Topo-Bisimulations are Coalgebraic

Christoph Schubert

ABSTRACT. *We show that the topological interpretation of the modal logic **S4** can be reformulated using a special kind of coalgebras for the filter functor. Thus the topological semantics is subsumed in coalgebraic semantics. Moreover, the relational notion of topo-bisimulation can be characterized via spans of open and continuous maps of topological spaces or via spans of coalgebras morphisms.*

1. Introduction

In this paper we are concerned with the semantics for the modal logic commonly known as **S4**. Nowadays the commonly used semantics for modal logics is based on relations, called Kripke semantics in honour of Saul Kripke. For the special logic **S4** another semantics based on topological spaces as given by Alfred Tarski in [10]. Hence, this semantics predates the one given by Kripke. For a detailed historical overview we refer to [7]. In Tarskis semantics the necessity-modality $\square$ is interpreted using the interior-operator of a topological space.

The notion of bisimulation was introduced by Johan van Benthem as a semantic criterion for states in possible different Kripke frames to satisfy the same formulas of a modal language. Since then, this notion has become fundamental in various areas such as process graphs, labelled transition systems, non-wellfounded set theory, and automata theory.

Bisimulations for topological models, so-called topo-bisimulations were introduced in [3] to give a notion of bisimilarity for topo-models, that is, topological spaces together with a valuation which interprets propositional variables. Thus, topo-bisimulations are a notion of bisimilarity for Tarskis semantics of **S4**.

In a separate line of research it has been shown that Kripke-models, graphs, labelled transition systems, automata as well as other types of systems commonly studied in theoretical computer science can be modelled as coalgebras. Here, the type of coalgebras is parametrized by a functor on the category of sets and functions. Choosing different functors allows us to obtain the differ-

ent kinds of structures mentioned above as coalgebras for this specific functor. Moreover, coalgebras allow us to give an encompassing definition of the notion of bisimulation in these fields: the coalgebraic view on bisimulations just states that we have a span of coalgebra-morphisms. This notion turns out to be equivalent to the special ones studied in the aforementioned fields.

We can interpret modal logics in general coalgebras using so-called predicate liftings. This gives us a way of modelling the different semantics of modal logic using coalgebras. For instance, the usual Kripke-semantics can be modelled using predicate liftings for the powerset-functor.

It has long been known that a topological spaces induces a coalgebra for the filter functor, essentially by the neighborhood mapping. We recall some results from the theory of lax algebras [9] which allow us to give a concise description of those coalgebras which arise from topological spaces in this way via a monad structure on the filter functor. Hence, the topological interpretation of **S4**-type modal logics can be seen as a branch of coalgebraic logic as well.

Moreover, we show that the notion of topo-bisimulation is in fact equivalent to the coalgebraic one. This allows us to derive some properties of topo-bisimulations from well-known facts of coalgebras and of open functions between topological spaces.

Acknowledgements. The author would like to thank the referee for the helpful suggestions.

2. Preliminaries

2.1. Modal Logic

We use the following basic language $\mathcal{L}$ of propositional modal logic. Formulas ϕ of $\mathcal{L}$ are given by the following grammar:

$$\phi ::= p \mid \top \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi$$

where $p \in \mathsf{Var}$ is a propositional variable. We make use of the usual abbreviations of propositional calculus, such as $\bot = \neg\top$ and $\phi \to \psi = \neg\phi \vee \psi$, plus $\Diamond\phi = \neg\Box\neg\phi$.

We are mainly concerned with the modal logic **S4**, which can be axiomatized by

$$\Box\top$$
$$(\Box\phi \wedge \Box\phi) \leftrightarrow \Box(\phi \wedge \psi)$$
$$\Box\phi \to \Box\Box\phi$$
$$\Box\phi \to \phi$$

together with the inference rules modus ponens (MP) and monotonicity (M):

$$\frac{\phi \to \psi \quad \phi}{\psi} \quad \text{(MP)} \qquad\qquad \frac{\phi \to \psi}{\Box\phi \to \Box\psi} \quad \text{(M)}$$

2.2. Topological Semantics of S4

We write **Top** for the category of topological spaces and continuous functions. A continuous function $f : X \longrightarrow Y$ between topological spaces is called *open* if the image of every open subset of X is open in Y. We write $\textbf{Top}_{\text{open}}$ for the category of topological spaces and continuous and open maps.

A *topo-model* with respect to some set Var of propositional variables is given by a topological space (X, τ) together with a valuation, that is, a function $V : X \to P\text{Var}$. We fix the set Var from now on and assume that it is non-empty.

We may interpret the formulas of the modal language $\mathcal{L}$ in a topo-model (X, τ, V) as follows [10]:

$$\begin{aligned}
[\![p]\!] &= \{ x \mid p \in V(x) \} \qquad \text{for } p \in \text{Var} \\
[\![\top]\!] &= X \\
[\![\neg\phi]\!] &= X \setminus [\![\phi]\!] \\
[\![\phi_1 \wedge \phi_2]\!] &= [\![\phi_1]\!] \cap [\![\phi_2]\!] \\
[\![\Box\phi]\!] &= \text{int}_\tau[\![\phi]\!],
\end{aligned} \qquad (1)$$

where $\text{int}_\tau A$ denotes the interior of the set A with respect to the topology τ, thus $\text{int}_\tau A = \bigcup \{ B \in \tau \mid B \subset A \}$. This semantics turns out to be sound and complete for the modal logic **S4**. Indeed, the validity of the first two axiom-schemes follows from the fact that int_τ preserves finite intersections. The validity of the remaining two axiom-schemes follows since int_τ is a (idempotent) interior operator and thus satisfies $\text{int}_\tau A \subset A$ and $\text{int}_\tau \cdot \text{int}_\tau = \text{int}_\tau$ for all $A \subset X$. We refer the reader to [4] for details.

2.3. Coalgebras

We fix an endofunctor T on the category **Set** of sets and functions. A *T-coalgebra* is given by a set X together with a function $a : X \longrightarrow TX$. A morphism $f : (X, a) \longrightarrow (Y, b)$ is given by a function $f : X \longrightarrow Y$ for which

$$\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
TX & \xrightarrow[\;Tf\;]{} & TY
\end{array}$$

commutes. We write **Coalg**T for the category of T-coalgebras and coalgebra morphisms. For further information on coalgebras we refer to [1].

A prominent example of coalgebras are Kripke-frames, which arise as coalgebras for the powerset functor P with $PX = \{\, A \subset X \,\}$ and $Pf(A) = f[A]$. Indeed, every relation $R \subset X \times X$ gives a P-coalgebra (X, a_R) with $a_R(x) = \{\, y \in X \mid (x,y) \in R \,\}$. Coalgebra morphisms are just bounded morphisms (sometimes called p-morphisms). Thus, the coalgebraic notion of morphisms for Kripke-frames is just the one suitable for modal logic [5].

Another example for coalgebras are Markov-chains, which arise via a functor determined by finite probability distributions.

2.3.1. Models

Fix a set Var of variables and a functor T on **Set**. A T-*model* is given by a T-coalgebra (X, a) together with a valuation, that is: a function $V : X \longrightarrow P\mathsf{Var}$. We we say that the model (X, a, V) *lives over* the coalgebra (X, a).

A *morphism of models* $(X, a, V) \longrightarrow (Y, b, W)$ is given by a coalgebra morphism $f : (X, a) \longrightarrow (Y, b)$ which satisfies $W \cdot f = V$. Observe that we may consider models as coalgebras for the functor given by $X \longmapsto TX \times P\mathsf{Var}$. Model morphisms correspond then to coalgebra morphisms with respect to the functor $T \times P\mathsf{Var}$. Hence, in a certain sense there is no need to talk about models at all. We find it useful to separate models and coalgebras in order to speak about the validity of modal formulas in coalgebras, a concept which we will recall now.

2.3.2. Modal Logics from Predicate Liftings

Coalgebras can be seen as a general form of semantics for modal logic. The link which allows us to interpret a formula from $\mathcal{L}$ in a T-coalgebra for an arbitrary functor T on **Set** is given by predicate liftings [8]. A *predicate lifting* for T is a natural transformation $\lambda : P^- \longrightarrow P^- \cdot T$, with P^- the contravariant powerset functor. Thus, $P^- X = PX$ and for a function $f : X \longrightarrow Y$ we have $P^- f : PY \longrightarrow PX$ via $P^- f(B) = f^{-1}[B]$.

Given a predicate lifting λ for an endofunctor T we can interpret the formulas of $\mathcal{L}$ in any T-model (X, a, V) as follows: for each $\phi \in \mathcal{L}$ we define the set *validity-set* $[\![\phi]\!] \subset X$ by induction on the structure of ϕ, here we proceed as in (1), we only need to modify the definition for $\Box\phi$ by setting:

$$[\![\Box\phi]\!] = P^- a \cdot \lambda_X([\![\phi]\!]). \tag{2}$$

As usual, we write $x \vDash \phi$ if, and only if $x \in [\![\phi]\!]$.

We say a modal formula is *valid* at a state of a coalgebra if it is valid in the corresponding state of *any model* over the coalgebra. We say a formula ϕ is *valid* in a coalgebra if it is valid at any state.

EXAMPLE 2.1. *Take for T the powerset-functor P and as predicate lifting the natural transformation (λ_X) given by*

$$\lambda_X(A) = \{\, B \subset X \mid B \subset A \,\}.$$

The semantics that λ induces for (X, a_R), R a relation on X, turns out to be

$$x \vDash \Box\phi \iff \forall y \in X : ((x, y) \in R \implies y \vDash \phi),$$

thus, the usual interpretation of $\Box$.

On the other hand, choosing $\kappa_X(A) = \{\, B \subset X \mid A \cap B \neq \emptyset \,\}$ as predicate lifting, we arrive at the following semantics:

$$x \vDash \Box\phi \iff \exists y \in X : ((x, y) \in R \land y \vDash \phi),$$

thus at the interpretation normally associated with $\Diamond$.

2.4. Filters

A *filter* on a set X is a family $\mathfrak{a}$ of subsets of X which satisfies:

- $X \in \mathfrak{a}$,

- $A \subset B$ and $A \in \mathfrak{a}$ imply $B \in \mathfrak{a}$,

- $A, A' \in \mathfrak{a}$ implies $A \cap A' \in \mathfrak{a}$.

We write FX for set of filters on a set X. FX is ordered by setting

$$\mathfrak{a} \leq \mathfrak{a}' \iff \mathfrak{a} \supset \mathfrak{a}'.$$

By setting $Ff(\mathfrak{a}) = \{\, B \subset Y \mid f^{-1}[B] \in \mathfrak{a} \,\}$ for a function $f : X \longrightarrow Y$ we can extend F to a functor on the category of sets. For every $x \in X$, $\eta_X(x) = \{\, A \subset X \mid x \in A \,\}$ is a filter.

LEMMA 2.2. *The family $\left((-)_X^{\#} : P^- X \to P^- FX \right)_X$ with $A_X^{\#} = \{\, \mathfrak{a} \in FX \mid A \in \mathfrak{a} \,\}$ is a natural transformation. Each $(-)_X^{\#}$ preserves finite intersections.*

Proof. Take any function $f : X \longrightarrow Y$. We have to show that

$$
\begin{array}{ccc}
PX & \xrightarrow{\;(\,)^{\#}\;} & PFX \\
{\scriptstyle P^- f}\Big\uparrow & & \Big\uparrow{\scriptstyle P^- Ff} \\
PY & \xrightarrow[\;(\,)^{\#}\;]{} & PFY
\end{array}
$$

commutes. For $B \subset Y$, we have:

$$\begin{aligned}
\mathfrak{a} \in P^- Ff(B^{\#}) &\iff Ff(\mathfrak{a}) \in B^{\#} \\
&\iff B \in Ff\mathfrak{a} \\
&\iff f^{-1}[B] \in \mathfrak{a} \\
&\iff \mathfrak{a} \in (f^{-1}[B])^{\#}.
\end{aligned}$$

$\square$

Thus, we have exhibited $(-)^{\#}$ as a predicate lifting for the filter functor F. Observe that the corresponding logic is *normal* (due to the fact that each $(-)_X^{\#}$ preserves finite intersections).

2.4.1. Kleisli Composition

We order the set of functions $X \longrightarrow FY$ pointwise. For $a : X \longrightarrow FX$, we define $\hat{a} : FX \longrightarrow FY$ by

$$\hat{a}(\mathfrak{x}) = \{\, B \subset Y \mid a^{-1}[B^{\#}] \in \mathfrak{x} \,\}.$$

Observe that $\hat{a}$ is monotone and that $\hat{}$ is monotone as well. One easily shows that $\hat{a} \cdot Ff = \widehat{a \cdot f}$ and $\widehat{Fg \cdot a} = Fg \cdot \hat{a}$ holds for $f : W \longrightarrow X$, $g : Y \longrightarrow Z$.

Given $a : X \longrightarrow FY$, $b : Y \longrightarrow FZ$, we a define $b \circ a : X \longrightarrow FZ$ as

$$b \circ a(x) = \hat{b}(a(x)) = \{\, C \subset Z \mid b^{-1}[C^{\#}] \in a(x) \,\}.$$

Thus $b \circ a(x) = \{\, C \subset Z \mid \{\, y \in Y \mid C \in b(y) \,\} \in a(x) \,\}$. The composition function $\circ$ is associative, and we have

$$\eta_Y \circ a = a = a \circ \eta_X$$

for each $a : X \longrightarrow FY$. In fact, $\circ$ is the Kleisli-composition of the monad structure on F given by unit η and multiplication given by $\mu_X(\mathfrak{A}) = \{\, A \subset X \mid A^{\#} \in \mathfrak{A} \,\}$.

2.5. Post-Fixpoints

Before we discuss the precise relationship between topological spaces and F-coalgebras, we state some elementary facts about monotone functions on pre-ordered sets. Let $t : V \longrightarrow V$ be a monotone function on a preordered set V. Call $x \in V$ a *post-fixpoint* of t if $x \leq t(x)$ holds. The following result on post-fixpoints is obvious.

LEMMA 2.3. *Let V be a complete lattice and $t : V \longrightarrow V$ be monotone.*

1. *The set of post-fixpoints of t is closed under suprema.*

2. *The set of post-fixpoints of t is closed under finite infima provided t preserves finite infima.*

LEMMA 2.4. *Let X and Y be preordered sets and assume that we have a commutative diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

of monotone functions. If x is a post-fixpoint of t, then $f(x)$ is a post-fixpoint of s.

Moreover, if f has a left-adjoint g and $y \in Y$ is a post-fixpoint of s, then $g(y)$ is a post-fixpoint of t.

Proof. The first claim is obvious. For the second result, fix $y \leq s(y)$. We have, by monotonicity of s:

$$ y \leq s(y) \leq s \cdot f \cdot g(y) = f \cdot t \cdot g(y) $$

which is equivalent to $g(y) \leq t(g(y))$ by $g \dashv f$. $\qquad\square$

3. Topological Spaces versus Filter-Coalgebras

For a F-coalgebra $(X, a : X \longrightarrow FX)$ we write t_a for the composite $P^- a \cdot (-)^{\#}_X :$ $PX \longrightarrow PX$. Clearly, t_a preserves finite intersections. Write $\mathcal{O}_a$ for the set of post-fixpoints of t_a. By Lemma 2.3, $\mathcal{O}_a$ is a topology on X.

PROPOSITION 3.1. *Let $(X, a) \xrightarrow{\ f\ } (Y, b)$ be a F-coalgebra morphism. Then $(X, \mathcal{O}_a) \xrightarrow{\ f\ } (Y, \mathcal{O}_b)$ is continuous and open; that is: $B \in \mathcal{O}_b$ implies $f^{-1}[B] \in \mathcal{O}_a$, and $A \in \mathcal{O}_a$ implies $f[A] \in \mathcal{O}_b$*

Proof. Commutativity of

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
FX & \xrightarrow[\ Ff\ ]{} & FY
\end{array}
$$

entails commutativity of

$$
\begin{array}{ccc}
PX & \xleftarrow{\ P^- f\ } & PY \\
{\scriptstyle(\ )^{\#}}\downarrow & & \downarrow{\scriptstyle(\ )^{\#}} \\
PFX & \xleftarrow{\ P^- Ff\ } & PFY \\
{\scriptstyle P^- a}\downarrow & & \downarrow{\scriptstyle P^- b} \\
PX & \xleftarrow{\ P^- f\ } & PY
\end{array}
$$

with outer arrows t_a (left) and t_b (right).

and thus the claim follows from Lemma 2.4 using the adjunction $Pf \dashv P^- f$. $\quad\square$

Thus, we have shown that the assignment $O : (X, a) \longmapsto (X, \mathcal{O}_a)$ defines a functor from $\mathbf{Coalg}F \longrightarrow \mathbf{Top}_{\mathrm{open}}$.

3.1. Topological Spaces among Filter-Coalgebras

Every topological space X gives rise to a F-coalgebra by letting $a(x)$ be the filter of all neighborhoods of the point x. Recall that a subset A of X is a *neighborhood* of x if there is an open subset U with $x \in U$ and $U \subset A$. Recall, e.g. from [6, I.1.2] that a function $a : X \longrightarrow FX$ arises in this way from a (unique) topology if, and only if,xs

(N1) $x \in A$ for all $A \in a(x)$,

(N2) $A \in a(x)$ implies that there exists $B \in a(x)$ such that $A \in a(y)$ for all $y \in B$.

LEMMA 3.2. *$a : X \longrightarrow FX$ arises as the neighborhood map of a (unique) topology if, and only if, $\eta_X \leq a$ and $a \circ a \leq a$ hold.*

Proof. We just need to show that $A \in a \circ a(x)$ is equivalent to the existence of $B \in a(x)$ such that $A \in a(y)$ for all $y \in B$. We have

$$
A \in a \circ a(x) \iff \{\, y \in X \mid A \in a(y) \,\} \in a(x)
$$

and the last property is equivalent to the existence of an arbitrary $B \in a(x)$ with $A \in a(y)$ for all $y \in B$ by virtue of $a(x)$ being upwards closed. $\quad\square$

If X is a topological space, we write NX for the F-coalgebra given by its neighborhoods.

LEMMA 3.3. *Let X, Y be topological spaces, $f : X \longrightarrow Y$ a continuous and open function. Then f is a coalgebra morphism $NX \longrightarrow NY$.*

Proof. Write (X, a) for NX and (Y, b) for NY and fix $x \in X$ and $B \subset Y$.

Assume $B \in b(f(x))$. There exists an open V with $f(x) \in V \subset B$. Since f is continuous $f^{-1}[V]$ is open in X and, since $x \in f^{-1}[V]$, $f^{-1}[V] \in a(x)$. Hence $V \in Ff \cdot a(x)$, and $B \in Ff \cdot a(x)$ follows.

Assume now that $B \in Ff \cdot a(x)$ holds. Then $f^{-1}[B] \in a(x)$ and we find $A \subset X$ open with $xinA$ and $A \subset f^{-1}[B]$. Since f is open, also $f[A]$ is open, and we obtain $f[A] \subset f[f^{-1}[B]] \subset B$. Moreover, we have $f(x) \in f[A]$, hence B is a neighborhood of $f(x)$, and $B \in b(f(x))$ follows. $\qquad\square$

Thus, N defines a concrete functor $\mathbf{Top}_{\text{open}} \longrightarrow \mathbf{Coalg}\, F$. By Lemma 3.2 the image of N can be characterized as those coalgebra $(X, a : X \longrightarrow FX)$ which are *reflexive* and *transitive* in the sense that they satisfy $\eta_X \leq a$ and $a \circ a \leq a$. Let us write $\mathbf{RTAlg}\, F$ for the full subcategory of $\mathbf{Coalg}\, F$ spanned by the reflexive and transitive coalgebras.

Observe that N actually factors through the embedding $\mathbf{RTAlg}\, F \to \mathbf{Coalg}\, F$. Hence, we obtain a diagram

$$
\begin{array}{ccc}
\mathbf{Top}_{\text{open}} & \underset{O}{\overset{N}{\rightleftarrows}} & \mathbf{RTAlg}\, F \\
\downarrow & & \uparrow{\scriptstyle E} \\
\mathbf{Top} & & \mathbf{Coalg}\, F
\end{array}
$$

where N and O on the top row are mutually inverse.

3.2. Fundamental Constructions on RTAlg F

We will study limits and colimits in $\mathbf{RTAlg}\, F$ and their relationship to the corresponding constructions in $\mathbf{Coalg}\, F$. Of course, this boils down to establishing properties of the category of topological spaces and open maps.

We call a family $(X_i \xrightarrow{e_i} X)_{i \in I}$ *jointly surjective* if $X = \bigcup_{i \in I} e_i[X_i]$ holds.

PROPOSITION 3.4. $\mathbf{RTAlg}\, F$ *is closed under jointly surjective sinks in* $\mathbf{Coalg}\, F$. *That is, if* $((X_i, a_i) \xrightarrow{e_i} (X, a))_{i \in I}$ *is a jointly surjective sink in* $\mathbf{Coalg}\, F$ *such that each* (X_i, a_i) *is reflexive and transitive, then* (X, a) *is reflexive and transitive.*

Proof. It suffices to show that $(a \circ a) \cdot e_i \leq a \cdot e_i$ and $\eta_X \cdot e_i \leq a \cdot e_i$ hold for all

$i \in I$.

$$
\begin{aligned}
(a \circ a) \cdot e_i &= \hat{a} \cdot a \cdot e_i \\
&= \hat{a} \cdot Fe_i \cdot a_i && \text{since } e_i \text{ is coalgebra morphism} \\
&= \widehat{a \cdot e_i} \cdot a_i \\
&= \widehat{Fe_i \cdot a_i} \cdot a_i && \text{since } e_i \text{ is coalgebra morphism} \\
&= Fe_i \cdot \hat{a}_i \cdot a_i \\
&\leq Fe_i \cdot a_i && \text{since } (X_i, a_i) \text{ is transitive and } Fe_i \text{ is monotone} \\
&= a \cdot e_i && \text{since } e_i \text{ is coalgebra morphism}
\end{aligned}
$$

Along the same lines:

$$
\eta_X \cdot e_i = Fe_i \cdot \eta_{X_i} \leq Fe_i \cdot a_i = a \cdot e_i,
$$

where the first equation is naturality of η and the inequality follows from monotonicity of Fe_i. $\qquad \square$

THEOREM 3.5. **RTAlg** F *is mono-coreflective in* **Coalg** F.

This means that, for each F-coalgebra (X, a), we may find a reflexive and transitive F-coalgebra $(\overline{X}, \overline{a})$ and an injective coalgebra morphism $m : (\overline{X}, \overline{a}) \longrightarrow (X, a)$ with the following property: for each (Y, b) in **RTAlg** F and each coalgebra morphisms $f : (Y, b) \longrightarrow (X, a)$, there exists a unique morphisms $\overline{f} : (Y, b) \longrightarrow (\overline{X}, \overline{a})$ with $f = m \cdot \overline{f}$.

Proof. We will only sketch a proof. Observe that we may factor any sink $(X_i \xrightarrow{f_i} X)_{i \in I}$ in the category of sets and function as $f_i = m \cdot e_i$ with m an injective function and $(e_i)_{i \in I}$ jointly surjective. This is in fact a factorization system for sinks in the sense of [2]. Since F preserve injectivity of functions, we can lift this factorization system to **Coalg** F. By (the dual of) [2, Theorem 16.8] and Proposition 3.4, **RTAlg** F is mono-coreflective in **Coalg** F. $\qquad \square$

COROLLARY 3.6. **RTAlg** F *is cocomplete. Hence,* $\mathbf{Top}_{\mathrm{open}}$ *is cocomplete as well.*

Proof. **Coalg** F is cocomplete since the category of sets and functions is so. Since **RTAlg** F is coreflective in **Coalg** F, it is cocomplete as well [2]. $\qquad \square$

4. Topo-Models and Coalgebras

We recall the following definition from [3]: Given two topo-models (X, τ, V) and (Y, σ, W), a *topo-bisimulation* is given by a relation $R \subset X \times Y$ such that the following conditions hold for each $x \in X$ and $y \in Y$:

(E) $xRy \implies V(x) = W(y)$;

(F) $xRy \implies \forall A \in \tau : [x \in A \implies \exists B \in \sigma : y \in B \wedge \forall y' \in B \exists x' \in A : x'Ry']$;

(B) $xRy \implies \forall B \in \sigma : [y \in B \implies \exists A \in \tau : x \in A \wedge \forall x' \in A \exists y' \in B : x'Ry']$.

Properties of topo-bisimulations are discussed in [3, 4]. For instance, topo-bisimilar states satisfy exactly the same formulas of the modal logic **S4**. We will give a characterization of topo-bisimulations in the spirit of coalgebras.

Given a relation $R \subset X \times Y$ as above, we write $X \xleftarrow{r_0} R \xrightarrow{r_1} Y$ for the projections. Observe that we can rewrite the second condition on B in (F) as $B \subset r_1[r_0^{-1}[A]]$, and that the second condition on A in (B) is equivalent to $A \subset r_0[r_1^{-1}[B]]$.

THEOREM 4.1. *Let (X, α) and (Y, β) be topological spaces, seen as F-coalgebras. Then a relation $R \subset X \times Y$ satisfies (F) and (B) from the definition of topo-bisimulation if, and only if, there exists a F-coalgebra structure $\gamma : R \longrightarrow FR$ such that*

$$\begin{array}{ccccc}
X & \xleftarrow{\quad r_0 \quad} & R & \xrightarrow{\quad r_1 \quad} & Y \\
\alpha \downarrow & & \gamma \downarrow & & \downarrow \beta \\
FX & \xleftarrow[Fr_0]{} & FR & \xrightarrow[Fr_1]{} & FY
\end{array} \qquad (3)$$

commutes. Moreover, γ can be chosen so that (R, γ) is reflexive and transitive.

Proof. Suppose there exists $\gamma : R \longrightarrow FR$ such that (3) commutes. Fix $(x, y) \in R$. We show that the forward-condition (F) holds. Take any α-open $A \subset X$ with $x \in A$. We have $A \in \alpha(x) = Fr_0 \cdot \gamma(x, y)$. Thus, $r_0^{-1}[A] \in \gamma(x, y)$. Write $B = r_1[r_0^{-1}[A]]$. By Proposition 3.1, B is β-open. We are left to show that $y \in B$.

We have $r_0^{-1}[A] \subset r_1^{-1} \cdot r_1 \cdot r_0^{-1}[A]$, thus $r_1^{-1} \cdot r_1 \cdot r_0^{-1}[A] \in \gamma(x, y)$. This implies that $B = r_1 \cdot r_0^{-1}[A] \in Fr_1 \cdot \gamma(x, y) = \beta(y)$. Finally, we can derive $y \in B$ since β is topological. The back condition (B) is shown in the same manner.

Now assume that R satisfies (F) and (B). Define $\gamma : R \longrightarrow FR$ as

$$\gamma(x, y) = \{\, C \subset R \mid \exists A \in \alpha(x), B \in \beta(y) : r_0^{-1}[A] \cap r_1^{-1}[B] \subset C \,\}.$$

It is straightforward to show that $\gamma(x, y)$ is a filter on R. Take any $(x, y) \in R$ and $A \in \alpha(x)$. Since α is topological, there exists an α-open U with $x \in U \subset A$. By (F), there exists B β-open with $y \in B$ and $B \subset r_1[r_0^{-1}[A]]$. Write $C = r_0^{-1}[A] \cap r_1^{-1}[B]$. We have $C \in \gamma(x, y)$. Since $C \subset r_0^{-1}[A]$, also $r_0^{-1}[A] \in \gamma(x, y)$, hence $A \in Fr_0 \cdot \gamma(x, y)$.

Take now $A \subset X$ with $r_0^{-1}[A] \in \gamma(x, y)$. We find $B \in \beta(y)$ and $A' \in \alpha(x)$ with $r_0^{-1}[A'] \cap r_1^{-1}[B] \subset r_0^{-1}[A]$. Since β is topological, we may assume that B is β-open and $y \in B$ holds. Thus, by (B), there is a α-open U with $x \in U$ and $U \subset r_0[r_1^{-1}[B]]$. We have

$$
\begin{aligned}
U \cap A' &\subset r_0[r_1^{-1}[B]] \cap A' \\
&= r_0[r_1^{-1}[B] \cap r_0^{-1}[A']] \\
&\subset r_0[r_0^{-1}[A]] \\
&\subset A.
\end{aligned}
$$

Thus $U \in \alpha(x)$ and $A' \in \alpha(x)$ imply $A \in \alpha(x)$, and we derive $Fr_0 \cdot \gamma \subset \alpha \cdot r_0$.

The equality $\beta \cdot r_1 = Fr_1 \cdot \gamma$ is shown in the same manner.

We shall now show that me may choose (R, γ) to be topological. It suffices to show that γ defined as above satisfies (N1) and (N2). Ad (N1): take any $(x, y) \in R$, $C \in \gamma(x, y)$. Thus there exists $A \in \alpha(x)$, $B \in \beta(y)$ with $r_0^{-1}[A] \cap r_1^{-1}[B] \subset C$. Both (X, α) and (Y, β) satisfy (N1), hence $x \in A$, $y \in B$. Thus $(x, y) \in r_0^{-1}[A]$ and $(x, y) \in r_1^{-1}[B]$, hence $(x, y) \in C$.

Ad (N2): take $C \in \gamma(x, y)$. There exists $A \in \alpha(x)$, $B \in \beta(y)$ with $r_0^{-1}[A] \cap r_1^{-1}[B] \subset C$. Since (X, α) and (Y, β) are topological, we find $A' \in \alpha(x)$, $B' \in \beta(y)$ with $A \in \alpha(x')$ for all $x' \in A'$ and $B \in \beta(y')$ for all $y' \in B'$.

Set $D = r_0^{-1}[A'] \cap r_1^{-1}[B'] \in \gamma(x, y)$. For any $(x', y') \in D$, we have $x' \in A'$, $y' \in B'$ and thus $A \in \alpha(x')$, $B \in \beta(y')$, whence $r_0^{-1}[A] \cap r_1^{-1}[B] \in \gamma(x', y')$, and thus $C \in \gamma(x', y')$. $\qquad\square$

COROLLARY 4.2. *Let $\mathcal{M} = (X, a, V)$ and $\mathcal{N} = (Y, b, W)$ be topo-models. Then a relation $R \subset X \times Y$ is a topo-bisimulation if, and only if,*

1. *$(x, y) \in R$ implies that $V(x) = W(y)$ holds, and*

2. *there is a topology on R such that the projections r_0 and r_1 become continuous and open functions.*

Proof. Obvious from Theorem 4.1 and the characterization of F-coalgebra morphisms as continuous and open functions. $\qquad\square$

Hence the relational notion of topo-bisimulation introduced in [3] may be rephrased using either coalgebraic notions or spans of continuous and open functions.

5. Modal Logic for Topological Spaces

Taking $T = F$ and $\lambda = (-)^{\#}$, we see that the validity-set transformer from (2) coincides with the "interior-operator" t_a. We write $\square$ for the modal operator induced on an F-coalgebra by $(\ )^{\#}$.

We will now show that one can use modal logic to axiomatize the topological spaces among the F-coalgebras:

THEOREM 5.1. *An F-coalgebra (X, a) arises from a topological space if, and only if, the formulas*

$$\Box\phi \to \phi, \qquad \Box\phi \to \Box\Box\phi$$

are valid in (X, a), where ϕ is a arbitrary formula in $\mathcal{L}$.

Proof. Assume that the formulas $\Box\phi \to \phi$ and $\Box\phi \to \Box\Box\phi$ are valid in (X, a), for any formula $\phi \in \mathcal{L}$; that is, for each model $\mathcal{M}$ over (X, a), we have $[\![\Box\phi]\!]_\mathcal{M} \subset [\![\phi]\!]_\mathcal{M}$ and $[\![\Box\phi]\!]_\mathcal{M} \subset [\![\Box\Box\phi]\!]_\mathcal{M}$ for each $\phi \in \mathcal{L}$.

We need to show that (X, a) satisfies (N1) and (N2). Take any $x \in X$ and $A \subset X$ a define a valuation $V : X \longrightarrow P\mathsf{Var}$ by

$$V(x) = \begin{cases} \mathsf{Var} & x \in A, \\ \emptyset & x \notin A. \end{cases}$$

Thus, in the model $\mathcal{M} = (X, a, V)$, we have $[\![p]\!]_\mathcal{M} = A$ for each $p \in \mathsf{Var}$. In the following we write $[\![\phi]\!]$ in place of $[\![\phi]\!]_\mathcal{M}$.

To establish (N1), assume $A \in a(x)$, that is, $[\![p]\!] \in a(x)$, which is equivalent to $x \in [\![\Box p]\!]$, thus $x \in [\![p]\!] = A$ by assumption.

To establish (N2), assume again $A \in a(x)$, thus $x \in [\![\Box p]\!]$, which implies $x \in [\![\Box\Box p]\!]$ by assumption. Write $B = [\![\Box p]\!]$, then $B \in a(x)$ follows from $x \in [\![\Box\Box p]\!]$. For any $y \in B$, we have $A = [\![p]\!] \in a(y)$ by construction. Thus, (N2) holds.

To show the converse implications, assume that (X, a) arises via a topological space. Assume $x \in [\![\Box\phi]\!]_\mathcal{M}$ for some model $\mathcal{M}$ over (X, a), where $x \in X$ and $\phi \in \mathcal{L}$ are arbitrary. Thus, $[\![\phi]\!]_\mathcal{M} \in a(x)$ holds, and $x \in [\![\phi]\!]_\mathcal{M}$ follows by (N1). Furthermore, $A[\![\phi]\!]_\mathcal{M} \in a(x)$ implies that there exists some $B \in a(x)$ such that $[\![\phi]\!]_\mathcal{M} \in a(y)$ for each $y \in B$. Since $a(x)$ is a filter, we obtain that the set $[\![\Box\phi]\!]_\mathcal{M}$ is in $a(x)$. This, in turn, is equivalent to $x \in [\![\Box\Box\phi]\!]_\mathcal{M}$. □

COROLLARY 5.2. *An F-coalgebra (X, a) arises from a topological space if, and only if, all the axioms of $\boldsymbol{S4}$ are valid in (X, a).* □

6. Further Work

We have seen that topo-bisimulations are in fact a special variant of coalgebraic bisimulations. In order to do so, we have characterized topological spaces among the coalgebras for the filter functor: topological spaces are the "reflexive" and "transitive" coalgebras. To formulate these notions of reflexivity and transitivity, we have made use of a monad structure on the filter functor.

From the viewpoint of coalgebraic logic, the following question arises: what happens when we replace the filter monad by an arbitrary monad on the category of sets? Of course choosing an arbitrary monad (T, η, μ) will not be very fruitful, one should be able to order the sets TX in such a way that the extension operator given by $\mu_Y \cdot T(-)$ is monotone with respect to the pointwise order on the set of functions $X \longrightarrow TY$.

In fact, most of results in this note can be transferred to a monad with these properties. This is work in progress.

References

[1] J. Adamék, *Introduction to coalgebra*, Theory Appl. Categ. **14** (2005), 157–199.

[2] J. Adamek, H. Herrlich and G. Strecker, *Abstract and concrete categories*, Wiley Interscience, U.S.A. (1990).

[3] M. Aiello and J. van Benthem, *Logical patterns in space*, in D. Barker-Plummer, D. Beaver, J. van Benthem, and P. Scotto di Luzio, *Words, Proofs, and Diagrams*, CSLI lecture notes volume 141, Stanford University Press, Stanford (2002), pag. 5–25.

[4] M. Aiello, J. van Benthem and G. Bezhanishvili, *Reasoning about space: the modal way*, J. Logic Comput. **13** (2003), 889.

[5] P. Blackburn, M. de Rijke and Y. Venema, *Modal logic*, Cambridge University Press, Cambridge (2001).

[6] N. Bourbaki, *Topological spaces*, Springer, Berlin (1968).

[7] R. Goldblatt, *Mathematical modal logic: a view of its evolution*, in D.M. Gabbay and J. Woods, *Handbook of the history of logic*, Elsevier (2006), pages 1–98.

[8] L. Schröder, *Expressivity of coalgebraic modal logics: The limits and beyond*, Theoret. Comp. Sci. **390** (2008), 230–247.

[9] C. Schubert and G. Seal, *Extensions in the theory of lax algebras*, Theory Appl. Categ. **21** (2008), 118–151.

[10] A. Tarski, *Der Aussagenkalkül und die Topologie*, Fund. Math. **31** (1938) 103–134.

Author's address:

Christoph Schubert
Chair of Software-Technology
Technische Universität Dortmund
44221 Dortmund Germany
E-mail: `christoph.schubert@tu-dortmund.de`