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Stating Infinity in Set/Hyperset Theory

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Abstract. *It is known that the Infinity Axiom can be expressed, even if the Axiom of Foundation is not assumed, in a logically simple form, by means of a formula involving only restricted universal quantifiers. Moreover, with Aczel's Anti-Foundation Axiom superseding von Neumann's Axiom of Foundation, a similar formula has recently emerged, which enjoys the additional property that it is satisfied only by (infinite) ill-founded sets. We give here new short proofs of both results.*

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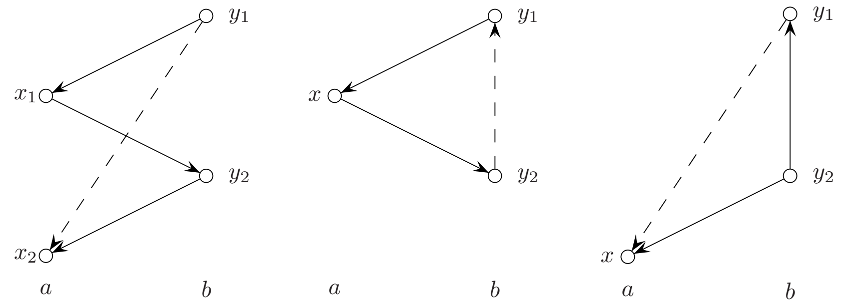
1. Introduction

Assuming the Foundation Axiom and the usual axioms of Zermelo-Fraenkel, except the Infinity Axiom, [7] pinpointed a $\exists\exists\forall\forall\forall\forall$-sentence, involving only restricted universal quantifiers, which entails the existence of infinite sets. A prenex sentence with two existential quantifiers in its leftmost position corresponds to a purely universal formula in two free variables, and the expressibility of infinitude in such a format (devoid of quantifier alternations) is peculiar of a set-theoretic setting. In [8], the formula in [7] was refined by exploiting the Axiom of Foundation: the sub-formula involving four universal quantifiers was replaced by one with two universal quantifiers; this resulted into the conjunction, to be called $\iota(a, b)$, of the following sub-formulae:

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii′) $(\forall x \in a)(\forall y \in x)(y \in b) \wedge (\forall x \in b)(\forall y \in x)(y \in a)$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$,



(a) Condition (iv′) of $\iota\iota(a, b)$ (b) Condition (v) of $\overline{\iota\iota}(a, b)$ (c) Condition (iv) of $\overline{\iota\iota}(a, b)$

Figure 1: Graphical representations of conditions (iv′), (v), and (iv).

where, as said, we are leaving the existential quantifiers binding a and b as understood.

When foundation is no longer assumed, infinity can still be expressed by means of an $\exists\exists\forall\forall\forall\forall$-sentence [9] by "merging" the results in [7] and [8], which amounts to adding the following condition to $\iota\iota(a, b)$:

(iv′) $(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_2 \in y_2 \in x_1 \in y_1 \rightarrow x_2 \in y_1)$.

Far from being a mere curiosity, $\iota\iota$ offered the clue (cf. [4]) for solving the satisfiability problem for $\exists^*\forall\forall$-sentences over the cumulative hierarchy often indicated as the intended model for ZF. A variant of $\iota\iota$ that involves $n + 2$ existential variables, where n can be any natural number, plays a similarly crucial role in [5], which addresses the satisfiability problem for the entire class of $\exists^*\forall^*$-sentences (still over the standard universe).

Variants of $\iota\iota$ have recently been shown, in connection with Aczel's ill-founded universe of sets [1], to state the existence of sets which, *besides being infinite, are also ill-founded*—cf. [6]. One of these formulae, to be indicated as $\overline{\iota\iota}$, results from $\iota\iota$ via replacement of (ii′) by

(ii) $(\forall x \in a)(\forall y \in x)(y \in b) \wedge$
$(\forall x \in b)(\forall y \in x)(y \in a \vee y \in b) \wedge (\forall y \in b)(y \notin y)$,

and by strengthening (iv′) into

(iv) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_1 \in y_2 \wedge x \in y_2 \rightarrow x \in y_1)$

(v) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_2 \in x \in y_1 \rightarrow y_1 \in y_2)$

(cf. Figure 1). We revisit here, striving to be brief, the facts that $\overline{\iota\iota}$ is satisfiable and that its satisfaction requires ill-founded, infinite values for a and b.

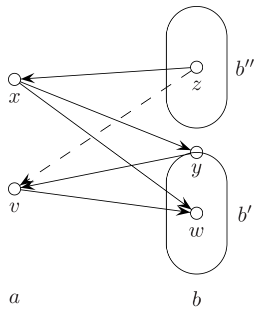


Figure 2: Graphical representation of the proof that $\mathbf{a}$ is infinite.

2. Infinite, Well-Founded Satisfiability

Before discussing, in Section 3, how to satisfy $\overline{\iota\iota}$, we turn our attention to $\iota\iota$. To see that $\iota\iota$ is satisfiable, consider the (well-founded) sets $\boldsymbol{\omega}_0, \boldsymbol{\omega}_1$:

$$\begin{array}{ll} \boldsymbol{\omega}_0 = \{\boldsymbol{\omega}_{1,i} : i \in \omega\}, & \boldsymbol{\omega}_{1,i} = \{\boldsymbol{\omega}_{0,j} : 0 \leqslant j \leqslant i\}, \\ \boldsymbol{\omega}_1 = \{\boldsymbol{\omega}_{0,i} : i \in \omega\}, & \boldsymbol{\omega}_{0,i} = \{\boldsymbol{\omega}_{1,j} : 0 \leqslant j \leqslant i-1\}, \quad i \in \omega \end{array}$$

(where, as customary, $\omega = \{0, 1, 2, \ldots\}$). It is plain that $\iota\iota(\boldsymbol{\omega}_1, \boldsymbol{\omega}_0)$ is true.

Next, assuming that $\mathbf{a}$ and $\mathbf{b}$ are sets such that $\iota\iota(\mathbf{a}, \mathbf{b})$ is true, we will now show that $\mathbf{a}$ is infinite. In its turn, $\mathbf{b}$ must be infinite, because $\bigcup \mathbf{a} \subseteq \mathbf{b}$ ensues from condition (ii$'$): trivially, in fact, for all sets x and a, the finiteness of $\mathscr{P}(x)$ ensues from x being finite and it holds that $a \subseteq \mathscr{P}(\bigcup a)$; therefore, if $\mathbf{b}$ were finite, then $\bigcup \mathbf{a}$ would be finite, $\mathscr{P}(\bigcup \mathbf{a})$ would be finite, and $\mathbf{a}$ would be finite.

Arguing by contradiction, assume that $\mathbf{a}$ is finite. As a preliminary, notice that $\mathbf{b} \neq \emptyset$, else (i) would imply the existence of a nonnull $c \in \mathbf{a}$, but then $c \subseteq \mathbf{b}$ would follow, by (ii$'$). For all $y \in \mathbf{b}$, $y \subsetneq \mathbf{a}$ holds by (ii$'$) and (i); hence, $\mathbf{a}$ being finite, we can consider an $\subseteq$-maximal $y \in \mathbf{b}$ and an $x \in \mathbf{a} \setminus y$, so that $y \in x$ holds by (iii). Let $b' = \{w \in \mathbf{b} \mid (\exists v \in y)(w \in v)\}$, and let $b'' = (\mathbf{b} \setminus b') \setminus \{y\}$. Conditions (iii) and (iv$'$) imply that $b' \subseteq x$, else $x \notin y$ would be violated. Since $\mathbf{b} \nsubseteq x$—else $\mathbf{b} \in \mathbf{a}$ would ensue, by (ii$'$)—, we can find a $z \in b'' \setminus x$, so that $x \in z$ holds by (iii). It will then follow, by the definition of b'', that $y \subsetneq z$ holds: indeed, since z belongs to no $v \in y$, and every $v \in y$ belongs to $\mathbf{a}$ by (ii$'$), (iii) entails $v \in z$ for all $v \in y$. But this strict inclusion contradicts the maximality of y.

Note: We could have avoided recourse to (iv$'$) in the argument of the preceding paragraph, had we placed ourselves under the Foundation Axiom.

3. Infinite, Ill-Founded Satisfiability

In order to get a pair $\boldsymbol{\omega}_0, \boldsymbol{\omega}_1$ of (ill-founded) sets such that $\overline{\iota\iota}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_0)$ is true, simply consider the system

$$\begin{array}{ll} \boldsymbol{\omega}_0 = \{\boldsymbol{\omega}_{1,i} : i \in \omega\}, & \boldsymbol{\omega}_{1,i} = \{\boldsymbol{\omega}_{0,j} : 0 \leqslant j \leqslant i\} \cup \{\boldsymbol{\omega}_{1,j} : j \in \omega \mid j > i\}, \\ \boldsymbol{\omega}_1 = \{\boldsymbol{\omega}_{0,i} : i \in \omega\}, & \boldsymbol{\omega}_{0,i} = \{\boldsymbol{\omega}_{1,j} : 0 \leqslant j \leqslant i-1\} \end{array}$$

$(i \in \omega)$, of equations and recall that, according to Aczel's anti-foundation axiom AFA, such a system admits one and only one solution over sets.

We will repeatedly exploit this plain consequence of AFA: If $\mathbf{A}, \mathbf{X}$ are sets such that $\mathbf{X} \neq \emptyset$, $\mathbf{A} \notin \mathbf{X}$, and for all $y \in \mathbf{X}$

$$\mathbf{A} \subseteq y \wedge y \setminus \mathbf{A} \subseteq \mathbf{X}$$

holds, then $\mathbf{X}$ is that unique set—to be designated as $\{\Omega_{\mathbf{A}}\}$ in what follows—which solves the equation $X = \{X \cup \mathbf{A}\}$.

We will henceforth assume that $\mathbf{a}, \mathbf{b}$ are sets such that $\overline{\iota\iota}(\mathbf{a}, \mathbf{b})$ holds. To see that $\mathbf{a} \cup \mathbf{b}$ cannot be well-founded, we argue as follows. Observe first that $\mathbf{a} \neq \emptyset$: indeed, if $\mathbf{a} = \emptyset$ held, then $\mathbf{b} \neq \emptyset \wedge \emptyset \notin \mathbf{b}$ and $\bigcup \mathbf{b} \subseteq \mathbf{b}$ would follow, by (i) and by (ii) respectively: but then $\mathbf{b}$ must be $\Omega_\emptyset$, against the subcondition $(\forall y \in b)(y \notin y)$ of (ii). Observe next that $\mathbf{a} \neq \{\emptyset\}$; for, assuming the contrary, we would have $\mathbf{b} \neq \emptyset$ and for every $y \in \mathbf{b}$, $\emptyset \in y$, by (i) and (iii) respectively. We would also have $y \setminus \{\emptyset\} \subseteq \mathbf{b}$ when $y \in \mathbf{b}$, by (ii); but then $\mathbf{b}$ would be the doubleton $\Omega_{\{\emptyset\}}$, against the condition $(\forall y \in b)(y \notin y)$. Therefore we can take an $x \in \mathbf{a} \setminus \{\emptyset\}$, and a $y \in x$; moreover, since $x \subsetneq \mathbf{b}$ by (ii) and (i), $y \in \mathbf{b}$ must hold, and we can take a $z \in \mathbf{b} \setminus x$. To conclude, observe that $x \in z$, because of (iii), and that consequently, by (v), $z \in y$. This membership cycle x, y, z witnesses that $\mathbf{a} \cup \mathbf{b}$ is ill-founded. (Incidentally, we have also seen that $\mathbf{b} \neq \emptyset$.)

Preliminary to proving that $\mathbf{a}$ and $\mathbf{b}$ are infinite, let us show that $(\forall y \in \mathbf{b})(\mathbf{a} \nsubseteq y)$. Arguing by contradiction, suppose that this is not the case, so that the set $\mathbf{X} = \{y \in \mathbf{b} \mid \mathbf{a} \subseteq y\}$ is nonnull. Then, by (ii) and (iv), we have $(\forall y \in \mathbf{X})(y \setminus \mathbf{a} \subseteq \mathbf{X})$. But then—since $\mathbf{a} \notin \mathbf{X}$ holds by (i)—we must have $\mathbf{X} = \{\Omega_{\mathbf{a}}\}$, contradicting the last conjunct of (ii).

We will now show the infinitude of $\mathbf{a}$, entailing the infinitude of $\mathbf{b}$ by the same argument given at the beginning of Section 2. Assume for a contradiction that $\mathbf{a}$ is finite, and consider a $y \in \mathbf{b}$ such that $|y \cap \mathbf{a}|$ is maximum. Since $\mathbf{a} \nsubseteq y$, there is an $x \in \mathbf{a} \setminus y$, so that $y \in x$ holds by (iii). Let $b' = \{w \in \mathbf{b} \mid (\exists v \in y \cap \mathbf{a})(w \in v)\}$, and let $b'' = \mathbf{b} \setminus b' \setminus \{y\}$. From (v) we get $(\forall w \in b')(y \in w)$. Note also that $b' \subseteq x$ holds, else, due to (iii) and (iv), $x \notin y$ would be violated. Since $x \subsetneq \mathbf{b}$ holds because of (ii) and (i), we can find a $z \in b'' \setminus x$, so that $x \in z$ holds by (iii). It hence follows, by (iii) and by the definition of b'', that $y \cap \mathbf{a} \subsetneq z$, contradicting the maximality of y.

4. Conclusions

A Soul admitted to Itself:
Finite Infinity.
Emily Dickinson

Is Ω (namely, the solution under AFA of the equation $X = \{X\}$) *infinite* in any sense? This question, addressed in [2, 3], is in fact intriguing. A negative answer—admittedly, philosophically insensitive—is based on the fact that, after all, Ω has cardinality one! While subscribing such an answer, the results presented in this paper—with special emphasis on the proving techniques—bring forth additional “evidence” by pinpointing the combinatorial aspects involved in the characterization of infinity by set-theoretic formulae.

Expressing infinity means characterizing some sort of unending element-generating process. This is usually done within syntactic constraints, measuring the expressiveness of the class of formulae in which the characterization has taken place. Set-theoretic formulae, as classically organized hierarchically by quantifier prefixes, offer one possible syntactic characterization, and the result in [7] proves that an infinite set can be described by a purely universal formula in two variables. This is surprising: a purely universal formula in the variables a and b, at first sight, should be capable of expressing ground properties of a and b with all the remaining elements in the universe, not to force the *size* of a and/or b to get any large. The result is in fact based on the analysis of the long range effects of the extensionality axiom that can be controlled even at the lowest (i.e. purely universal) level of the above mentioned syntactic hierarchy. This can be done even in absence of foundation (cf. [9]) or in presence of AFA (cf. [6]) and the proofs given in this paper illustrate that, essentially, the mechanism at work is always the same. Such mechanism consists in a usage of the principle that a set is completely characterized by the collection of its elements, properly declined depending upon whether foundation is assumed or not.

Hence, even though the Ω-characterizing formula

$$x \neq \emptyset \wedge (\forall y \in x)(y = x)$$

is a purely universal formula, the technical difficulties to express infinity in its simplest possible form are still there and the proofs in this paper should help clarify the machinery that can do the job.

We conclude by mentioning that the decidability of the satisfaction problem for the class of purely universal formulae (the so-called Bernays-Shönfinkel class), cf. [5], is as yet very little explored for the non-well-founded case. The results presented here could help in closing this problem.

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References

[1] P. Aczel, *Non-well-founded sets*, CSLI Lecture Notes volume 14, Stanford University Press, Stanford (1988).
[2] J. Barwise and L. Moss, *Hypersets*, Math. Intelligencer **13** (1991), 31–41.
[3] J. Barwise and L. Moss, *Vicious circles*, CSLI Lecture Notes volume 60, Stanford University Press, Stanford (1996).
[4] D. Bellè and F. Parlamento, *Truth in V for $\exists^*\forall\forall$-sentences is decidable*, J. Symbolic Logic **71** (2006), 1200–1222.
[5] E.G. Omodeo and A. Policriti, *The Bernays-Schönfinkel-Ramsey class for set theory: semidecidability*, J. Symbolic Logic **75** (2010), 459–480.
[6] E.G. Omodeo, A. Policriti, and A.I. Tomescu, *Statements of ill-founded infinity in set theory*, in P. Cegielski, *Studies in weak arithmetics*, CLSI Lecture Notes volume 196, Stanford University Press, Stanford (2010), pages 173–199.
[7] F. Parlamento and A. Policriti, *The logically simplest form of the infinity axiom*, Proc. Amer. Math. Soc. **103** (1988), 274–276.
[8] F. Parlamento and A. Policriti, *Note on "The logically simplest form of the infinity axiom"*, Proc. Amer. Math. Soc. **108** (1990), 285–286.
[9] F. Parlamento and A. Policriti, *Expressing infinity without foundation*, J. Symb. Logic **56** (1991), 1230–1235.

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