

Spectral Properties of an Operator Associated with Hartree Type Equations with External Coulomb Potential

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ABSTRACT. *The work treats the nonexistence of zero resonance for the linearization of the Hartree type equation with external Coulomb potential around the solitary type solutions.*

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1. Introduction

Solitary waves associated with the Hartree type equation in external Coulomb potential are solutions of type

$$\psi_s(t, x) = \chi(x)e^{-i\omega t}, \quad x \in \mathbf{R}^3, t \in \mathbf{R},$$

where χ satisfies the equation

$$\Delta\chi + \omega\chi = e^2 \left(q(|\chi|^2) - \frac{Z}{|x|} \right) \chi, \quad x \in \mathbf{R}^3, \quad (1)$$

$$\int_{\mathbf{R}^3} \chi^2 = N. \quad (2)$$

Here $e > 0$ is the electron charge,

$$V(x) = -\frac{e^2 Z}{|x|}, \quad (3)$$

is the external potential and $Z > 0$ is the charge of the (external) nucleus.

Here and below

$$q(f)(x) = \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|} dy. \quad (4)$$

The solution ψ satisfies the Hartree type equation

$$\begin{aligned} i\partial_t\psi(t, x) &= -\Delta\psi(t, x) \\ &+ \left(e^2 \int_{\mathbf{R}^3} \frac{|\psi(t, y)|^2}{|x-y|} dy + V(x) \right) \psi(t, x), \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^3. \end{aligned} \quad (5)$$

The existence of solitary type solutions to the Hartree equation (5) is a well-studied problem (see [4], [5]). Similar models where the external potential is replaced by self-interacting term of type $|\chi|^{p-1}$ are studied in [3].

The natural functional associated with this problem is (see [4])

$$\begin{aligned} \mathcal{E}(\chi) &= \frac{1}{2} \|\nabla\chi\|_{L^2}^2 \\ &+ \frac{e^2}{4} \int_{\mathbf{R}^3} q(|\chi|^2)(x) |\chi(x)|^2 dx - \frac{e^2 Z}{2} \int_{\mathbf{R}^3} \frac{|\chi(x)|^2}{|x|} dx. \end{aligned} \quad (6)$$

The corresponding minimization problem is associated with the quantity

$$I_N = \inf\{\mathcal{E}(\chi); \chi \in H^1, \|\chi\|_{L^2}^2 = N\}. \quad (7)$$

Denote by $\omega_1 < 0$ the first eigenvalue of the operator

$$-\Delta - \frac{Ze^2}{|x|}.$$

Assuming

$$\omega_1 < \omega \leq 0 \quad (8)$$

we have the following result due to Lions-Cazenave.

LEMMA 1.1. (*Theorem III.1 in [4]*) For any ω satisfying (8) there exists a unique positive solution $\chi(x) = \chi_\omega(x) \in H^1$ of the equation (1), such that

i) for $\omega = 0$ the corresponding positive solution $\chi_0(x)$ satisfies

$$\Delta\chi_0 = e^2 \left(q(|\chi_0|^2) - \frac{Z}{|x|} \right) \chi_0, \quad x \in \mathbf{R}^3, \quad (9)$$

and

$$\|\chi_0\|_{L^2}^2 > Z;$$

ii) we have

$$\lim_{\omega \rightarrow \omega_1^+} \|\chi_\omega\|_{H^1} = 0;$$

iii) the function $\omega \in (\omega_1, 0) \rightarrow \|\chi_\omega\|_{L^2}$ is a strictly increasing one and belongs to $C^1(\omega_1, 0)$.

REMARK 1.2. *The result of Theorem III.1 in [4] treats more general cases of potentials of type*

$$V(x) = - \sum_{j=1}^K \frac{Z}{|x - x_j|},$$

while in our case we have

$$V(x) = - \frac{Z}{|x|}.$$

Therefore the energy functional $\mathcal{E}(\chi)$ is rotationally invariant in our case. This fact combined with the uniqueness of the positive solution to (1) implies that the solutions $\chi_\omega(x)$ are radially symmetric.

REMARK 1.3. *As we shall see in the next section (see also section 7 in [5]) the solution $\chi_\omega(x)$ is rapidly decreasing in x as $|x| \rightarrow \infty$.*

In particular the property iii) of the above lemma guarantees that to given $Z > 0$ one can find a unique χ^* satisfying the properties

$$\left\{ \begin{array}{l} \chi^* \text{ is radial positive function and } \int_{\mathbf{R}^3} |\chi^*|^2 = Z; \\ \chi^* = \chi^*(|x|) \text{ is a minimizer of } I_Z; \\ \text{there exists a unique } \omega^* \in (\omega_1, 0), \text{ such that} \\ \Delta \chi^* + \omega^* \chi^* = e^2 \left(q(|\chi^*|^2) - \frac{Z}{|x|} \right) \chi^*. \end{array} \right.$$

The asymptotic stability of solitary waves are studied in several works (see for example [8]). The starting point is a standard linearization around the solitary solution. Making a similar linearization of the Hartree equation (5) around the solitary solution leads to the necessity to use some spectral properties of the following operator

$$H_0 = -\Delta + bQ(|\chi^*|^2), \quad Q(f)(x) = q(f)(x) - \frac{Z}{|x|}.$$

More precisely, we need the following two properties:

$$(H1) \left\{ \begin{array}{l} 0 \text{ is not an eigenvalue of } -\Delta + bQ(|\chi^*|^2), \\ 0 \text{ is not a resonance of } -\Delta + bQ(|\chi^*|^2). \end{array} \right.$$

It is well known (see [6] for example) that zero resonances are the main obstacle to have dispersive and Strichartz type estimate. Moreover, some resolvent estimates depend essentially on the existence of the resonance at zero. Otherwise, we need smallness assumptions on the perturbation (see [7] for example). There are no positive eigenvalues of $-\Delta + bQ(|\chi^*|^2)$ due to [2] for example. Somehow, this fact is not sufficient to assure that 0 is not an eigenvalue.

The main goal of this work is to show that the assumptions (H1) can be verified the special case of Hartree type equation in external Coulomb potential.

2. Solitary waves: qualitative properties

Decay properties of the radial solutions to (1) satisfying the regularity property

$$\chi \in H^1(\mathbf{R}^3), \quad (10)$$

can be established after the substitution

$$\mathbf{U}(r) = r\chi(r), \quad \mathbf{W}(r) = -rq(|\chi|^2)(r). \quad (11)$$

From the first equation in (1) we have

$$\mathbf{U}''(r) + \omega\mathbf{U}(r) = e^2\mathbf{K}(r), \quad (12)$$

where

$$\mathbf{K}(r) = \frac{\mathbf{W}}{r}\mathbf{U} - \frac{Z}{r}\mathbf{U}. \quad (13)$$

The function $\mathbf{W}(r)$ satisfies

$$\mathbf{W}''(r) = \frac{\mathbf{U}^2(r)}{4\pi r}. \quad (14)$$

Then we have the following statements.

LEMMA 2.1. *If the assumption (10) is satisfied, then*

$$\mathbf{U} \in H^k((1, +\infty)), \quad \mathbf{W}' \in H^{k-1}((1, +\infty)), \quad (15)$$

and

$$\sum_{j=0}^k |\mathbf{U}^{(j)}(r)| \leq \frac{C}{r^N}, \quad \sum_{j=1}^k |\mathbf{W}^{(j)}(r)| \leq \frac{C}{r^N} \quad (16)$$

for each integers $k \geq 2, N \geq 2$, and for any real $r \geq 1$.

PROOF. The Sobolev embedding implies that

$$\begin{aligned} \int_0^{+\infty} |\mathbf{U}(r)|^2 dr + \int_0^{+\infty} |\mathbf{U}'(r)|^2 dr &\leq C\|\chi\|_{H^1(\mathbf{R}^3)}^2, \\ \int_0^{+\infty} |\mathbf{W}'(r)|^2 dr &\leq C\|q(|\chi|^2)\|_{H^1(\mathbf{R}^3)}^2 \leq C\|\chi\|_{H^1(\mathbf{R}^3)}^4. \end{aligned} \quad (17)$$

Note that we have used the Hardy inequality

$$\int_0^{+\infty} |f(r)|^2 dr \leq C \int_0^{+\infty} |f'(r)|^2 r^2 dr \quad (18)$$

in the above estimates. The inequality

$$\|q(|\chi|^2)\|_{\dot{H}^1(\mathbf{R}^3)} \leq C\|\chi\|_{H^1(\mathbf{R}^3)}^2$$

follows from the relations

$$\|q(f)\|_{\dot{H}^1(\mathbf{R}^3)} = c\|\cdot|\cdot|^{-2} * f\|_{L^2(\mathbf{R}^3)},$$

the Young inequality

$$\|\cdot|\cdot|^{-2} * f\|_{L^2(\mathbf{R}^3)} \leq C\|f\|_{L^{6/5}(\mathbf{R}^3)}$$

and the Sobolev inequality

$$\|\chi\|_{L^{6/5}(\mathbf{R}^3)}^2 \leq C\|\chi\|_{H^1(\mathbf{R}^3)}^2.$$

Hence,

$$\mathbf{U} \in H^1((0, +\infty)), \quad \mathbf{W}' \in L^2((0, +\infty)).$$

Proceeding further inductively we find (15).

The above properties and the Sobolev embedding imply

$$\lim_{r \rightarrow +\infty} |\mathbf{U}(r)| = 0, \quad \lim_{r \rightarrow +\infty} |\mathbf{U}'(r)| = 0. \quad (19)$$

In a similar way we get

$$\lim_{r \rightarrow +\infty} |\mathbf{W}'(r)| = 0. \quad (20)$$

We can improve the last property. Indeed, integrating (14) we find

$$\mathbf{W}'(r) = \int_r^\infty \frac{\mathbf{U}^2(\tau)}{\tau} d\tau. \quad (21)$$

Since

$$\int_r^\infty \mathbf{U}^2(\tau) d\tau \leq C, \quad (22)$$

we get

$$0 \leq \mathbf{W}'(r) \leq \frac{C}{r}. \quad (23)$$

Our next step is to obtain weighted Sobolev estimates. Since the initial data for \mathbf{U} are

$$\mathbf{U}(0) = 0, \quad \mathbf{U}'(0) = a_1, \quad (24)$$

we have the following integral equation satisfied by \mathbf{U}

$$\mathbf{U}(r) = \sinh(\sqrt{-\omega} r) \frac{a_1}{\sqrt{-\omega}} + e^2 \int_0^r \sinh(\sqrt{-\omega} (r - \rho)) \mathbf{K}(\rho) d\rho. \quad (25)$$

It is easy to see that the function \mathbf{K} satisfies the estimate

$$\mathbf{K}(r) = O(r^{-1}), \quad r \geq 1, \quad (26)$$

due to (13). Then the condition (19) and a simple qualitative study of the integral equation in (25) guarantee that

$$\frac{a_1}{\sqrt{-\omega}} + e^2 \int_0^\infty e^{\sqrt{-\omega}\rho} \mathbf{K}(\rho) d\rho = 0.$$

This fact enables one to represent \mathbf{U} as follows:

$$\begin{aligned} \mathbf{U}(r) = & e^{-\sqrt{-\omega}r} b_1 \\ & - e^2 \int_r^\infty e^{\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho - e^2 \int_0^r e^{-\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho. \end{aligned} \quad (27)$$

The first term in the right side of (27) is exponentially decaying. The second term we can represent as the following sum

$$\int_r^{2r} e^{\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho + \int_{2r}^\infty e^{\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho.$$

It is clear that

$$\int_{2r}^\infty e^{\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho$$

is decaying exponentially, while

$$\int_r^{2r} e^{\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho \leq \frac{C}{r} \int_r^{2r} e^{\sqrt{-\omega}(r-\rho)} d\rho = \frac{C_1}{r}$$

due to (26). In a similar way we can treat the last term

$$\int_0^r e^{-\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho$$

in (27). This term now is a sum of type

$$\int_0^{r/2} e^{-\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho + \int_{r/2}^r e^{-\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho.$$

The term

$$\int_0^{r/2} e^{-\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho$$

decays exponentially in r and the property (27) implies that

$$\int_{r/2}^r e^{-\sqrt{-\omega}(r-\rho)} \mathbf{K}(\rho) d\rho = O(r^{-1}).$$

The above observation and (27) imply that

$$\mathbf{U} = O(r^{-1})$$

and

$$\mathbf{K}(r) = c \left(\frac{\mathbf{W}}{r} \mathbf{U} - \frac{Z}{r} \mathbf{U} \right) = O(r^{-2}).$$

This estimate implies a stronger version of (22)

$$\int_r^\infty \mathbf{U}^2(\tau) d\tau \leq \frac{C}{r}, \quad (28)$$

and from (21) we improve (23) as follows:

$$0 \leq \mathbf{W}'(r) \leq \frac{C}{r^2}. \quad (29)$$

This argument shows that combining (23) and (13) we can obtain inductively

$$\sum_{j=0}^k |\mathbf{U}^{(j)}(r)| \leq \frac{C}{r^N} \quad (30)$$

and

$$\sum_{j=1}^k |\mathbf{W}^{(j)}(r)| \leq \frac{C}{r^N} \quad (31)$$

for any integers $k \geq 1$ and $N \geq 2$. \square

3. The case of half line: 0 is not an eigenvalue.

Consider the operator

$$P(u)(r) = u''(r) + W(r)u(r), \quad r \in (0, \infty). \quad (32)$$

We shall assume that $W(r) = W_1(r) + W_2(r)$, where

$$W_1(r) = -\frac{\alpha(\alpha+1)}{r^2}, \quad \alpha \geq 0, \quad (33)$$

while $W_2(r)$ is a $C^1(0, \infty)$ positive strictly decreasing function satisfying the estimates

$$\begin{cases} |W_2(r)| < C/r, & \text{for } 0 < r < 1; \\ |W_2(r)| < C/r^a, & \text{for } r > 1 \text{ and some } a > 2\alpha + 3. \end{cases} \quad (34)$$

DEFINITION 3.1. A real number λ is called an eigenvalue of P if there exists $u \in H^1(0, \infty)$ such that $u(0) = 0$, $u(r)$ is not identically zero and $P(u) = \lambda u$ in the distributional sense in $(0, \infty)$.

The first step is to obtain an asymptotic expansion of the solution.

LEMMA 3.2. Assume $a > 2\alpha + 3$ in the assumption (34). If 0 is an eigenvalue of P and $Pu = 0$ in the sense of Definition 3.1, then one can find a real number C_1 so that

$$u(r) = \frac{C_1}{r^\alpha} + O(r^{-a+\alpha+3}) \quad (35)$$

and

$$u'(r) = -\frac{C_1\alpha}{r^{\alpha+1}} + O(r^{-a+\alpha+2}) \quad (36)$$

as $r > 1$ tends to infinity.

Proof. One can rewrite the equation $Pu = 0$ as

$$\left[r^{-\alpha} \left(u'(r) + \frac{\alpha}{r} u(r) \right) \right]' + r^{-\alpha} W_2(r) u(r) = 0. \quad (37)$$

Note that the assumption $W_2(r) \in C^1(0, \infty)$ combined with the equation $Pu = 0$ imply that $u \in C^2(R_1, R_2)$ for any $0 < R_1 < R_2$. Integrating (37) in the interval (R_1, R_2) , we find

$$\begin{aligned} & \left| R_2^{-2\alpha} (R_2^\alpha u(R_2))' - R_1^{-2\alpha} (R_1^\alpha u(R_1))' \right| \\ &= \left| \int_{R_1}^{R_2} \tau^{-\alpha} W_2(\tau) u(\tau) d\tau \right|, \end{aligned} \quad (38)$$

so using the assumption (34) together with the fact that u is bounded and taking $R_1 > 1$, we find

$$\left| \int_{R_1}^{R_2} \tau^{-\alpha} W_2(\tau) u(\tau) d\tau \right| \leq CR_1^{-a+2}.$$

In this way we conclude that the limit exists

$$\exists C_0 \equiv \lim_{r \rightarrow \infty} r^{-2\alpha} (r^\alpha u(r))' \quad (39)$$

and we have the expansion

$$(r^\alpha u(r))' = C_0 r^{2\alpha} + O(r^{-a+2\alpha+2}). \quad (40)$$

Consider the function

$$g(r) = r^\alpha u(r) - C_0 \frac{r^{2\alpha+1}}{2\alpha+1}.$$

Then (40) implies that $g'(r) = O(r^{-a+2\alpha+2}) \in L^1(1, \infty)$, since $a > 2\alpha + 3$ and we see that $g(r)$ has a limit (say C_1) as r goes to ∞ and

$$g(r) = C_1 - \int_r^\infty g'(\tau) d\tau = C_1 + O(r^{-a+2\alpha+3}).$$

Then we obtain

$$u(r) = C_0 \frac{r^{\alpha+1}}{2\alpha+1} + \frac{C_1}{r^\alpha} + O(r^{-a+\alpha+3}) \quad (41)$$

and

$$u'(r) = C_0 \frac{(\alpha+1)r^\alpha}{2\alpha+1} - \frac{C_1\alpha}{r^{\alpha+1}} + O(r^{-a+\alpha+2}) \quad (42)$$

as $r > 1$ tends to infinity.

Comparing these asymptotic developments with the fact that u is bounded, we see that $C_0 = 0$ and this completes the proof of the lemma.

The next step is to show that 0 is not an eigenvalue.

LEMMA 3.3. *Suppose $a > 3\alpha + 4$ in the assumption (34). Then 0 is not an eigenvalue of P .*

Proof. Suppose that there exists a real valued function $u(r) \in H^1(0, \infty)$, so that $P(u) = 0$. Our goal shall be to show that u is identically zero. The Sobolev embedding on $(0, \infty)$ implies that $u(r) \in C([0, \infty))$. Then the equation $Pu = 0$ guarantees that $u \in H^2(R, \infty) \subset C^1([R, \infty))$ for any $R > 0$. To analyze the behavior of the solution at infinity, we integrate the equation $Pu = 0$ in the interval (R, R_1) and find

$$|u'(R) - u'(R_1)| \leq \left(\int_R^{R_1} W(r)^2 dr \right)^{1/2} \|u\|_{L^2} \leq \frac{C}{R^{3/2}}.$$

The assumption $u \in H^1(0, \infty)$ easily yields

$$|u'(R)| \leq \frac{C}{R^{3/2}}, \quad |u(R)| \leq \frac{C}{R^{1/2}}. \quad (43)$$

From the asymptotic expansion obtained in the previous lemma we have also

$$\left| u'(R) + \frac{\alpha}{R} u(R) \right| \leq \frac{C}{R^{-a+\alpha+3}}. \quad (44)$$

Make the change of variables

$$r \rightarrow s = \frac{1}{r}, \quad u(r) \rightarrow v(s) = su(1/s). \quad (45)$$

If $u \in C^1([R, \infty))$, then $v \in C^1([0, 1/R])$ and the inequalities (43) imply

$$|v'(s)| \leq Cs^{1/2}, \quad |v(s)| \leq Cs^{3/2}. \quad (46)$$

We have also

$$\left| v'(s) - \frac{(\alpha+1)v(s)}{s} \right| \leq Cs^{a-\alpha-3}.$$

Hence,

$$v(0) = 0, v'(0) = 0. \quad (47)$$

Note that

$$|v(s)|^2 s^{-2(\alpha+1)} = 2 \int_0^s \tau^{-2(\alpha+1)} \left(v'(\tau) - \frac{(\alpha+1)v(\tau)}{\tau} \right) v(\tau) d\tau = O(s^{a-3\alpha-3})$$

due to the assumption $a > 3\alpha + 4$. Hence,

$$|v(s)|^2 \leq Cs^{a-2\alpha-1}, \quad |v'(s)|^2 \leq Cs^{a-2\alpha-2}. \quad (48)$$

The equation for v is the following one:

$$v''(s) + W^*(s)v(s) = 0, \quad (49)$$

where here and below

$$W^*(s) = \frac{W(1/s)}{s^4}. \quad (50)$$

The assumption $W(r) = W_1(r) + W_2(r)$, $W_1(r) = -\alpha(\alpha+1)/r^2$ shows that

$$W^*(s) = W_1^*(s) + W_2^*(s), \quad W_1^*(s) = -\frac{\alpha(\alpha+1)}{s^2}.$$

One can rewrite (49) as follows:

$$\left(v'(s) + \frac{\alpha}{s}v \right)' - \frac{\alpha}{s} \left(v'(s) + \frac{\alpha}{s}v \right) + W_2^*(s)v(s) = 0. \quad (51)$$

Recall that $v(s)$ is a real-valued function. Multiplying this equation by $s^{-2\alpha}(v' + \alpha v/s)$, one can use the relation

$$\left(f'(s) - \frac{\alpha}{s}f(s) \right) s^{-2\alpha}f(s) = \frac{1}{2} \left(s^{-2\alpha}f^2(s) \right)'$$

with $f(s) = (v' + \alpha v/s)$; so setting

$$Y(s) = \sup_{0 \leq \tau \leq s} \frac{1}{2} \tau^{-2\alpha} \left(v'(\tau) + \frac{\alpha}{\tau}v(\tau) \right)^2$$

and integrate from 0 to s . Using the estimates (48), we obtain $Y(0) = 0$, and

$$Y(s) \leq \int_0^s W_2^*(\tau) v(\tau) Y^{1/2}(\tau) d\tau. \quad (52)$$

Note also that $v(s)$ is real-valued function and we have the inequalities

$$\begin{aligned} |v(s)|^2 &= 2 \int_0^s v'(\tau) v(\tau) d\tau = 2 \int_0^s \left(v'(\tau) + \frac{\alpha}{\tau} v(\tau) \right) v(\tau) d\tau \\ &\quad - 2(\alpha) \int_0^s \frac{v(\tau)^2}{\tau} d\tau \leq 4s^\alpha \int_0^s \sqrt{Y(\tau)} |v(\tau)| d\tau \end{aligned}$$

These inequalities easily imply

$$|v(s)|^2 \leq C s^{2+2\alpha} Y(s). \quad (53)$$

Turning back to the estimate (52) we get the inequality

$$Y(s) \leq C \int_0^s W_2^*(\tau) Y(\tau) d\tau. \quad (54)$$

The assumption (34) shows that

$$|W_2^*(s)| = s^{-4} |W_2(s^{-1})| \leq C s^{a-4}$$

and integrability of this function on any interval $0 < s < R$ is guaranteed by the fact that $a > 3$.

An application of the Gronwall inequality shows that $Y(s) \equiv 0$, so from (53) we deduce $v \equiv 0$. This completes the proof of the lemma.

4. The case of half line: asymptotic expansions of resonance solution.

Our next step is to see if 0 is a resonance.

DEFINITION 4.1. *A real number λ is called resonance of*

$$P = \left(\frac{d}{dr} \right)^2 + W(r)$$

if there exists $u \in C([0, \infty))$, such that $u(0) = 0$, $u(r)$ is not identically zero, $P(u) = \lambda u$ in the distributional sense in $(0, \infty)$ and the solution u satisfies the inequality

$$|u(r)| \leq C r^\beta, \quad r \geq 1 \quad (55)$$

with some $0 \leq \beta < 1$.

LEMMA 4.2. *Assume $a > 2\alpha + 3$ in the assumption (34). If 0 is a resonance of P and $Pu = 0$ in the sense of Definition 4.1, then one can find a real number C_1 so that*

$$u(r) = \frac{C_1}{r^\alpha} + O(r^{-a+\alpha+3}) \quad (56)$$

and

$$u'(r) = -\frac{C_1\alpha}{r^{\alpha+1}} + O(r^{-a+\alpha+2}) \quad (57)$$

as $r > 1$ tends to infinity.

Proof. One can rewrite the equation $Pu = 0$ as

$$\left[r^{-\alpha} \left(u'(r) + \frac{\alpha}{r} u(r) \right) \right]' + r^{-\alpha} W_2(r) u(r) = 0. \quad (58)$$

Note that the assumption $W_2(r) \in C^1(0, \infty)$ combined with the equation $Pu = 0$ imply that $u \in C^2(R_1, R_2)$ for any $0 < R_1 < R_2$. Integrating (58) in the interval (R_1, R_2) , we find

$$\begin{aligned} & \left| R_2^{-2\alpha} (R_2^\alpha u(R_2))' - R_1^{-2\alpha} (R_1^\alpha u(R_1))' \right| \\ &= \left| \int_{R_1}^{R_2} \tau^{-\alpha} W_2(\tau) u(\tau) d\tau \right|, \end{aligned} \quad (59)$$

so using the assumption (34) together with (4.2) and taking $R_1 > 1$ we find

$$\left| \int_{R_1}^{R_2} \tau^{-\alpha} W_2(\tau) u(\tau) d\tau \right| \leq CR_1^{-a+2}.$$

In this way we conclude that the limit exists

$$\exists C_0 \equiv \lim_{r \rightarrow \infty} r^{-2\alpha} (r^\alpha u(r))' \quad (60)$$

and we have the expansion

$$(r^\alpha u(r))' = C_0 r^{2\alpha} + O(r^{-a+2\alpha+2}). \quad (61)$$

Consider the function

$$g(r) = r^\alpha u(r) - C_0 \frac{r^{2\alpha+1}}{2\alpha+1}.$$

Then (61) implies that $g'(r) = O(r^{-a+2\alpha+2}) \in L^1(1, \infty)$, since $a > 2\alpha + 3$ and we see that $g(r)$ has a limit (say C_1) as r goes to ∞ and

$$g(r) = C_1 - \int_r^\infty g'(\tau) d\tau = C_1 + O(r^{-a+2\alpha+3}).$$

Then we obtain

$$u(r) = C_0 \frac{r^{\alpha+1}}{2\alpha+1} + \frac{C_1}{r^\alpha} + O(r^{-a+\alpha+3}) \quad (62)$$

and

$$u'(r) = C_0 \frac{(\alpha+1)r^\alpha}{2\alpha+1} - \frac{C_1\alpha}{r^{\alpha+1}} + O(r^{-a+\alpha+2}) \quad (63)$$

as $r > 1$ tends to infinity.

Comparing these asymptotic developments with the assumption (55) we see that $C_0 = 0$ and this completes the proof of the lemma.

LEMMA 4.3. *Assume $a > 2\alpha + 3$ in the assumption (34) and $\alpha > 1/2$, then 0 is not a resonance of P in the sense of Definition 4.1.*

Proof. The assertion is trivial, since the asymptotic expansions (56) and (57) with $\alpha > 1/2$ guarantee that the solution u of $Pu = 0$ belongs to H^1 and the Lemma 3.3 implies $u = 0$.

More interesting case is $\alpha = 0$. Then we have the following eigenvalue problem

$$u''(r) + W_2(r)u = \mu^2 u, \quad \mu \geq 0, \quad r > 0, \quad (64)$$

satisfying the boundary condition

$$u(0) = 0. \quad (65)$$

For the case $\alpha = 0$ we shall assume that

$$W_2(r) = \int_r^\infty \left(\frac{1}{r} - \frac{1}{s} \right) \phi(s) ds, \quad (66)$$

where $\phi(s)$ is a smooth exponentially decaying positive function, such that

$$\int_0^\infty \phi(s) ds = 1. \quad (67)$$

One can use Lemma 4.1 in [1] to verify that (66) is indeed satisfied for the Hartree model studied here.

LEMMA 4.4. *Assume $a > 3$ in the assumption (34), $\alpha = 0$ and $W_2(r)$ is defined by (66). Then 0 is not a resonance of P in the sense of Definition 4.1.*

Proof. We shall need the asymptotic expansions (62) and (63). Take any smooth function $g(r)$ on $(0, \infty)$ such that $g(r)$ has at most polynomial growth at infinity, is integrable near 0 and define $G(r) = \int_0^r g(\tau) d\tau$. Multiplying the equation

$$P(u) \equiv u''(r) + W_2(r)u = 0$$

by u' and integrating over $s \in (r, \infty)$ and then multiplying again by the weight function $g(r)$ and integrating over $r \in (0, \infty)$, we find

$$\begin{aligned} \int_0^\infty \frac{g(r)|u'(r)|^2}{2} dr + \int_0^\infty \frac{g(r)W_2(r)|u(r)|^2}{2} dr \\ + \int_0^\infty \frac{G(r)W_2'(r)|u(r)|^2}{2} dr = 0. \end{aligned} \quad (68)$$

Taking a smooth function $h(r)$ on $(0, \infty)$ such that $h(r)$ has at most polynomial growth at infinity, is integrable near 0, $h''(r)$ is integrable over $(1, \infty)$, we multiply the equation $Pu = 0$ by hu and integrate over $(0, \infty)$, so we get

$$\begin{aligned} \int_0^\infty \frac{h''(r)|u(r)|^2}{2} dr - \int_0^\infty h(r)|u'(r)|^2 dr \\ + \int_0^\infty h(r)W_2(r)|u(r)|^2 dr = 0. \end{aligned} \quad (69)$$

Choosing $h = g/2$ and summing the above two relations we obtain (modulo factor $1/2$)

$$\begin{aligned} \int_0^\infty (2g(r)W_2(r) + G(r)W_2'(r)) |u(r)|^2 dr \\ + \int_0^\infty \frac{g''(r)|u(r)|^2}{2} dr = 0. \end{aligned} \quad (70)$$

In the case $g(r) = r^{N-1}$ for $r < N^3/3$, where $N > 2$ is a large parameter, we have $G(r) = r^N/N$ for $r < N^3/3$, so we can rewrite (70) as

$$\begin{aligned} \int_0^{N^3/3} \left(\int_r^\infty \left(\frac{(N-2)(N-1)}{2r} - \frac{1}{N} + 2 - 2\frac{r}{s} \right) \phi(s) ds \right) r^{N-2} |u(r)|^2 dr \\ + O(N^{-M}) = 0, \end{aligned} \quad (71)$$

where $M > 2$ can be chosen arbitrary. Note that

$$\left(\frac{(N-2)(N-1)}{2r} - \frac{1}{N} + 2 - 2\frac{r}{s} \right) \geq \frac{C}{N}, \quad r < N^3/3, s \geq r.$$

So the integral in the left side of (71) can be estimated from below by

$$\frac{C}{N} \int_0^1 \int_1^\infty \phi(s) ds r^{N-3} |u(r)|^2 dr \geq \frac{C_1}{N} \int_0^1 \int_1^\infty \phi(s) ds r^{N-1} dr \geq \frac{C_2}{N^2},$$

where C, C_1, C_2 are positive constants independent of N . Comparing with (71) we arrive at a contradiction. \square

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