently utilizes for numerous non-linear problems if only in a somewhat obscured way, probably finds his most transparent description in the framework presented here.

Since for non-linear problems there is the issue of local-in-time versus global-in-time solutions, it is worth noting that, although, we have presented a global solution theory it is easy to interpret the results as local-in-time. This is due to causality since the solutions do not change up to a point in time if the operator is modified after this point in time. Still, this observation needs to be explored and employed.

A final direction of further investigation we would like to mention is partial differential-algebraic equations (PDAE). Such equations appear in their simplest form already if a material law operator $M\left(\partial_0^{-1}\right)$ is merely a first order polynomial in ∂_0^{-1} and the zero order coefficient M_0 has a non-trivial null space. Such cases have been called (P)-degenerate in [5]. There, however, only a few special cases have been studied.

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