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On Wave Equations with Dissipation II

Hideo Nakazawa

Dedicated to Professor Kiyoshi Mochizuki on his 70th birthday

Abstract. *Our recent results on wave equations with dissipation are surveyed and resolvent estimates for stationary dissipative wave equations in an exterior domain are also proved.*

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1. Introduction

Scattering theory for wave equations with dissipative terms has been established by professor Kiyoshi Mochizuki [7] in 1976, and since then many works have appeared. In this report, we continue our paper [10] and survey some results on dissipative wave equations based on [3] and [5]. We also give a slight extension of our previous result on the principle of limiting absorption [9]. For the background of these problems we refer to [4], for instance.

Firstly, we consider the following wave equation with a dissipative perturbation of rank one:

$$w_{tt}(x,t) - w_{xx}(x,t) + (w_t(\cdot,t), \psi(\cdot))\, \psi(x) = 0, \quad (x,t) \in \boldsymbol{R}^1 \times (0,\infty), \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the usual L^2- inner product, and $\psi = \psi(\cdot) \in L^{2,s}(\boldsymbol{R}^1)$ for some $s > 1/2$ and with the usual weighted L^2-space defined by

$$L^{2,s}(\boldsymbol{R}^1) = \{f\,;\, ||f||_s < \infty\}, \quad ||f||_s^2 = \int_{\boldsymbol{R}^1} (1+|x|^2)^s |f(x)|^2 dx.$$

For this equation, we want to characterize the function ψ for which the solutions behave like free waves. The results are as follows:

Theorem 1.1. *Consider the equation* (1.1).

(1) If $\left| \int_{\boldsymbol{R}^1} \psi(x) dx \right| \leq \sqrt{2}$, *then the solutions become asymptotically free.*

(2) If $\left|\int_{\boldsymbol{R}^1}\psi(x)dx\right| > \sqrt{2}$*, then the total energy of some solutions decays to zero as t tends to infinity.*

Next, we consider the following initial-boundary value problems:

$$\begin{cases} w_{tt}(x,y,t) - \Delta w(x,y,t) + b(x,y)w_t(x,y,t) = 0, & (x,y,t) \in \Omega \times (0,\infty), \\ w(x,y,0) = w_1(x,y), \quad w_t(x,y,0) = w_2(x,y), & (x,y) \in \Omega, \\ w(x,0,t) = w(x,\pi,t) = 0, & (x,t) \in \boldsymbol{R}^N \times (0,\infty), \end{cases} \tag{1.2}$$

where Ω is given by

$$\Omega = \boldsymbol{R}^N \times (0,\pi) = \left\{(x,y);\, x \in \boldsymbol{R}^N,\, 0 < y < \pi\right\}$$

for $N \geq 1$ and where $b(x,y)$ is a measurable function decaying as $|x| \to \infty$. Here the domain Ω is called wave guide or layered domain.

Under these settings, we shall study the behavior of solutions, i.e., the total energy decay and the existence of scattering states. For the function $b(x,y)$ we consider the following two conditions:

- (some kind of long-range condition)

$$(L) \quad : \quad b_1\left[\prod_{k=0}^{m}\log^{[k]}(e_m+r)\right]^{-1} \leq b(x,y) \leq b_2$$

  for some $b_1, b_2 > 0$ and $r = |x|$,

- (some kind of short-range condition)

$$(S) \quad : \quad 0 \leq b(x,y) \leq b_3\left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]^{-1}\left[\log^{[m]}(e_m+r)\right]^{-1-\delta}$$

  for some $0 < \delta \leq 1$, $b_3 > 0$, and $r = |x|$.

Here, m is non-negative integer and

$$e_0 = 1, \quad e_m = e^{e_{m-1}}, \quad \log^{[0]} s = s, \quad \log^{[m]} s = \log\log^{[m-1]} s \quad (m \geq 1).$$

Our second result is given by the following theorem:

Theorem 1.2. *Consider the equation* (1.2).

(1) Under the assumption (L)*, the total energy of solutions decays to zero as t goes to infinity.*

(2) Under the assumption (S)*, the solutions become asymptotically free.*

It would be interesting to note whether the same approach could be used to prove the above results if Ω is changed into wave guides of the form $\mathcal{O} \times \boldsymbol{R}^N$ in the above settings, where $\mathcal{O} \subset \boldsymbol{R}^k$ is a bounded subset for some $k \in \boldsymbol{N}$ with sufficiently smooth boundary. However, it is unsolved at present.

Finally, we show the resolvent estimates for stationary dissipative wave equations in an exterior domain:

$$\begin{cases} -\Delta u(x) - i\kappa b(x)u(x) - \kappa^2 u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is an exterior domain outside the bounded obstacle $\mathcal{O} \subset \boldsymbol{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ and $\kappa = \sigma + i\tau$ with $\tau = \mathrm{Im}\kappa \geq 0$. We assume that $\mathcal{O} = \boldsymbol{R}^N \setminus \Omega$ is star shaped with respect to the origin $x = 0$ and the function $b(x)$ satisfies

$$0 \leq b(x) \leq b_0 \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right]^{-1} \left[\log^{[m]}(e_m + r)\right]^{-1-2\delta} \tag{1.4}$$

for some $b_0 > 0$ and $0 < \delta < 1$.

Theorem 1.3. *Assume that $N \geq 3$ and* (1.4). *If u is a solution of* (1.3), *then there exists a positive constant C independent of κ such that the following inequality holds*:

$$|\kappa| \|u\|_{a_0^{-1}} \leq C\|f\|_{a_0},$$

where

$$||f||_a = ||a^{1/2} f||_{L^2(\Omega)}$$

and

$$a_0(r) = \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right] \left[\log^{[m]}(e_m + r)\right]^{1+\delta}.$$

Note that we do not assume smallness of $b(x)$ because under assumption (1.4) the complex upper-half-plane is contained in the set of $\kappa \in \boldsymbol{C}$ where the bounded inverse of operator pencil $-\Delta - i\kappa b(x) - \kappa^2$ exists. As corollary, we can follow the same line as in [9] to prove the limiting absorption principle for an exterior problem as $\tau = \mathrm{Im}\kappa \downarrow 0$.

Corollary 1.4. *Let $\kappa = \sigma + i\tau \in \boldsymbol{C}_+$. Consider the equation* (1.3). *Then there exists the limit*

$$\lim_{\tau \downarrow 0} u(\sigma, \tau) = \lim_{\tau \downarrow 0} (-\Delta - i\kappa b(x) - \kappa^2)^{-1} f(x) \quad \in \quad L^2_{a_0^{-1}}$$

for $f \in L^2_{a_0}$, where $a_0 = a_0(r)$ is the same as in Theorem 1.3 and

$$L^2_a = \{f \, ; \, ||f||_a < \infty\}.$$

Note that our result also holds for the case $b(x) \equiv 0$ or $\Omega = \boldsymbol{R}^N$ $(N \neq 2)$. So this result is an extension of the well-known resolvent estimate for Helmholtz equations in an exterior domain.

The contents of the present paper will be outlined as follows. In section 2 we shall discuss the problem (1.1) and Theorem 1.1. In section 3 the problem (1.2) is considered and explain Theorem 1.2. In the final section we shall prove Theorem 1.3.

2. Wave equations with dissipative perturbation of rank one

In this section, we consider the scattering problem for a wave equation with a rank one dissipation (1.1) and we deal with this equation as a perturbation of

$$u_{tt}(x,t) - u_{xx}(x,t) = 0, \qquad (x,t) \in \boldsymbol{R}^1 \times (0,\infty). \tag{2.1}$$

The equations (1.1) and (2.1) can be reduced to the ordinary differential equation of Schrödinger type

$$i\frac{dv(t)}{dt} = Hv(t), \quad v(0) \in \mathcal{E} \tag{2.2}$$

in the energy space $\mathcal{E}$ as follows. The energy space $\mathcal{E} = \mathcal{E}(\boldsymbol{R}^1)$ is a Hilbert space associated with energy conservation law, its inner product is given by

$$\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right)_{\mathcal{E}} = \int_{\boldsymbol{R}^1} \left\{ \partial_x f_1(x) \overline{\partial_x g_1(x)} + f_2(x)\overline{g_2(x)} \right\} dx.$$

The norm derived from this is denoted by $||\cdot||_{\mathcal{E}}$. For equation (1.1), the perturbed operator H is defined by

$$H = i \begin{pmatrix} 0 & 1 \\ \partial_x^2 & -(\cdot,\psi)\psi \end{pmatrix}$$

with domain

$$\mathcal{D}(H) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E};\, \partial_x^2 f_1 \in L^2(\boldsymbol{R}^1),\, f_2 \in H^1(\boldsymbol{R}^1) \right\}.$$

Similarly, for the free equation (2.1), unperturbed operator H_0 is defined by

$$H_0 = i \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix}$$

with the same domain $\mathcal{D}(H_0) = \mathcal{D}(H)$. It is well-known that $\sigma(H_0) = \sigma_c(H_0) = \sigma_{ess}(H_0) = \boldsymbol{R}$ and $\sigma_p(H_0) = \sigma_r(H_0) = \emptyset$ hold. Since H is maximal dissipative and H_0 is selfadjoint in $\mathcal{E}$, we find by Reed and Simon [12, Theorem X-50] that H and H_0 generate a contraction semi-group $\{e^{-itH}\}_{t\geq 0}$ and unitary group $\{e^{-itH_0}\}_{t\in \boldsymbol{R}}$, respectively. We want to know more detailed spectrum structure of H under some assumption for the function ψ. To answer this, we assume that the function ψ satisfies

(A1) $\psi \in L^{2,s+1}(\boldsymbol{R}^1)$ with $s > 1/2$,

(A2) $\Psi(\alpha) \leq \Psi(\beta)$ if $0 \leq \beta \leq \alpha$,

where

$$\Psi(\alpha) = |\hat{\psi}(\alpha)|^2 + |\hat{\psi}(-\alpha)|^2.$$

For example, the function $\psi(x) = e^{-|x|^2/2}$ satisfies $(A1)$ and $(A2)$ since $\hat{\psi}(\alpha) = e^{-|\alpha|^2/2}$. These assumptions are the condition to guarantee that the singularity of the resolvent of H is simple.

We are interested in scattering theory and the resolution of modes for dissipative systems (see e.g., [4]). To do so, we need to characterize some singular points of the resolvent and after that, we have to show Parseval formula (Proposition 2.7, below). Firstly, the following can be derived from applying the proof in [7]:

Theorem 2.1. *Under the assumption* $(A1)$ *and* $(A2)$*, the operator* H *does not have real eigenvalues and the wave operator*

$$W = s\text{-}\lim_{t\to+\infty} e^{itH_0}e^{-itH}$$

exists in $\mathcal{E}$ *as a non-trivial operator.*

For $z \in \boldsymbol{C} \setminus \boldsymbol{R}$, we define

$$r_0(z) = \left(\partial_x^2 - z^2\right)^{-1}.$$

For this, the principle of limiting absorption is valid, that is, for any $\lambda \in \boldsymbol{R}$ the limits

$$\lambda^j \partial_x^k r_0(\lambda \pm i0) = \lim_{\kappa\downarrow 0}(\lambda \pm i\kappa)^j \partial_x^k r_0(\lambda \pm i\kappa) \qquad \Big(\text{for } (j,k) = (0,1),(1,0)\Big)$$

exist in the uniform operator topology from $L^{2,s}(\boldsymbol{R}^1)$ to $L^{2,-s}(\boldsymbol{R}^1)$ if $s > 1/2$.

Using this, we find that the free resolvent of H_0 is represented as

$$R_0(z)f = (H_0 - z)^{-1} f = \begin{pmatrix} r_0(z)(zf_1 + if_2) \\ i\partial_x r_0(z)\partial_x f_1 + zr_0(z)f_2 \end{pmatrix} \quad \left(f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E}\right)$$

for $z \in \boldsymbol{C} \setminus \boldsymbol{R}$. To obtain the perturbed resolvent of H, we define functions as

$$\varGamma(z) = 1 - iz\big(r_0(z)\psi, \psi\big), \quad \varGamma(\lambda \pm i0) = 1 - i\lambda\big(r_0(\lambda \pm i0)\psi, \psi\big),$$

where $(\cdot,\cdot)$ is interpreted as dual coupling of $L^{2,s}$ and $L^{2,-s}$. The difficulty is caused by the zero of these functions which is

$$\sum\nolimits_{\pm} = \{z \in \boldsymbol{C}_{\pm}; \Gamma(z) = 0\}, \quad \sum\nolimits_{\pm}^{0} = \{z \in \boldsymbol{R}; \Gamma(\lambda \pm i0) = 0\}.$$

Then it holds that

$$z \notin \sum\nolimits_{+} \ (\text{resp.} \sum\nolimits_{-}) \ \Leftrightarrow \ z \in \rho(H) \cap \boldsymbol{C}_{+} \ (\text{resp.}\ \rho(H) \cap \boldsymbol{C}_{-})$$

and the perturbed resolvent $R(z)$ has the representation

$$R(z)f = R_0(z)f + \frac{i\big(f, v(\overline{z})\big)_{\mathcal{E}}}{\Gamma(z)} v(z) \quad \left(f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E}\right)$$

where

$$v(z) = \begin{pmatrix} ir_0(z)\psi \\ zr_0(z)\psi \end{pmatrix}.$$

Thus we obtain

Proposition 2.2. *Assume (A1) and (A2). Then we have*

(1)

$$\sum\nolimits_{+} = \sum\nolimits_{+}^{0} = \emptyset.$$

(2) There exists some $\kappa_0 < 0$ such that

$$\sum\nolimits_{-} = \begin{cases} \emptyset, & (\Gamma(-i0) \geq 0), \\ \{i\kappa_0\}, & (\Gamma(-i0) < 0), \end{cases}$$

and

$$\sum\nolimits_{-}^{0} = \begin{cases} \emptyset, & (\Gamma(-i0) \neq 0), \\ \{0\}, & (\Gamma(-i0) = 0), \end{cases}$$

where

$$\Gamma(-i0) = 1 - \frac{1}{2}\left|\int_{\boldsymbol{R}^1} \psi(x)dx\right|^2.$$

Moreover, in the case $\Gamma(-i0) < 0$ and $\Gamma(-i0) = 0$, it holds that $\Gamma'(i\kappa_0) \neq 0$ and $\Gamma(-i0) \neq 0$, respectively.

So we find out that the spectral structure depends on the size of ψ as follows.

Theorem 2.3. *Assume (A1) and (A2) for function ψ. Then we have*

$$\sigma(H) \cap \boldsymbol{C}_{-} = \begin{cases} \emptyset & \left(\left|\int_{\boldsymbol{R}^1} \psi(x)dx\right| \leq \sqrt{2}\right), \\ \{i\kappa_0\} & \left(\left|\int_{\boldsymbol{R}^1} \psi(x)dx\right| > \sqrt{2}\right) \end{cases}$$

for some $\kappa_0 < 0$. Moreover, $i\kappa_0$ is an eigenvalue and its multiplicity is one.

For example, the function $\psi(x) = \varepsilon e^{-|x|^2/2}$ satisfies $\left|\int_{\boldsymbol{R}^1} \psi(x)dx\right| = \varepsilon\sqrt{2\pi}$. Thus, only one eigenvalue appears in the case of $\varepsilon > (1/\pi)^2$, while no such eigenvalue exists for $\varepsilon \leq (1/\pi)^2$.

If complex eigenvalue exists, we can define a projection $P_{i\kappa_0}$ with respect to this eigenvalue $i\kappa_0$ as follows.

$$(P_{i\kappa_0} f, g)_{\mathcal{E}} = -\frac{1}{2\pi i}\int_\Gamma (R(z)f, g)_{\mathcal{E}} dz \quad \text{for any} \quad f, g \in \mathcal{E},$$

where $R(z) = (H - z)^{-1}$ is the resolvent of H and $\Gamma(\subset \boldsymbol{C}_-)$ is a closed curve around $i\kappa_0$.

Corollary 2.4. $\mathcal{R}(P) \subset \ker W$.

Now we shall state the construction of a spectral representation for the free (unperturbed) operator H_0.

Proposition 2.5. *For* $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E}_s$ $(s > 1/2)$, *where*

$$\mathcal{E}_s = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} ; \int_{\boldsymbol{R}^1} (1 + |x|^2)^s \left\{|\partial_x f_1(x)|^2 + |f_2(x)|^2\right\} dx < \infty \right\},$$

we define the operator $\mathcal{F}_0$ *by*

$$(\mathcal{F}_0 f)(\lambda) = \begin{cases} \begin{pmatrix} \dfrac{\lambda \hat{f}_1(\lambda) + i\hat{f}_2(\lambda)}{\sqrt{2}} \\ \dfrac{\lambda \hat{f}_1(-\lambda) + i\hat{f}_2(-\lambda)}{\sqrt{2}} \end{pmatrix} & (\lambda > 0), \\ \begin{pmatrix} \dfrac{-\lambda \hat{f}_1(-\lambda) - i\hat{f}_2(-\lambda)}{\sqrt{2}} \\ \dfrac{-\lambda \hat{f}_1(\lambda) - i\hat{f}_2(\lambda)}{\sqrt{2}} \end{pmatrix} & (\lambda < 0). \end{cases}$$

Then

(1) $\mathcal{F}_0$ *is extended to a unitary operator from* $\mathcal{E}$ *onto* $L^2(\boldsymbol{R}; \boldsymbol{C}^2)$.

(2) *For any* $f \in \mathcal{D}(H_0)$ *and* $g \in \mathcal{E}$,

$$(H_0 f, g)_{\mathcal{E}} = \int_{-\infty}^{\infty} \lambda \Big((\mathcal{F}_0 f)(\lambda), (\mathcal{F}_0 g)(\lambda) \Big)_{\boldsymbol{C}^2} d\lambda$$

holds, where $(\cdot, \cdot)_{\boldsymbol{C}^2}$ *denotes the usual inner product in* $\boldsymbol{C}^2$.

We call $\mathcal{F}_0$ the spectral representation for H_0. For the spectral representation of the perturbed operator H we have the following proposition.

PROPOSITION 2.6. *For* $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E}_s$ $(s > 1/2)$, *define the two operators* $\mathcal{F}$ *and* $\mathcal{G}$ *by*

$$(\mathcal{F}f)(\lambda) = (\mathcal{F}_0 f)(\lambda) + \frac{i\big(f, v(\lambda - i0)\big)_{\mathcal{E}}}{\Gamma(\lambda + i0)} \left(\mathcal{F}_0 \begin{pmatrix} 0 \\ \psi \end{pmatrix}\right)(\lambda),$$
$$(\mathcal{G}f)(\lambda) = (\mathcal{F}_0 f)(\lambda) - \frac{i\big(f, v(\lambda - i0)\big)_{\mathcal{E}}}{\overline{\Gamma(\lambda - i0)}} \left(\mathcal{F}_0 \begin{pmatrix} 0 \\ \psi \end{pmatrix}\right)(\lambda).$$

Then $\mathcal{F}$ *is extended to a bounded operator from* $\mathcal{E}$ *to* $L^2(\boldsymbol{R}; \boldsymbol{C}^2)$ *and satisfies* $\mathcal{F} = \mathcal{F}_0 W$. *Moreover, we have*

$$\int_{-\infty}^{\infty} \Big((\mathcal{F}Hf)(\lambda), \tilde{g}(\lambda)\Big)_{\boldsymbol{C}^2} d\lambda = \int_{-\infty}^{\infty} \lambda \Big((\mathcal{F}f)(\lambda), \tilde{g}(\lambda)\Big)_{\boldsymbol{C}^2} d\lambda \qquad (2.3)$$

for any $f \in \mathcal{D}(H)$ *and* $\tilde{g} \in L^2(\boldsymbol{R}; \boldsymbol{C}^2)$.

Therefore, we call the operator $\mathcal{F}$ the spectral representation of H. The operator $\mathcal{G}$ is the formal spectral representation of the adjoint operator H^*. Now we shall state the generalized Parseval formula.

PROPOSITION 2.7. *Assume (A1) and (A2).*

(1) If $\left| \int_{\boldsymbol{R}} \psi(x) dx \right| < \sqrt{2}$, *then*

$$(f, g)_{\mathcal{E}} = \int_{-\infty}^{\infty} \big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\boldsymbol{C}^2} d\lambda$$

for any $f, g \in \mathcal{E}$.

(2) If $\left| \int_{\boldsymbol{R}} \psi(x) dx \right| > \sqrt{2}$, *then*

$$(f, g)_{\mathcal{E}} = \int_{-\infty}^{\infty} \big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\boldsymbol{C}^2} d\lambda + (Pf, g)_{\mathcal{E}}$$

for any $f, g \in \mathcal{E}$.

(3) If $\left| \int_{\boldsymbol{R}} \psi(x) dx \right| = \sqrt{2}$, *then*

$$(f, g)_{\mathcal{E}} = \int_{-\infty}^{\infty} \big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\boldsymbol{C}^2} d\lambda$$

for any $f \in \mathcal{H}, g \in \tilde{\mathcal{E}}$, *where* $\tilde{\mathcal{E}} \subset \mathcal{E}$ *is defined by*

$$\tilde{\mathcal{E}} = \Big\{ g \in \mathcal{S}(\boldsymbol{R}) \times \mathcal{S}(\boldsymbol{R}) \,\Big|\, (v(-i0), g)_{\mathcal{E}} = 0 \Big\}.$$

Remark 2.8. We may consider the above Proposition 2.7 as the spectral decomposition theorem for the dissipative operator H. For instance, in the case $\left| \int_{\boldsymbol{R}} \psi(x)dx \right| > \sqrt{2}$, it holds by (2.3) that

$$\big(Hf, g\big)_{\mathcal{E}} = \int_{-\infty}^{\infty} \lambda\big((\mathcal{F}f)(\lambda), (\mathcal{G}g)(\lambda)\big)_{\boldsymbol{C}^2} d\lambda + i\kappa_0 \big(Pf, g\big)_{\mathcal{E}}$$

for any $f \in \mathcal{D}(H)$ and $g \in \mathcal{E}$.

Theorem 2.9. *Assume (A1) and (A2).*

(1) If $\left| \int_{\boldsymbol{R}} \psi(x)dx \right| \leq \sqrt{2}$, *then* $\ker W = \{0\}$.

(2) If $\left| \int_{\boldsymbol{R}} \psi(x)dx \right| > \sqrt{2}$, *then* $\ker W = \mathcal{R}(P)$.

Proof. (1) Consider the case $\left| \int_{\boldsymbol{R}} \psi(x)dx \right| < \sqrt{2}$. By Corollary 2.4, we may show $Wf = 0$ implies $f = 0$. Since $\mathcal{F}_0$ is unitary in $\mathcal{E}$, we have $\mathcal{F}f = 0$ by $\mathcal{F} = \mathcal{F}_0 W$ in Proposition 2.6. By Proposition 2.7 (1), we have $(f, g)_{\mathcal{E}} = 0$ for any $g \in \mathcal{E}$, hence we obtain the desired result. In the case $\left| \int_{\boldsymbol{R}} \psi(x)dx \right| = \sqrt{2}$, similar arguments show that $Wf = 0$ implies $(f, g)_{\mathcal{E}} = 0$ by Proposition 2.7 (3) since the space $\tilde{\mathcal{E}}$ is dense in $\mathcal{E}$. This shows $f = 0$.
(2) In the same way, if $\left| \int_{\boldsymbol{R}} \psi(x)dx \right| > \sqrt{2}$, we find $Wf = 0$ implies $\mathcal{F}f = 0$. Hence $(f, g)_{\mathcal{E}} = (Pf, g)_{\mathcal{E}}$ by Proposition 2.7 (2), from which the desired result follows. □

From the last theorem, we easily obtain the statements of Theorem 1.1.

3. Decay and scattering for wave equations with dissipations in layered media

In this section we shall describe a result on wave equations with dissipation in some layered media [5]. Consider the initial-boundary value problem (1.2) and assume conditions (L) or (S). Under these assumptions, we consider (1.2) as perturbation of the free equation:

$$\begin{cases} u_{tt}(x,y,t) - \Delta u(x,y,t) = 0, & (x,y,t) \in \Omega \times \boldsymbol{R}, \\ u(x,0,t) = u(x,\pi,t) = 0, & (x,t) \in \boldsymbol{R}^N \times \boldsymbol{R}. \end{cases} \tag{3.1}$$

Then (1.2) and (3.1) can be reduced to the ordinary differential equation (2.2) in the energy space $\mathcal{E}$, where $\mathcal{E} = H_0^1(\Omega) \times L^2(\Omega)$. Now we define two operators H and H_0 by

$$H = H_0 + V, \tag{3.2}$$

$$H_0 = i\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \qquad V = i\begin{pmatrix} 0 & 0 \\ 0 & -b(x,y) \end{pmatrix} \tag{3.3}$$

with domain

$$\mathcal{D}(H) = \mathcal{D}(H_0) = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{E}; \Delta f_1 \in L^2(\Omega), f_2 \in H_0^1(\Omega) \right\}.$$

Then the operators H and H_0 are maximal dissipative and selfadjoint in $\mathcal{E}$, respectively, therefore, H and H_0 generate a contraction semi-group $\{e^{-itH}\}_{t\geq 0}$ and unitary group $\{e^{-itH_0}\}_{t\in\boldsymbol{R}}$, respectively.

Now we shall explain the difficulties of our problem. To do so, we define a selfadjoint operator L_0 in $L^2(\Omega)$ by

$$L_0 u = -\Delta u, \quad \mathcal{D}(L_0) = \left\{ u \in H_0^1(\Omega); \Delta u \in L^2(\Omega) \right\}.$$

For $z \notin \boldsymbol{R}$, we define its resolvent by $R_0(z) = (L_0 - z^2)^{-1}$. Then this has the following integral representation:

$$(R_0(z)\varphi)(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \sin ny \int_0^{\pi} \sin ny' (r_n(z)\varphi)(x,y')dy'$$

for $\varphi \in C_0^\infty(\Omega)$, where $r_n(z) = \left(-\Delta_x - (z^2 - n^2)\right)^{-1}$ with $\Delta_x = \sum_{j=1}^N \partial^2/\partial x_j^2$.

By this, we find $\sigma(L_0) = \sigma_c(L_0) = \cup_{n=1}^\infty [n^2, \infty) = [1,\infty)$. The operator $r_n(z)$ (therefore $R_0(z)$) has a singularity at $z^2 = n^2$. The end point n^2 of each interval $[n^2,\infty)$ is called thresholds of the operator L_0. Then the free solution of (3.1) is represented by

$$u(x,y,t) = \sum_{n=1}^{\infty} u_n(x,t) \sin ny,$$

where $u_n(x,t)$ is the solution of

$$u_{ntt}(x,t) - \Delta_x u_n(x,t) + n^2 u_n(x,t) = 0, \quad (x,t) \in \boldsymbol{R}^N \times \boldsymbol{R}.$$

In the above settings, the operator $\sqrt{iV}$ is in general not H_0-smooth near the threshold $\pm k$, where $k \in \boldsymbol{N}$. To relax these singularities we choose the operator $\sqrt{iV}(H_0 - i)^{-2}(H_0 \pm k)$ as smooth operators near the thresholds $\pm k$. In addition, we use the operator $\prod_{k=1}^{n}(H - i)^{-4}(H^2 - k^2)$ instead of Simon's approximate operator $(H - i)^{-2}H$ [13] and we employ the density argument using some approximate operators.

Now we shall start with Theorem 1.2 (1) on energy decay. As for the proof, we have only to show the following

Proposition 3.1. *Assume (L) for fixed m and the initial data $w_0 = (w_1, w_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Let ε be a positive number satisfying $0 < \varepsilon \le \min\{1, b_1/2\}$. Then*

$$\|e^{-itH}w_0\|_{\mathcal{E}}^2 \le C_1 \left\{\log^{[m]}(e_m + t)\right\}^{-\varepsilon}$$

holds for some positive constant $C_1 = C_1(w_1, w_2, b_1, b_2, \varepsilon) > 0$.

We omit the proof of the above proposition since it follows from the same arguments as in [8].

On the other hand, to show Theorem 1.2 (2) on scattering, we need some abstract results. Let $\mathcal{H}$ be a separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $||\cdot||$, and let iH be a maximal dissipative operator in $\mathcal{H}$, so that $U(t) = e^{-itH}$ $(t \ge 0)$ is the contraction semi-group generated by iH. Finally, let $\mathcal{H}_b$ be the space generated by the eigenvectors of H with real eigenvalues. The next lemma follows from a slight modification of the proof by Petkov [11, Lemma 1.1.6].

Lemma 3.2. *For a finite sequence $\{\alpha_n\} \subset \boldsymbol{R}$, the following two sets Ψ_n $(n = 0, 1)$ are dense in $\mathcal{H}_b^\perp$, the orthogonal complement of $\mathcal{H}_b$.*

$$\begin{aligned}
\Psi_0 &= \left\{\prod_{k=1}^{n}(H - \alpha_k)\left\{(H - i)^{-1}\right\}^2 f; f \in \mathcal{D}(H) \cap \mathcal{H}_b^\perp\right\}, \\
\Psi_1 &= \left\{\prod_{k=1}^{n}(H - \alpha_k)(H - i)^{-1} f; f \in \mathcal{H}_b^\perp\right\}.
\end{aligned}$$

The results above are needed to prove Theorem 3.3 below. Especially, denseness of Ψ_1 in $\mathcal{H}_b^\perp$ means that the weak limit of $U(t)f$ as t goes to infinity vanishes. By this fact, the non-existence of real eigenvalues of H is shown by the similar argument in Kadowaki [2].

Let H_0 and V be selfadjoint operators in $\mathcal{H}$ and $\{U_0(t)\}_{t\in\boldsymbol{R}}$ be the unitary group generated by H_0. Let $E(\lambda)$ be the spectral family of H_0. Put

$$F_n(\lambda) = \left\{(\lambda - a_n)(\lambda - i)^{-1}\right\}\left\{(\lambda + a_n)(\lambda - i)^{-1}\right\} = (\lambda^2 - a_n^2)(\lambda - i)^{-2}.$$

Assume the following three conditions:

(A1) $\sigma(H_0) = \sigma_{ac}(H_0) = (-\infty, -m] \cup [m, \infty)$ for some $m \geq 0$,

(A2) iV is non-negative and H_0-compact,

(A3) there exists a sequence $\{a_n\}$ satisfying $m < a_1 < a_2 < \cdots < a_n < \cdots$, $\lim_{n\to\infty} a_n = \infty$ such that the operator $\sqrt{iV} F_n(H_0) E_{a_n, a_{n+1}}(H_0)$ is H_0-smooth.

Here the operator K is called H_0-smooth (Kato [6]) if there exists a positive constant C such that

$$\int_{\boldsymbol{R}} ||KU_0(t)||^2 dt \leq C||f||^2$$

and we put $E_{\alpha,\beta}(A_0) = E\left((-\beta, -\alpha) \cup (\alpha, \beta)\right)$ for $0 < \alpha < \beta$.

To prove Theorem 1.2 (2), we need the following

Theorem 3.3. *Assume (A1), (A2) and (A3) and put $H = H_0 + V$. Then the operator H does not have real eigenvalues and the wave operator*

$$W = s\text{-}\lim_{t\to+\infty} U_0(-t)U(t)$$

exists in $\mathcal{H}$ as a non-trivial operator.

This theorem is proven by the method similar to the arguments in Kadowaki [2] (see the description just behind Lemma 3.2).

Now we give the outline of the proof of Theorem 1.2 (2). We only have to show that the two operators H and H_0 defined by (3.2) and (3.3) satisfy $(A1)$ and $(A3)$, since assumption $(A2)$ follows from Rellich's theorem. Let us start with following results on smooth operators which follow from the well-known resolvent estimate for $-\Delta_x$ by Agmon [1] and the inclusion relations on weighted L^2-space having weights like Y_m in Lemma 3.4 below and Besov spaces [10: Corollary 3.1].

Lemma 3.4. *Let $n \in \boldsymbol{N}$ and $m \in \{0\} \cup \boldsymbol{N}$. Then for every $\lambda \in (-\infty, -n) \cup (n, \infty)$, there exist the limits*

$$(r_n(\lambda \pm i0)Y_m u, Y_m v) = \lim_{\mathrm{Im} z\to\pm 0} (r_n(z)Y_m u, Y_m v), \tag{3.4}$$

where

$$Y_m = \left[\prod_{k=0}^{m} \log^{[k]}(e_m + |x|)\right]^{-1/2} \left[\log^{[m]}(e_m + |x|)\right]^{-\delta/2}$$

and $u, v \in L^2(\boldsymbol{R}^N)$. Moreover, there exists a positive constant C independent of z such that

$$||Y_m r_n(z) Y_m f|| \leq C|z^2 - n^2|^{-1/2}||f|| \tag{3.5}$$

for z satisfying $Re(z^2 - n^2) > 0$.

Using this, we obtain

LEMMA 3.5. *Let $n \in \boldsymbol{N}$ and $m \in \{0\} \cup \boldsymbol{N}$. Assume that $z = \lambda + i\kappa$ is a complex number with $\lambda \in (-n-1, -n) \cup (n, n+1)$. Then there exists a positive constant C independent of z such that*

$$||Y_m R_0(z) Y_m f||^2 \leq C \left(|\lambda^2 - n^2|^{-1} + |\lambda^2 - (n+1)^2|^{-1} \right) ||f||^2.$$

Noting this lemma, we have

PROPOSITION 3.6. *For the operator H_0 defined by (3.3), assumption $(A1)$ with $m = 1$ is satisfied:*

$$\sigma(H_0) = \sigma_{ac}(H_0) = (-\infty, -1] \cup [1, \infty).$$

To show that the operator H_0 satisfies $(A3)$, we need

PROPOSITION 3.7. *Let $n \in \boldsymbol{N}$, $\varepsilon \in (0,1)$ and $f \in \mathcal{E}$.*

(1) For any $\lambda \in (-n-\varepsilon, -n) \cup (n, n+\varepsilon)$, there exists a positive constant C_1 independent of λ such that

$$\frac{d}{d\lambda} ||E(\lambda)\sqrt{iV} f||^2 \leq C_1 (\lambda^2 - n^2)^{-1/2} ||f||^2.$$

(2) For any $\lambda \in (-n-1, -n-\varepsilon) \cup (n+\varepsilon, n+1)$, there exists a positive constant C_2 independent of λ such that

$$\frac{d}{d\lambda} ||E(\lambda)\sqrt{iV} f||^2 \leq C_2 ||f||^2.$$

[Proof of (A3)] By the above proposition, we may put $a_n = n$. Now we choose $\varepsilon \in (0,1)$ and put $z = \mu + i\kappa \in \boldsymbol{C}_+$ and we define the operator $K_\varepsilon(n)$ by

$$K_\varepsilon(n) = \sqrt{iV} F_n(H_0) E_{n,n+\varepsilon}(H_0).$$

Noting the following formulas

$$\begin{aligned}
&\operatorname{Im} \left((H_0 - z)^{-1} K_\varepsilon(n)^* f, K_\varepsilon(n)^* f \right) \\
&= \left(\int_{-n-\varepsilon}^{-n} + \int_n^{n+\varepsilon} \right) \\
&\qquad \times \frac{\kappa}{(\lambda - \mu)^2 + \kappa^2} |(\lambda - i)^{-2} (\lambda^2 - n^2)|^2 \frac{d}{d\lambda} ||E(\lambda)\sqrt{iV} f||^2 d\lambda, \\
&\int_{-\infty}^{+\infty} \frac{\kappa}{(\lambda - \mu)^2 + \kappa^2} d\lambda = \pi,
\end{aligned}$$

we have

$$\sup_{\mathrm{Im} z\neq 0,\, f\in\mathcal{E}} \left|\mathrm{Im}\left((H_0-z)^{-1}K_\varepsilon(n)^*f, K_\varepsilon(n)^*f\right)\right| \le C||f||^2$$

for some positive constant C. So, the operator $K_\varepsilon(n)$ becomes H_0-smooth. Similarly with this, we can show that the operators $\sqrt{iV}E_{n,n+\varepsilon}(H_0)$, therefore $\sqrt{iV}F_n(H_0)E_{n+\varepsilon,n+1}(H_0)$ also become H_0-smooth. Using the fact $E_{\pm(n+\varepsilon)}=0$ $(\sigma_p(H_0)=\emptyset)$, we conclude that (A3) holds. □

4. Resolvent estimate for stationary dissipative wave equations in an exterior domain

In the final section, we shall state some revised result of the resolvent estimate in [9]. Let us consider the stationary problem (1.3) under the condition (1.4).

Similarly to [9], we define two operators $\mathcal{D}^+$ and $\mathcal{D}^+_r$ as follows:

$$\begin{aligned}\mathcal{D}^+u &= \nabla u+\frac{N-1}{2r}\frac{x}{r}u-i\kappa\frac{x}{r}u \quad (\mathrm{Im}\kappa\ge 0)\,,\\ \mathcal{D}^+_r u &= \frac{x}{r}\cdot\mathcal{D}^+u=u_r+\frac{N-1}{2r}u-i\kappa u \quad (\mathrm{Im}\kappa\ge 0)\,.\end{aligned}$$

Proposition 4.1. *If u is a solution of* (1.3), *then the following inequality holds*:

$$\begin{aligned}&\int_\Omega \left[\log^{[m]}(e_m+r)\right]^{-1-\delta}\left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]^{-1}\\ &\qquad\times\left\{|\kappa|^2|u|^2+\left|u_r+\frac{N-1}{2r}u\right|^2\right\}dx \qquad (4.1)\\ &\le \int_\Omega \left[\log^{[m]}(e_m+r)\right]^{-1-\delta}\left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]^{-1}|\mathcal{D}^+_r u|^2dx\\ &\qquad+\frac{2}{\delta}\int_\Omega\left[\log^{[m]}(e_m+r)\right]^{-\delta}|f\overline{i\kappa u}|dx.\end{aligned}$$

Proof. (cf.[9: Lemma 3.1, 3.2]) Let g be the function defined by

$$g(r)=\left[\log^{[m]}(e_m+r)\right]^{-\delta}.$$

Multiplying both sides of the first equation of (1.3) by $g\overline{i\kappa u}$, we have

$$-\nabla\cdot(g\overline{i\kappa u}\nabla u)+g_r\overline{i\kappa u}u_r+\overline{i\kappa}g|\nabla u|^2-|\kappa|^2gb(x)|u|^2+i\kappa|\kappa|^2g|u|^2=gf\overline{i\kappa u}. \quad (4.2)$$

Integrating both sides of (4.2) by parts on Ω, we have

$$\int_\Omega g_r \overline{i\kappa u} u_r dx + \int_\Omega \left\{ \overline{i\kappa} g |\nabla u|^2 - |\kappa|^2 g b(x) |u|^2 + i\kappa |\kappa|^2 g |u|^2 \right\} dx$$
$$= \int_\Omega g f \overline{i\kappa u} dx. \tag{4.3}$$

Note that the identity

$$\mathrm{Re}(-\overline{i\kappa u} u_r) = \frac{1}{2}|\mathcal{D}_r^+ u|^2 - \frac{1}{2}\left(\left| u_r + \frac{N-1}{2r} u \right|^2 + |\kappa|^2 |u|^2 \right) - \frac{\tau(N-1)}{2r}|u|^2,$$

where Re means the real part and $\tau = \mathrm{Im}\kappa$. Taking the real part of both sides of (4.3) and using the assumption $b(x) \geq 0$ and $\tau \geq 0$, we obtain the desired inequality since

$$g_r(r) = -\delta \left[\log^{[m]}(e_m + r) \right]^{-1-\delta} \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r) \right]^{-1}$$

holds. □

Lemma 4.2. *Let $\varphi = \varphi(r)$ be a non-negative real-valued C^1-function. If u is a solution of* (1.3)*, then the following inequality holds*:

$$\int_\Omega \left\{ \left(\tau\varphi + \frac{\varphi_r}{2} \right) |\mathcal{D}^+ u|^2 + \left(\frac{\varphi}{r} - \varphi_r \right) \left(|\mathcal{D}^+ u|^2 - |\mathcal{D}_r^+ u|^2 \right) \right\} dx$$
$$+ \int_\Omega \frac{b(x)\varphi}{2} \left(|\mathcal{D}_r^+ u|^2 + |\kappa|^2 |u|^2 - \left| u_r + \frac{N-1}{2r} u \right|^2 \right) dx$$
$$+ \int_\Omega \left\{ -\frac{(N-1)(N-3)}{8} \left(\frac{\varphi}{r^2} \right)_r |u|^2 + \frac{\tau(N-1)(N-3)}{4r^2} \varphi |u|^2 \right\} dx$$
$$\leq \int_\Omega \left| f \varphi \overline{\mathcal{D}_r^+ u} \right| dx.$$

Proof. We can apply an almost similar argument as in [9: Lemma 3.3]. The only difference is that the boundary integral appears. But noting the boundary condition $u = 0$ on $\partial\Omega$, we have

$$\mathcal{D}_n^+ u = \left(\mathcal{D}^+ u \cdot n \right) = u_n, \quad \mathcal{D}_r^+ u = u_r = \left(\frac{x}{r} \cdot n \right) u_n, \quad \mathcal{D}^+ u = \nabla u = u_n n,$$

where n is unit outer normal to the boundary $\partial\Omega$. The boundary integral is non-negative (see (3.11) in [9]) :

$$\int_{\partial\Omega} \varphi \left\{ \mathcal{D}_n^+ u \overline{\mathcal{D}_r^+ u} - \frac{|\mathcal{D}^+ u|^2}{2} \left(\frac{x}{r} \cdot n \right) \right\} dS = -\frac{1}{2} \int_{\partial\Omega} \varphi |u_n|^2 \left(\frac{x}{r} \cdot n \right) dS \geq 0$$

since $\left(\frac{x}{r} \cdot n\right) \leq 0$ holds by the assumption that the obstacle $\mathcal{O}$ is star shaped with respect to the origin. Thus we have the desired inequality. $\square$

By this lemma, we obtain

PROPOSITION 4.3. *If u is a solution of* (1.3), *then the following inequality holds*:

$$\begin{aligned}\int_\Omega \left[\log^{[m]}(e_m + r)\right]^{-1+\delta} \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right]^{-1} |\mathcal{D}^+ u|^2 \\ -\delta \int_\Omega b(x) \left[\log^{[m]}(e_m + r)\right]^\delta \left|u_r + \frac{N-1}{2r} u\right|^2 dx \\ \leq \frac{2}{\delta} \int_\Omega \left[\log^{[m]}(e_m + r)\right]^\delta \left|f\overline{\mathcal{D}_r^+ u}\right| dx.\end{aligned} \tag{4.4}$$

Proof. Put

$$\varphi = \varphi(r) = \left[\log^{[m]}(e_m + r)\right]^\delta$$

in Lemma 4.2. Then since

$$\varphi_r = \delta \left[\log^{[m]}(e_m + r)\right]^{-1+\delta} \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right]^{-1},$$

we have $\frac{\varphi}{r} - \varphi_r \geq 0$ and $\left(\frac{\varphi}{r^2}\right)_r \leq 0$. Noting these relations, $N \neq 2$, $b(x) \geq 0$ and $|\mathcal{D}_r^\pm u| \leq |\mathcal{D}^\pm u|$ we obtain the desired result. $\square$

THEOREM 4.4. *If u is a solution of* (1.3), *then there exist positive constants C_1, C_2 and C_3 independent of κ such that the following inequalities hold*:

$$|\kappa| \|u\|_{a_0^{-1}} \leq C_1 \|f\|_{a_0}, \tag{4.5}$$

$$\|\mathcal{D}^+ u\|_{a_1} \leq C_2 \|f\|_{a_0}, \tag{4.6}$$

$$\left\|u_r + \frac{N-1}{2r} u\right\|_{a_0^{-1}} \leq C_3 ||f||_{a_0}, \tag{4.7}$$

where

$$||f||_a = ||a^{1/2} f||_{L^2(\Omega)}$$

and

$$\begin{aligned} a_0(r) &= \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right] \left[\log^{[m]}(e_m + r)\right]^{1+\delta}, \\ a_1(r) &= \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right]^{-1} \left[\log^{[m]}(e_m + r)\right]^{-1+\delta}. \end{aligned}$$

Proof. We may assume that

$$0 < \delta < \min\{1, 1/b_0\}.$$

Then, we can choose a positive constant φ_0 as

$$1 < \varphi_0 < \frac{1}{b_0\delta}. \tag{4.8}$$

Since $\varphi_0 > 1$, we can choose $\varepsilon > 0$ as

$$0 < \varepsilon < \min\{\varphi_0 - 1, 1\}. \tag{4.9}$$

Now adding (4.1) to (4.4) multiplied by φ_0, we obtain

$$\begin{aligned}
&|\kappa|^2 \int_\Omega \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right]^{-1} \left[\log^{[m]}(e_m + r)\right]^{-1-\delta} |u|^2 dx \\
&\quad + \int_\Omega \left(\left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right]^{-1} \left[\log^{[m]}(e_m + r)\right]^{-1-\delta} \right. \\
&\qquad \left. -\varphi_0 \delta b(x) \left[\log^{[m]}(e_m + r)\right]^{\delta} \right) \left| u_r + \frac{N-1}{2r} u \right|^2 dx \qquad (4.10) \\
&\quad + \int_\Omega \left[\prod_{k=0}^{m} \log^{[k]}(e_m + r)\right]^{-1} \left(\varphi_0 \left[\log^{[m]}(e_m + r)\right]^{\delta} \right. \\
&\qquad \left. - \left[\log^{[m]}(e_m + r)\right]^{-\delta} \right) \left|\mathcal{D}^+ u\right|^2 dx \\
&\le \frac{2}{\delta} \int_\Omega \left[\log^{[m]}(e_m + r)\right]^{-\delta} |f\overline{i\kappa u}| dx + \frac{2\varphi_0}{\delta} \int_\Omega \left[\log^{[m]}(e_m + r)\right]^{\delta} |f\overline{\mathcal{D}_r^+ u}| dx.
\end{aligned}$$

Note that by the Schwarz inequality, we have the following two inequalities:

$$\begin{aligned}
&\frac{2}{\delta} \int_\Omega \left[\log^{[m]}(e_m + r)\right]^{-\delta} |f\overline{i\kappa u}| dx \\
&\le \frac{1}{\varepsilon\delta^2} \int_\Omega \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right] \left[\log^{[m]}(e_m + r)\right]^{1+\delta} |f|^2 dx \qquad (4.11) \\
&\quad + \varepsilon|\kappa|^2 \int_\Omega \left[\prod_{k=0}^{m-1} \log^{[k]}(e_m + r)\right]^{-1} \left[\log^{[m]}(e_m + r)\right]^{-1-\delta} |u|^2 dx,
\end{aligned}$$

$$\begin{aligned}
&\frac{2\varphi_0}{\delta}\int_\Omega \left[\log^{[m]}(e_m+r)\right]^\delta |f\overline{\mathcal{D}_r^+ u}|dx \\
&\le \frac{\varphi_0^2}{\varepsilon\delta^2}\int_\Omega \left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]\left[\log^{[m]}(e_m+r)\right]^{1+\delta}|f|^2dx \qquad (4.12)\\
&\quad +\varepsilon\int_\Omega \left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]^{-1}\left[\log^{[m]}(e_m+r)\right]^{-1-\delta}|\mathcal{D}^+u|^2dx.
\end{aligned}$$

Then (4.10), (4.11) and (4.12) give

$$\begin{aligned}
&(1-\varepsilon)|\kappa|^2\int_\Omega \left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]^{-1}\left[\log^{[m]}(e_m+r)\right]^{-1-\delta}|u|^2dx \\
&\quad +\int_\Omega\left(\left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]^{-1}\left[\log^{[m]}(e_m+r)\right]^{-1-\delta}\right. \\
&\qquad \left. -\varphi_0\delta b(x)\left[\log^{[m]}(e_m+r)\right]^\delta\right)\left|u_r+\frac{N-1}{2r}u\right|^2dx \qquad (4.13)\\
&\quad +\int_\Omega \left[\prod_{k=0}^{m}\log^{[k]}(e_m+r)\right]^{-1}\left\{(\varphi_0-\varepsilon)\left[\log^{[m]}(e_m+r)\right]^\delta\right. \\
&\qquad \left. -\left[\log^{[m]}(e_m+r)\right]^{-\delta}\right\}\left|\mathcal{D}^+u\right|^2dx \\
&\le \frac{1+\varphi_0^2}{\varepsilon\delta^2}\int_\Omega \left[\prod_{k=0}^{m-1}\log^{[k]}(e_m+r)\right]\left[\log^{[m]}(e_m+r)\right]^{1+\delta}|f|^2dx.
\end{aligned}$$

Using (1.4), (4.8) and (4.9) for (4.13), we have the desired inequalities (4.5), (4.6) and (4.7). □

REMARK 4.5. If we assume the smallness for the function $b(x)$, then we can show the same results as in Theorem 4.4 and Corollary 1.4 in the case $\kappa \in \boldsymbol{C}_-$. Therefore, we can obtain the principle of limiting absorption for the matrix type operator $H = i\begin{pmatrix} 0 & 1 \\ \Delta & -b(x)\end{pmatrix}$ (see [9: Theorem 5.1]).

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Author's address:

Hideo Nakazawa
Department of Mathematics
Chiba Institute of Technology

2-1-1, Shibazono, Narashino, Chiba 275-0023, Japan
E-mail: nakazawa.hideo@it-chiba.ac.jp

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