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The Kirchhoff Equation with Global Solutions in Unbounded Domains

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Abstract. *The aim of this paper is to find a general class of data in which the global well-posedness for the initial-boundary value problem to the Kirchhoff equation in unbounded domains is assured. The result obtained in the present paper will be applied to the existence of scattering operators. Some examples of function spaces contained in this class will be presented.*

Keywords: Kirchhoff equation; unbounded domains; generalized Fourier transform.
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1. Introduction

This paper is devoted to the investigation of several aspects of nonlinear hyperbolic equations, in particular, the global existence in time of solutions to the Kirchhoff equation in unbounded domains. The Fourier transform is the powerful tool to understand the behaviour of solutions to the Cauchy problem, such as the dispersive estimates, Strichartz estimates and scattering theory. If one shifts these problems to the initial-boundary value problem in unbounded domains, one would be confronted with some difficulties. To overcome these hurdles, we can use the generalized Fourier transform. It is one of good tools; it will provide some information on the illumination of these kinds of problems. For example, the Strichartz estimates for wave equation with a potential are established by using this transform (see [19]). There, the kernel of propagator can be analyzed well. Referring to this idea, we can also prove the existence of global solutions to the Kirchhoff equation in an exterior domain (see Theorem 1.4 from [18]). The main point is to prove the decay estimates for some oscillatory integrals associated to the Kirchhoff equation. To prove this theorem, we rely on the existence theorem established in Theorem 1.1 from [18], where the data belong to a class described by oscillatory integrals. The aim in this paper is to give a wider class of the initial data than [18], and to provide an unified treatment of the Kirchhoff equation in unbounded domains.

Let Ω be an unbounded domain of $\mathbb{R}^n$ $(n \geq 1)$ such that its boundary $\partial\Omega$ is of C^∞. We consider the initial-boundary value problem to the Kirchhoff equation, for a function $u=u(t,x)$:

$$\partial_t^2 u - \left(1+\int_\Omega |\nabla u|^2\,dx\right)\Delta u = 0, \quad t\neq 0, \quad x\in\Omega, \tag{1}$$

with the data

$$u(0,x)=f_0(x), \quad \partial_t u(0,x)=f_1(x), \quad x\in\Omega, \tag{2}$$

and the boundary condition

$$u(t,x)=0, \quad t\in\mathbb{R}, \quad x\in\partial\Omega. \tag{3}$$

As is well known, the Kirchhoff equation was proposed by Kirchhoff in 1883, as a model of the vibrating string with fixed ends. There are extensive works on the global well-posedness (see [21, 22] for bounded domains, and [2, 3, 4, 5, 6, 7, 10, 14, 13, 15, 16, 24, 27] for the Cauchy problem in $\mathbb{R}^n$). In the results of [3, 4, 5, 6, 7, 10, 14, 24] on the global existence to the Cauchy problem, the polynomial weight conditions are imposed on small data. On the other hand, Yamazaki found a more general class of data which ensures the global existence (see [27]), and Kajitani succeeded to obtain global solutions with small data in a wider class than the Yamazaki class (see [13]). Let us explain briefly the relation among the classes of Yamazaki and Kajitani. The Yamazaki class consists of a pair of functions such that a certain oscillatory integral decays, and the Kajitani class is replaced by the integrability condition. This means that the Yamazaki class is a subclass of the Kajitani one.

As for the initial-boundary value problem in exterior domains, we should refer to the works [11, 18, 23, 28, 29]. When the supports of the generalized Fourier transform of the initial data are away from the origin, Racke proved the global existence of the Kirchhoff equation in an arbitrary exterior domain under the assumption that the supports of the generalized Fourier transform of data are compact (see [23]). After him, Heiming improved his result (see [11]); she removed the upper bound of the generalized Fourier transform of data if $\mathbb{R}^n\setminus\Omega$ is star-shaped. From a different viewpoint, Yamazaki found the new class of data to ensure the global existence, which is the Sobolev spaces without any weight (see [28, 29]). Recently, the present author obtained two remarkable results (see [18]); the first one is that the Yamazaki class is made in an arbitrary exterior domain, or even in unbounded domains (see Theorem 1.1 from [18]), and the second one removes the additional assumption in [11, 23] that the supports of the generalized Fourier transforms of data are away from the origin (see Theorem 1.4 from [18]). Thus, in this paper we will extend the Yamazaki class of Theorem 1.1 in [18] to the Kajitani class in unbounded domains (see Theorem 2.1 below). Thanks to this class, we can develop the scattering for the Kirchhoff equation (see [9, 14, 27] for the Cauchy problem, cf. [17]). The existence of scattering states is obtained in Theorem 1.5 from [18], whereas the existence of wave operators will be discussed elsewhere. In the final part of this paper we will review the space in [18, Theroem 1.4] that is contained in the class of Theorem 2.1.

2. Results

We shall introduce several notations in order to state the results. For a non-negative integer m and real number κ, we define the weighted Sobolev space over a domain G of $\mathbb{R}^n$:

$$H^m_\kappa(G) = \{ f : \langle x\rangle^\kappa \partial^\alpha_x f \in L^2(G),\ |\alpha| \le m \},$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$ and we put $L^2_\kappa(G) = H^0_\kappa(G)$. We define also the weighted Sobolev space $H^\sigma_\kappa(G)$ of fractional order $\sigma \ge 0$ by the complex interpolation method:

$$H^\sigma_\kappa(G) = \left[L^2_\kappa(G), H^m_\kappa(G)\right]_\theta, \quad \sigma \le m, \quad \sigma = \theta m \quad \text{with } 0 \le \theta \le 1,$$

where m is an integer. $H^\sigma(G)$ (or even $H^1_0(G)$) is the usual Sobolev space of order σ over G. Let A be a self-adjoint realization of $-\Delta$ on $L^2(\Omega)$ with the Dirichlet boundary condition in the unbounded domain Ω, i.e.,

$$\begin{cases} \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega), \\ Au = -\Delta u, \quad u \in C^\infty_0(\Omega). \end{cases} \tag{4}$$

Since A is the non-negative self-adjoint operator on $L^2(\Omega)$, we can define the square root $A^{1/2}$ of A. In what follows, we put $H = A^{1/2}$.

In order to state the global well-posedness for the problem (1)–(3), we introduce a class $Y(\Omega)$:

$$Y(\Omega) := \left\{(f,g) \in (H^{3/2}(\Omega)\cap H^1_0(\Omega)) \times H^{1/2}(\Omega) \,:\, |(f,g)|_{Y(\Omega)} < +\infty\right\},$$

with

$$\begin{aligned} |(f,g)|_{Y(\Omega)} = \int_{-\infty}^{+\infty} \Bigg\{ &\left|\left(e^{i\tau H} H^{3/2} f, H^{3/2} f\right)_{L^2(\Omega)}\right| \\ &+ \left|\left(e^{i\tau H} H^{3/2} f, H^{1/2} g\right)_{L^2(\Omega)}\right| + \left|\left(e^{i\tau H} H^{1/2} g, H^{1/2} g\right)_{L^2(\Omega)}\right| \Bigg\}\, d\tau, \end{aligned}$$

where $(f,g)_{L^2(\Omega)}$ denotes the $L^2(\Omega)$-inner product of f and g.

We are now in a position to state the results. The result is as follows:

Theorem 2.1. *Let $n \geq 1$. If the data $f_0(x)$ and $f_1(x)$ satisfy $(f_0, f_1) \in Y(\Omega)$ and*

$$\|\nabla f_0\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)} + |(f_0, f_1)|_{Y(\Omega)} \ll 1,$$

then the initial-boundary value problem (1)–(3) *admits a unique solution $u \in C(\mathbb{R}; H^{3/2}(\Omega) \cap H^1_0(\Omega)) \cap C^1(\mathbb{R}; H^{1/2}(\Omega))$.*

The class $Y(\mathbb{R}^n)$ in $\mathbb{R}^n$ is introduced by Kajitani (see [13], cf. [24]), and generalizes the class of Yamazaki (see [27]). We note that the method in the proof of [13, Theorem 1.1] would effective also on the initial-boundary value problem in unbounded domains. However, we will prove Theorem 2.1 in an alternative way. In the unbounded domain Ω we can also consider the class of Yamazaki as follows:

$$Y_k(\Omega) := \left\{(f, g) \in (H^{3/2}(\Omega) \cap H^1_0(\Omega)) \times H^{1/2}(\Omega) \, : \, |(f, g)|_{Y_k(\Omega)} < +\infty\right\}$$

for $k > 1$ with

$$|(f, g)|_{Y_k(\Omega)} = \sup_{\tau \in \mathbb{R}} \langle \tau \rangle^k \left\{ \left| \left(e^{i\tau H} H^{3/2} f, H^{3/2} f \right)_{L^2(\Omega)} \right| + \left| \left(e^{i\tau H} H^{3/2} f, H^{1/2} g \right)_{L^2(\Omega)} \right| + \left| \left(e^{i\tau H} H^{1/2} g, H^{1/2} g \right)_{L^2(\Omega)} \right| \right\}.$$

The inclusions among the classes $Y(\Omega)$ and $Y_k(\Omega)$ are as follows:

$$Y_k(\Omega) \subset Y(\Omega) \quad \text{for any } k > 1. \tag{5}$$

In [18, Theorem 1.1] we assumed that $(f_0, f_1) \in Y_k(\Omega)$ for some $k > 1$. Hence it follows from (5) that Theorem 2.1 generalizes [18, Theorem 1.1].

The examples of spaces contained in $Y(\Omega)$ are the function spaces in [28, Theorem 4] and [29]. However, we have one more example:

Example 2.2. *Let Ω be a domain of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is compact and its boundary $\partial\Omega$ is of C^∞. For $\sigma \geq 0$ and $\varkappa \in \mathbb{R}$, let $H^\sigma_{\varkappa,0}(\Omega)$ be the completion of $C^\infty_0(\Omega)$ in the norm $\|\cdot\|_{H^\sigma_\varkappa(\Omega)}$. Then it is proved in Example 1.3 from [18] that if $n \geq 3$ and $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to the origin, then the inclusion*

$$H^{s_0+1}_{s(k),0}(\Omega) \times H^{s_0}_{s(k),0}(\Omega) \subset Y_k(\Omega)$$

holds for any $s_0 > (n+1)/2$, $s(k) > \max(n + 1/2, k + n/2)$ and $k > 1$.

As a consequence of Theorem 2.1, Example 2.2 and the inclusion (5), we have Theorem 1.4 from [18]:

Theorem 2.3. *Let $\Omega, n, s_0, s(k)$ be as in Example 2.2. If the data $f_0(x)$ and $f_1(x)$ satisfy*

$$f_0(x) \in H^{s_0+1}_{s(k),0}(\Omega), \quad f_1(x) \in H^{s_0}_{s(k),0}(\Omega),$$

and

$$\|f_0\|_{H^{s_0+1}_{s(k)}(\Omega)} + \|f_1\|_{H^{s_0}_{s(k)}(\Omega)} \ll 1,$$

then the initial-boundary value problem (1)–(3) *admits a unique solution* $u \in \cap_{j=0,1,2} C^j(\mathbb{R}; H^{s_0+1-j}(\Omega))$.

3. Proof of Theorem 2.1

The idea of proof comes from D'Ancona & Spagnolo [6]. Let us consider the *linear* problem:

$$\partial_t^2 u - c(t)^2 \Delta u = 0, \quad x \in \Omega, \tag{6}$$

for $t \neq 0$, with the initial condition

$$u(0,x) = f_0(x), \quad \partial_t u(0,x) = f_1(x), \tag{7}$$

and the boundary condition

$$u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \partial\Omega. \tag{8}$$

Here $c(t)$ satisfies a suitable condition introduced later. We define a new function

$$\tilde{c}(t)^2 = 1 + \int_\Omega |\nabla u|^2 \, dx. \tag{9}$$

This defines a map

$$\Theta : c \mapsto \tilde{c}.$$

If we can find a fixed point of Θ in a suitable space, the solution $u(t,x)$ to (6)–(8) will be a solution to the original problem (1)–(3).

Now let us introduce a set $\mathcal{K}$ as follows:

A set $\mathcal{K}$. *Given $\Lambda > 1$ and $K > 0$, the function $c(t) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ belongs to $\mathcal{K} = \mathcal{K}(\Lambda, K)$ if the following two conditions are satisfied*:

$$1 \le c(t) \le \Lambda,$$

$$\int_{-\infty}^{+\infty} |c'(t)| \, dt \le K.$$

The following proposition is crucial in the argument.

Proposition 3.1. *Let $c(t) \in \mathcal{K}$. Then there exists a constant $M > 0$ such that if K satisfies $K < 1/\sqrt{2}$, then*

$$1 \leq \tilde{c}(t) \leq 1 + \|\nabla f_0\|_{L^2(\Omega)} + M|(f_0, f_1)|_{Y(\Omega)}, \tag{10}$$

$$\int_{-\infty}^{+\infty} |\tilde{c}'(t)|\, dt \leq M|(f_0, f_1)|_{Y(\Omega)}. \tag{11}$$

Proof. The proof is essentially based on the idea of [6, Theorem 1.1] and [29, Thoerem 4]. For the solution $u(t,x)$ to (6)–(8), we define two functions

$$v_{\pm}(t) = \frac{e^{\pm i\vartheta(t)H}}{\sqrt{c(t)}}(\partial_t u \mp ic(t)Hu),$$

where we put

$$\vartheta(t) = \int_0^t c(s)\, ds.$$

We need two functionals

$$I(r,t) = \left(He^{2irH}v_-(t), v_+(t)\right)_{L^2(\Omega)},$$
$$J(r,t) = \left(He^{2irH}v_+(t), v_+(t)\right)_{L^2(\Omega)} + \left(He^{2irH}v_-(t), v_-(t)\right)_{L^2(\Omega)}$$

for $r, t \in \mathbb{R}$. Then it can be checked that

$$2\tilde{c}(t)\tilde{c}'(t) = \Im I(\vartheta(t), t). \tag{12}$$

We define

$$[f](r) = \sup_{t\in\mathbb{R}} |f(r,t)|$$

for every function $f = f(r,t)$ on $\mathbb{R} \times \mathbb{R}$. If we prove the following estimate:

$$\int_{-\infty}^{+\infty} [I](r)\, dr \leq 2M|(f_0, f_1)|_{Y(\Omega)} \tag{13}$$

for a suitable constant M depending only on K and Λ, we conclude the inequality (11). Indeed, by using (12) we have

$$|\tilde{c}'(t)| \leq \frac{1}{2}|I(\vartheta(t), t)| \leq \frac{1}{2}[I](\vartheta(t)).$$

Changing the variable $\tau = \vartheta(t)$ and using (13), we have

$$\int_{-\infty}^{+\infty} \frac{1}{2}[I](\vartheta(t))\, dt = \int_{-\infty}^{+\infty} \frac{1}{2c(\vartheta^{-1}(\tau))}[I](\tau)\, d\tau \leq M|(f_0, f_1)|_{Y(\Omega)},$$

which proves (11). Hence we will pay attention to show the inequality (13).

To begin with, we observe that there exists a constant C_1 such that

$$\int_{-\infty}^{+\infty} \{|I(r,0)| + |J(r,0)|\}\, dr \le C_1 |(f_0, f_1)|_{Y(\Omega)}. \tag{14}$$

It follows from the definition of $|(f_0, f_1)|_{Y(\Omega)}$ that

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left| \left(He^{2irH} v_-(0), v_+(0)\right)_{L^2(\Omega)} \right| dr \\ \le\ & \int_{-\infty}^{+\infty} \left\{ \left| -c(0) \left(e^{2irH} H^{3/2} f_0, H^{3/2} f_0\right)_{L^2(\Omega)} \right| \right. \\ & + \left| c(0)^{-1} \left(e^{2irH} H^{1/2} f_1, H^{1/2} f_1\right)_{L^2(\Omega)} \right| \\ & \left. + \left| i \left(e^{2irH} H^{3/2} f_0, H^{1/2} f_1\right)_{L^2(\Omega)} + i \left(e^{2irH} H^{1/2} f_1, H^{3/2} f_0\right)_{L^2(\Omega)} \right| \right\} dr \\ \le\ & C |(f_0, f_1)|_{Y(\Omega)}, \end{aligned}$$

which implies that

$$\int_{-\infty}^{+\infty} |I(r,0)|\, dr \le C |(f_0, f_1)|_{Y(\Omega)}.$$

In a similar way, we have the same type estimate for $J(\cdot, 0)$. Hence we obtain (14).

Let us prove (13). By the definition of $v_\pm(t)$ we have

$$v'_\pm(t) = -\frac{c'(t)}{2c(t)} e^{\pm 2i\vartheta(t)H} v_\mp(t). \tag{15}$$

Differentiating $I(r,t)$ and $J(r,t)$ with respect to t and plugging (15) into the resulting ones, we get

$$\partial_t I(r,t) = -\frac{c'(t)}{2c(t)} J(r - \vartheta(t), t),$$

$$\partial_t J(r,t) = -\frac{c'(t)}{c(t)} \left(I(r + \vartheta(t), t) + \overline{I(-r + \vartheta(t), t)} \right).$$

Write these equations into integral equation:

$$\begin{aligned} I(r,t) = I(r,0) - \frac{1}{2} \int_0^t \frac{c'(s)}{c(s)} J(r - \vartheta(s), 0)\, ds \\ + \frac{1}{2} \int_0^t \frac{c'(s)}{c(s)} \int_0^s \frac{c'(\sigma)}{c(\sigma)} \left(I(r - \vartheta(s) + \vartheta(\sigma), \sigma) + \overline{I(-r + \vartheta(s) + \vartheta(\sigma), \sigma)} \right) d\sigma ds. \end{aligned} \tag{16}$$

Then we see from (16) that

$$\int_{-\infty}^{+\infty} [I](r)\,dr \le C|(f_0,f_1)|_{Y(\Omega)} + I_1 + I_2,$$

where we used the inequality (14) and put

$$I_1 = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{|c'(s)|}{c(s)} \left(\int_{-\infty}^{+\infty} |J(r-\vartheta(s),0)|\,dr\right) ds,$$

$$I_2 = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{|c'(s)|}{c(s)} \int_{-\infty}^{+\infty} \frac{|c'(\sigma)|}{c(\sigma)}$$
$$\left(\int_{-\infty}^{+\infty} \left(|I(r-\vartheta(s)+\vartheta(\sigma),\sigma)| + \left|\overline{I(-r+\vartheta(s)+\vartheta(\sigma),\sigma)}\right|\right) dr\right) d\sigma ds.$$

As to the integral I_1, using $\int_{-\infty}^{+\infty} |c'(s)|\,ds \le K$ and the inequality (14) together with the invariance property of the Lebesgue integral, we get

$$I_1 \le C_1 K|(f_0,f_1)|_{Y(\Omega)}.$$

In a similar way, we can estimate

$$\begin{aligned} &\int_{-\infty}^{+\infty} \{|I(r-\vartheta(s)+\vartheta(\sigma),\sigma)| + |I(-r+\vartheta(s)+\vartheta(\sigma),\sigma)|\}\ dr \\ \le &\int_{-\infty}^{+\infty} \{[I](r-\vartheta(s)+\vartheta(\sigma)) + [I](-r+\vartheta(s)+\vartheta(\sigma))\}\ dr \\ =&2\int_{-\infty}^{+\infty} [I](r)\,dr. \end{aligned}$$

Hence using this estimate we get

$$I_2 \le \left(\int_{-\infty}^{+\infty} \frac{|c'(s)|}{c(s)}\,ds\right)^2 \int_{-\infty}^{+\infty} [I](r)\,dr \le K^2 \int_{-\infty}^{+\infty} [I](r)\,dr.$$

Thus we arrive at

$$\int_{-\infty}^{+\infty} [I](r)\,dr \le C_1|(f_0,f_1)|_{Y(\Omega)} + C_1K|(f_0,f_1)|_{Y(\Omega)} + K^2 \int_{-\infty}^{+\infty} [I](r)\,dr.$$

If K satisfies $K < 1/\sqrt{2}$, then

$$\int_{-\infty}^{+\infty} [I](r)\,dr \le 2(C_1 + C_1K)|(f_0,f_1)|_{Y(\Omega)} \equiv 2M|(f_0,f_1)|_{Y(\Omega)},$$

which proves (13).

It remains to prove the estimate (10). The first inequality is obvious, if we recall the definition (9) of $\tilde{c}(t)$. As to the second inequality, we observe that

$$\tilde{c}(t) \le \tilde{c}(0) + \int_0^\infty |\tilde{c}'(\tau)|\, d\tau.$$

Thus it is sufficient to integrate (11) and use the fact $\tilde{c}^2(0) = 1 + \|\nabla f_0\|^2_{L^2(\Omega)}$. The proof of Proposition 3.1 is now complete. □

Proof of Theorem 2.1. We employ the Schauder-Tychonoff fixed point theorem. Let $c(t) \in \mathcal{K}$, and we fix the data $(f_0, f_1) \in Y(\Omega)$. Then it follows from Proposition 3.1 that the map

$$\Theta : c(t) \mapsto \tilde{c}(t)$$

maps $\mathcal{K}$ into itself provided that the quantity $\|\nabla f_0\|^2_{L^2(\Omega)} + |(f_0, f_1)|_{Y(\Omega)}$ is sufficiently small. Now $\mathcal{K}$ may be regarded as the convex subset of the Fréchet space $L^\infty_{\mathrm{loc}}(\mathbb{R})$, and we endow $\mathcal{K}$ with the induced topology.

Compactness of $\mathcal{K}$. Since $\mathcal{K}$ is uniformly bounded and equi-continuous on every compact t-interval, one can deduce from the Ascoli-Arzelà theorem that $\mathcal{K}$ is relatively compact in $L^\infty_{\mathrm{loc}}(\mathbb{R})$, and it is sequentially compact. This means that every sequence $\{c_j(t)\}_{j=1}^\infty$ in $\mathcal{K}$ has a subsequence, denoted by the same, converging to some $c(\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$:

$$c_j(t) \underset{(j\to\infty)}{\longrightarrow} c(t) \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}), \qquad \|c(\cdot)\|_{L^\infty(\mathbb{R})} \le \Lambda,$$

where we used the fact that the absolute continuity of $\{c_j(t)\}$ is uniform in j on account of Vitali-Hahn-Saks theorem (see e.g., §2 in Chapter II from [30, Yosida]), since the finite limit $\lim_{j\to\infty} \int_s^t c_j'(\tau)\, d\tau$ exists for every interval (s, t). Moreover, the derivative $c'(t)$ exists almost everywhere on $\mathbb{R}$. Now, for the derivative $c'(t)$, if we prove that

$$\int_{-\infty}^{+\infty} |c'(t)|\, dt \le K, \tag{17}$$

then $c(t) \in \mathcal{K}$, which proves the compactness of $\mathcal{K}$.

For the proof of (17), we observe from Theorem 4 in §1 of Chapter V of [30] that $\{c_j'(\cdot)\}$ converges weakly to some function $d(\cdot) \in L^1(\mathbb{R})$ as $j \to \infty$, since the finite limit $\lim_{j\to\infty} \int_s^t c_j'(\tau)\, d\tau$ exists for every interval (s, t) and $\{c_j(\cdot)\}$ is uniformly bounded in $L^1(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} |c_j'(t)|\, dt \le K. \tag{18}$$

By standard arguments we can conclude that $c'(t) = d(t)$ for a.e. $t \in \mathbb{R}$. Hence (17) is true, since

$$\int_{-\infty}^{+\infty} |c'(t)|\, dt \leq \liminf_{j\to\infty} \int_{-\infty}^{+\infty} |c_j'(t)|\, dt \leq K,$$

where we used (18).

Continuity of Θ *on* $\mathcal{K}$. We may consider the case $t > 0$, since the case $t < 0$ can be treated in the same way. Let us take a sequence $\{c_m(t)\}$ in $\mathcal{K}$ such that

$$c_m(t) \to c(t) \in \mathcal{K} \quad \text{in } L^\infty_{\mathrm{loc}}(0,\infty) \quad (m \to \infty),$$

and let $u_m(t,x)$ and $u(t,x)$ be corresponding solutions to $c_m(t)$ and $c(t)$, respectively, with fixed data $(f_0, f_1) \in Y(\Omega)$. Then we prove that the images $\tilde{c}_m(t) := \Theta(c_m(t))$ and $\tilde{c}(t) := \Theta(c(t))$ satisfy

$$\tilde{c}_m(t) \to \tilde{c}(t) \quad \text{in } L^\infty_{\mathrm{loc}}(0,\infty) \quad (m \to \infty). \tag{19}$$

The functions $v_m := u_m - u$, $m = 1, 2, \ldots$, solve the following initial-boundary value problem:

$$\begin{cases} \partial_t^2 v_m - c(t)^2 \Delta v_m = \left\{c_m(t)^2 - c(t)^2\right\} \Delta u_m, & (t,x) \in \mathbb{R}\times\Omega, \\ v_m(0,x) = 0, \quad \partial_t v_m(0,x) = 0, & x \in \Omega, \\ v_m(t,x) = 0, \quad (t,x) \in \mathbb{R}\times\partial\Omega. \end{cases}$$

Differentiate the energy $E(v_m(t))$ for v_m with respect to t, where

$$E(v_m(t)) = \|v_m'(t)\|^2_{L^2(\Omega)} + c(t)^2 \|\nabla v_m(t)\|^2_{L^2(\Omega)}, \quad (' = \partial_t).$$

Then we get

$$\begin{aligned} E'(v_m(t)) = &-2\left\{c_m(t)^2 - c(t)^2\right\} \Re\left(\Delta u_m(t), v_m'(t)\right)_{L^2(\Omega)} \\ &+ 2c(t)c'(t)\,\|\nabla v_m(t)\|^2_{L^2(\Omega)} \\ \leq &2\left|c_m(t)^2 - c(t)^2\right| \|u_m(t)\|_{H^{3/2}(\Omega)} \|v_m'(t)\|_{H^{1/2}(\Omega)} + 2\frac{c'(t)}{c(t)} E(v_m(t)). \end{aligned} \tag{20}$$

By standard arguments we see that $\|u_m(t)\|_{H^{3/2}(\Omega)}$ and $\|v_m'(t)\|_{H^{1/2}(\Omega)}$ are bounded by $\|f_0\|_{H^{3/2}(\Omega)} + \|f_1\|_{H^{1/2}(\Omega)}$, we integrate (20) and apply Gronwall's lemma to obtain, for all $t \geq 0$,

$$E(v_m(t)) \leq C\left(\int_0^t \left|c_m(\tau)^2 - c(\tau)^2\right| d\tau\right)\left(\|f_0\|_{H^{3/2}(\Omega)} + \|f_1\|_{H^{1/2}(\Omega)}\right)^2 \times e^{2\int_0^\infty \frac{|c'(\tau)|}{c(\tau)}\, d\tau},$$

which implies that

$$\left.\begin{aligned}\nabla u_m(t) \to \nabla u(t)\\ u_m'(t) \to u'(t)\end{aligned}\right\} \quad \text{in } L^\infty_{\mathrm{loc}}(0,\infty;L^2(\Omega)) \text{ as } m \to \infty.$$

Hence we get (19), which proves the continuity of Θ. □

Completion of the proof of Theorem 2.1. The Schauder-Tychonoff fixed point theorem implies that Θ has a fixed point in $\mathcal{K}$, and hence, we conclude that the solution $u(t,x)$ to (6)–(8) is the solution to (1)–(3). The uniqueness of solutions is obvious. This proves Theorem 2.1. □

4. Generalized Fourier transform

We prepare the notion of the generalized Fourier transform to prove Example 2.2. Let Ω be an exterior domain such that $\mathbb{R}^n \setminus \Omega$ is compact and its boundary $\partial\Omega$ is of C^∞. Consider the Helmholtz equation with a parameter $z \in \mathbb{C}$ in Ω:

$$\begin{cases}(-\Delta - z)u = f & \text{in} \quad \Omega,\\ \qquad\quad u = 0 & \text{on} \quad \partial\Omega.\end{cases} \tag{21}$$

It is well known that 0 is not an eigenvalue of A. The spectrum $\sigma(A)$ of A is absolutely continuous and coincides with $[0,\infty)$. We denote by $R(z) = (A-z)^{-1}$ the resolvent of A.

Following Wilcox [26], let us define the generalized Fourier transforms in an arbitrary exterior domain. The existence of the limits $R(|\xi|^2 \pm i0) = \lim_{\varepsilon\to 0} R(|\xi|^2 \pm i\varepsilon)$ is proved by Mochizuki [20], which is called the limiting absorption principle. Introducing a function $j(x) \in C^\infty(\mathbb{R}^n)$ vanishing in a neighbourhood of $\mathbb{R}^n \setminus \Omega$ and equal to one for large $|x|$, let us define the generalized Fourier transform as follows:

$$(\mathcal{F}_\pm f)(\xi) = \lim_{R\to\infty} (2\pi)^{-n/2} \int_{\Omega_R} \overline{\psi_\pm(x,\xi)} f(x)\,dx \quad \text{in} \quad L^2(\mathbb{R}^n),$$

where we put

$$\psi_\pm(x,\xi) = j(x)e^{ix\cdot\xi} + [R(|\xi|^2 \pm i0)M_\xi(\cdot)](x)$$
$$\text{with} \quad M_\xi(x) = (A - |\xi|^2)(j(x)e^{ix\cdot\xi}).$$

Notice that we can write formally

$$M_\xi(x) = -(\Delta j(x) + 2i\xi\cdot\nabla j(x))e^{ix\cdot\xi}, \tag{22}$$

hence, $\operatorname{supp} M_\xi(\cdot) \subset B_{r_0+1}(0)\setminus B_{r_0}(0)$ for any fixed $\xi \in \mathbb{R}^n$. The kernel $\psi_\pm(x,\xi)$ is called eigenfunction of the operator A with eigenvalue $|\xi|^2$ in the sense that, formally, $(A-|\xi|^2)\psi_\pm(x,\xi)=0$, but $\psi(x,\xi)\notin L^2(\Omega)$. Similarly, the inverse transform is defined by

$$(\mathcal{F}_\pm^* g)(x)=\lim_{R\to\infty}(2\pi)^{-n/2}\int_{B_R(0)}\psi_\pm(x,\xi)g(\xi)\,d\xi \quad \text{in } L^2(\Omega).$$

We treat $\mathcal{F}_+f$ only and drop the subscipt $+$, since $\mathcal{F}_-f$ can be dealt with by essentially the same method. The transform $\mathcal{F}f$ thus defined obeys the following properties (see, e.g., Shenk II [25, Theorem 1 and Corollary 5.1]):

- $\mathcal{F}$ is a unitary mapping
$$\mathcal{F}: L^2(\Omega)\to L^2(\mathbb{R}^n).$$
Hence
$$\mathcal{F}\mathcal{F}^*=I.$$
- $\mathcal{F}$ is fulfilled with the generalized Parseval equality:
$$(\mathcal{F}f,\mathcal{F}g)_{L^2(\mathbb{R}^n)}=(f,g)_{L^2(\Omega)},\quad f,g\in L^2(\Omega). \tag{23}$$
- $\mathcal{F}$ diagonalizes the operator A in the sense that
$$\mathcal{F}(\varphi(A)f)(\xi)=\varphi(|\xi|^2)(\mathcal{F}f)(\xi), \tag{24}$$
where $\varphi(A)$ is the operator defined by the spectral representation theorem for self-adjoint operators.

The following lemma is concerning with the differentiability properties of the generalized Fourier transform $(\mathcal{F}f)(\xi)$.

Lemma 4.1 (Lemma 2.8 from [18]). *Let $n\ge 3$ and ε_0 be a sufficiently small number. Then the following estimates hold*:

(i) (*High frequency estimates*). *Assume that $\mathbb{R}^n\setminus\Omega$ is star-shaped with respect to the origin. Let $s>1/2$. If $f\in L^2_{s+|\alpha|}(\Omega)$ for some multi-index α, then*

$$\left|\partial_\xi^\alpha(\mathcal{F}f)(\xi)\right|\le\left|\partial_\xi^\alpha(\mathcal{F}_0(jf))(\xi)\right|+C_{\alpha,\varepsilon_0}\|f\|_{L^2_{s+|\alpha|}(\Omega)} \tag{25}$$

for all $|\xi|\ge\varepsilon_0$, where $(\mathcal{F}_0g)(\xi)$ denotes the Fourier transform of $g(x)$ on $\mathbb{R}^n$.

(ii) (*Low frequency estimates*). *Let $s>n+1/2$. Then the following estimates hold for all $0<|\xi|\le\varepsilon_0$:*

$$\left|\partial_\xi^\alpha(\mathcal{F}f)(\xi)\right|\le\left|\partial_\xi^\alpha(\mathcal{F}_0(jf))(\xi)\right|+C_{\alpha,\varepsilon_0}\left\{1+|\xi|^{n-2-|\alpha|}\left|(\log|\xi|)^{\varepsilon(n)}\right|\right\}\|f\|_{L^2_s(\Omega)} \tag{26}$$

for all $0<|\xi|<\varepsilon_0$ and $|\alpha|\le n-2$, and

$$\left|\partial_\xi^\alpha(\mathcal{F}f)(\xi)\right|\le\left|\partial_\xi^\alpha(\mathcal{F}_0(jf))(\xi)\right|+C_{\alpha,\varepsilon_0}\left(1+|\xi|^{n-2-|\alpha|}\right)\|f\|_{L^2_s(\Omega)} \tag{27}$$

for all $0<|\xi|<\varepsilon_0$ and $|\alpha|=n-1,n$.

Let us make a few remarks on Lemma 4.1. The proof of the high frequency part is based on the results of Heiming [11] whose method is similar to Isozaki [12]. The low frequency part can be obtained by using [18, Proposition 2.5] which states the resolvent expansion around the origin. We also note that the geometrical condition is not needed in low frequencies.

5. Proof of Example 2.2

We need a decay estimate of some oscillatory integrals whose proof can be found in [18, Lemma 3.2]. For the completeness, we will give the outline of proof.

LEMMA 5.1. *Let $n \geq 3$. Assume that $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to the origin. Let $f_1 \in H^{\gamma_1+1/2}_{s(k),0}(\Omega)$ and $f_2 \in H^{\gamma_2+1/2}_{s(k),0}(\Omega)$ for some $s(k) > \max(n+1/2, k+n/2)$, $k \in (1, n]$, and for some $\gamma_1, \gamma_2 > n/2$. Consider the oscillatory integral of the form*

$$F(\tau) = \int_{\mathbb{R}^n} e^{i\tau|\xi|}(\mathcal{F}f_1)(\xi)(\mathcal{F}f_2)(\xi)|\xi|\, d\xi, \quad (\tau \in \mathbb{R}).$$

Then

$$|F(\tau)| \leq C(1+|\tau|)^{-k}\|f_1\|_{H^{\gamma_1+1/2}_{s(k)}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s(k)}(\Omega)}.$$

Proof. Let us overview the outline of the proof of Lemma 5.1. First, we observe that $F(\tau)$ is bounded in $\tau \in \mathbb{R}$, provided that $f_1 \in H^{1/2}(\Omega)$ and $f_2 \in H^{1/2}(\Omega)$. In fact, by using the generalized Parseval identity (23) and diagonalization property (24), we have

$$|F(\tau)| \leq \int_{\mathbb{R}^n} |(\mathcal{F}f_1)(\xi)||(\mathcal{F}f_2)(\xi)||\xi|\, d\xi \leq \|f_1\|_{H^{1/2}(\Omega)}\|f_2\|_{H^{1/2}(\Omega)}.$$

Hence we have only to prove the case $|\tau| \geq 1$. Inserting the cut-off function $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ equal to one for $|\xi| \geq \varepsilon_0$ and 0 for $|\xi| \leq \varepsilon_0/2$, we write

$$\begin{aligned} &F(\tau) = F_1(\tau) + F_2(\tau) \\ =&\int_{\mathbb{R}^n} e^{i\tau|\xi|}\chi(\xi)(\mathcal{F}f_1)(\xi)(\mathcal{F}f_2)(\xi)|\xi|\, d\xi \\ &+\int_{\mathbb{R}^n} e^{i\tau|\xi|}(1-\chi(\xi))(\mathcal{F}f_1)(\xi)(\mathcal{F}f_2)(\xi)|\xi|\, d\xi. \end{aligned}$$

Since the support of the amplitude function in $F_1(\tau)$ is away from the origin, and since k-fold ξ-derivatives of the amplitude function decay as $|\xi| \to \infty$ for any integer k, we can perform k-fold integration by parts with an operator $P = \frac{\nabla_\xi(\tau|\xi|)}{i|\nabla_\xi(\tau|\xi|)|^2} \cdot \nabla_\xi$; thus we find that

$$F_1(\tau) = \int_{\mathbb{R}^n} e^{i\tau|\xi|}(P^*)^k \left[\chi(\xi)|\xi|^{-\gamma_1-\gamma_2}(\mathcal{F}H^{\gamma_1+1/2}f_1)(\xi)(\mathcal{F}H^{\gamma_1+1/2}f_2)(\xi)\right] d\xi,$$

where we used the diagonalization property (24). Then, by using (25) from Lemma 4.1,we get

$$\begin{aligned} &|F_1(\tau)| \\ \le& C_k|\tau|^{-k}\sum_{a\le k}\Big(\|f_1\|_{H^{\gamma_1+1/2}_{k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{a}(\Omega)} + \|f_1\|_{H^{\gamma_1+1/2}_{k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s+a}(\Omega)} \\ &+ \|f_1\|_{H^{\gamma_1+1/2}_{s+k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{a}(\Omega)} + \|f_1\|_{H^{\gamma_1+1/2}_{s+k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s+a}(\Omega)}\Big) \qquad (28) \\ \le& C_k|\tau|^{-k}\sum_{a\le k}\|f_1\|_{H^{\gamma_1+1/2}_{s+k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s+a}(\Omega)} \end{aligned}$$

for any $k \in \mathbb{N}$ and $s > 1/2$.

We now turn to the estimate of $F_2(\tau)$. For brevity, we denote the symbol in the integral $F_2(\tau)$ by

$$A(\xi) = (1-\chi(\xi))(\mathcal{F}f_1)(\xi)(\mathcal{F}f_2)(\xi)|\xi|.$$

Making change of variable $\xi = \lambda\omega$ ($\lambda = |\xi|, \omega \in \mathbb{S}^{n-1}$), we have

$$F_2(\tau) = \int_{\mathbb{S}^{n-1}}\int_0^\infty e^{i\lambda\tau}A(\lambda\omega)\lambda^{n-1}\,d\lambda d\omega.$$

Then integrating by parts, we get

$$\left|\int_{\mathbb{S}^{n-1}}\int_0^\infty e^{i\lambda\tau}A(\lambda\omega)\lambda^{n-1}\,d\lambda d\omega\right| \le C_k|\tau|^{-k}\int_{\mathbb{S}^{n-1}}\int_0^\infty \left|\partial_\lambda^k(A(\lambda\omega)\lambda^{n-1})\right| d\lambda d\omega$$

for $k = 0, 1, \dots, n$, where we used the fact that integration by parts is possible up to n-times, since the boundary term appears for $k \ge n+1$. Hence we have only to estimate $\partial_\lambda^k(A(\lambda\omega)\lambda^{n-1})$. However, this can be done by using (26)–(27) in Lemma 4.1, and hence, we get the following:

$$|F_2(\tau)| \le C_k|\tau|^{-k}\|f_1\|_{L^2_{s(k)}(\Omega)}\|f_2\|_{L^2_{s(k)}(\Omega)}, \qquad (29)$$

for $s(k) > \max(n+1/2, k+n/2)$ and $k = 1, \dots, n$.

Combining the estimates (28)–(29), we arrive at the estimate

$$|F(\tau)| \le C_k(1+|\tau|)^{-k}\,\|f_1\|_{H^{\gamma_1+1/2}_{s(k)}(\Omega)}\,\|f_2\|_{H^{\gamma_2+1/2}_{s(k)}(\Omega)}\,, \qquad (30)$$

for any $s(k) > \max(n+1/2, k+n/2)$ and $k = 1, \dots, n$. But then (30) holds also for any *real* $k \in [1, n]$, if we use the interpolation argument. This ends the proof. □

Put $\gamma_1 = \gamma_2 = s_0 - 1/2$ in Lemma 5.1. Recall the definition of $Y_k(\Omega)$. If we choose $(\mathcal{F}f_1)(\xi), (\mathcal{F}f_2)(\xi)$ as $|\xi|(\mathcal{F}f_0)(\xi)$ in Lemma 5.1, one has

$$\left| \left(e^{i\tau|\xi|}|\xi|^{3/2}\mathcal{F}f_0, |\xi|^{3/2}\mathcal{F}f_0 \right)_{L^2(\mathbb{R}^n)} \right| \le C(1+|\tau|)^{-k}\|Hf_0\|^2_{H^{s_0}_{s(k)}(\Omega)}.$$

If we choose $(\mathcal{F}f_1)(\xi), (\mathcal{F}f_2)(\xi)$ as $(\mathcal{F}f_1)(\xi)$ in Lemma 5.1, one has

$$\left| \left(e^{i\tau|\xi|}|\xi|^{1/2}\mathcal{F}f_1, |\xi|^{1/2}\mathcal{F}f_1 \right)_{L^2(\mathbb{R}^n)} \right| \le C(1+|\tau|)^{-k}\|f_1\|^2_{H^{s_0}_{s(k)}(\Omega)}.$$

If we choose $(\mathcal{F}f_1)(\xi)$ as $|\xi|(\mathcal{F}f_0)(\xi)$, and $(\mathcal{F}f_2)(\xi)$ as $(\mathcal{F}f_1)(\xi)$ in Lemma 5.1, respectively, one has

$$\left| \left(e^{i\tau|\xi|}|\xi|^{3/2}\mathcal{F}f_0, |\xi|^{1/2}\mathcal{F}f_1 \right)_{L^2(\mathbb{R}^n)} \right| \le C(1+|\tau|)^{-k}\|Hf_0\|_{H^{s_0}_{s(k)}(\Omega)}\|f_1\|_{H^{s_0}_{s(k)}}.$$

These estimates imply Example 2.2:

$$H^{s_0+1}_{s(k),0}(\Omega) \times H^{s_0}_{s(k),0}(\Omega) \subset Y_k(\Omega).$$

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