Rend. Istit. Mat. Univ. Trieste
Volume 42 (2010), 191–203.

A Note on the Coalgebraic Interpretation of Game Logic

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Abstract. *We propose a coalgebraic interpretation of game logic, making the results of coalgebraic logic available for this context. We study some properties of a coalgebraic interpretation, showing among others that Aczel's Theorem on the characterization of bisimilar models through spans of morphisms is valid here. We investigate also congruences as those equivalences on the state space which preserve the structure of the model.*

Keywords: Coalgebras, Game Logic, Kripke Models, Bisimulations, Congruences.
MS Classification 2010: 03B45, 68Q85, 68N30

1. Introduction

It is well known that there is an intensive interplay between games and modal logic, witnessed, e.g., by the work of Bonanno [2, 3], see also [7, 12]. Game Logic [8, 14] is an approach which captures the dynamics of games very similar to dynamic logics such as propositional dynamic logics (PDL), hereby modelling the interaction of players (which are typically called Angel and Demon). Syntactically, this is modelled through the introduction of dual games with Angel and Demon as players.

A typical formula in game logic then would be $\langle\gamma\rangle\varphi$ with the informal interpretation that Angel has a strategy such that after playing γ formula φ may hold; a similar interpretation is given for the formula $\langle\gamma^d\rangle\varphi$, only that in game γ players Angel and Demon are switching their rôles. PDL is viewed as the fragment of this logic in which the demon is not present, and in fact it can be shown that many properties of PDL carry over; see [8] for a discussion.

Game logic is interpreted through Kripke models. This is not particularly surprising, given the similarities of modal logics and game logic; Kripke models permit to translate many concepts from modal logics to game logic, carefully delineating the differences as well. The discussions pertaining to modal logics show that coalgebraic logics are an appropriate and very promising way of finding a general unifying theory for these logics, see, e.g, Y. Venema's survey [15] or [6] for a very recent collection. It would be helpful to move game logic under this umbrella as well, if not in its entirety, then at least those parts of game logic which can be treated coalgebraically in a sensible way.

The present note puts forward a proposal for doing just this. We look at models for game logics as coalgebras decorated with valuations, i.e., additional information on how to treat atomic sentences. This opens the road for game morphisms as a way of transporting information between game models, in this way comparing them with respect to their expressivity. Bisimulations can be treated coalgebraically in this context as well, so that we are in a good position to establish a variant of Aczel's Theorem for game models. Having morphisms available, giving the definition of congruences becomes easy (because we now have diagrams). The logic gives us generously a congruence for free, which actually turns out to be the coarsest congruence for a model. Taking a look at their kernel, morphisms are shown to spawn congruences as well. Finally it is shown that factoring a factor space does not expand the structural information: the factor's factor space may be obtained through a congruence on the base space as well (this rings an algebraic bell: it is just a close cousin of the second isomorphism theorem in group theory, adapted to the context).

The discussion is organized as follows: Section 2 follows [8] in introducing game logics and their interpretation through Kripke models, section 3 proposes a coalgebraic interpretation, and establishes that bisimilar models are characterized through a span of morphisms. Section 4 looks at congruences and indicates that the concept is interesting in studying game logics and their models; section 5 wraps it all up and suggests further work.

2. The Syntax of Game Logic

We define game logic and discuss some of its properties, initially following [8, Section 2] fairly closely. Assume that Γ_0 is a set of atomic games, and that Φ_0 is a set of primitive propositions; these sets are fixed for the rest of the paper. The sets of games resp. formulas are given through this syntax

$$\begin{aligned}\gamma &::= g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d\\ \varphi &::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi\end{aligned}$$

with $g \in \Gamma_0$ and $p \in \Phi_0$. Define as usual $[\gamma]\varphi := \neg\langle\gamma\rangle\neg\varphi$. The dual game γ^d is the same as playing γ, reversing the rôles of Angel and Demon, however, i.e., any choice being made by Angel is made by Demon, and vice versa.

$\gamma_1 \cup \gamma_2$ is interpreted as angelic choice: Angel may choose between the games γ_1 and γ_2. Demonic choice is modelled through $\gamma_1 \cap \gamma_2 := \left(\gamma_1^d \cup \gamma_2^d\right)^d$. Thus Demon chooses which subgame to play, leaving the roles of the players in the subgames intact.

Angelic iteration is modelled through γ^*. Thus playing γ^*, Angel may choose how often to play γ (if at all); demonic iteration is defined as $\gamma^\times := \big((\gamma^d)^*\big)^d$. The game $\varphi?$ tests whether formula φ holds, hence $p?;\gamma$ is the game which is played after testing whether the primitive formula p holds, and if this is the case proceeds with γ. Because this note's focus is coalgebraic rather than game theoretic, we will not discuss these formulas further and refer the reader to [8].

Given the sets Γ_0 and Φ_0 of atomic games resp. of atomic statements, $\mathfrak{M} = \big(S, (E_g)_{g\in\Gamma_0}, V\big)$ is called a *game model* iff $E_g : S \to \mathcal{P}^2(S)$ is a map with $E_g(s)$ upward closed for each $s \in S$ (thus $X \in E_g(s)$ and $X \subseteq X'$ implies $X' \in E_g(s)$) and $V : \Phi_0 \to \mathcal{P}(S)$ is a map which assigns to each atomic statement a set of states. $X \in E_g(s)$ is interpreted as player Angel having a strategy to achieve X upon playing atomic game $g \in \Gamma_0$ in state $s \in S$ [8, Definition 2].

Each map $E_g : S \to \mathcal{P}^2(S)$ induces a map $\widehat{E}_g : \mathcal{P}(S) \to \mathcal{P}(S)$ upon setting

$$\widehat{E}_g(A) := \{s \in S \mid A \in E_g(s)\},$$

thus $\widehat{E}_g(A)$ is the set of all states for which Angel has a strategy for the atomic game g to achieve A. This map is extended inductively for games upon setting

$$\begin{aligned}
(\widehat{E}_{\gamma_1;\gamma_2})(A) &:= \big(\widehat{E}_{\gamma_1} \circ \widehat{E}_{\gamma_2}\big)(A),\\
(\widehat{E}_{\gamma_1\cup\gamma_2})(A) &:= \big(\widehat{E}_{\gamma_1} \cup \widehat{E}_{\gamma_2}\big)(A),\\
\widehat{E}_{\gamma^d}(A) &:= S \setminus \widehat{E}_{\gamma}(S \setminus A),\\
\widehat{E}_{\gamma^*}(A) &:= \bigcup_{n\geq 0} \widehat{E}_{\gamma^n}(A)
\end{aligned}$$

(here $\gamma^0 := \texttt{skip}$ with $\widehat{E}_{\texttt{skip}}(A) := A$ and $\gamma^{n+1} := \gamma^n;\gamma$). Given a model $\mathfrak{M}$, one finally defines $\widehat{E}_{\varphi?}(A) := [\![\phi]\!]_{\mathfrak{M}} \cap A$, where $[\![\phi]\!]_{\mathfrak{M}} := \{s \in S \mid \mathfrak{M}, s \models \varphi\}$ is the set of states in which formula φ holds. This will be defined in a moment.

Note that [8] defines the semantics for the iteration as a smallest fixed point. To be specific, for γ^* the function $A \mapsto \mu Y.A \cup E_\gamma(Y)$ is used, μ indicating as usual the smallest fixed point. The reason for modifying this definition for the present scenario is the observation that the definition given here fits smoother into the coalgebraic scenario, given the problems with transporting fixed points along maps.

The relation $\models$ is defined inductively as follows:

$$\mathfrak{M}, s \models \bot \Leftrightarrow s \in \emptyset$$
$$\mathfrak{M}, s \models p \Leftrightarrow s \in V(p) \text{ if } p \in \Phi_0$$
$$\mathfrak{M}, s \models \neg\varphi \Leftrightarrow \mathfrak{M}, s \not\models \varphi$$
$$\mathfrak{M}, s \models \varphi_1 \vee \varphi_2 \Leftrightarrow \mathfrak{M}, s \models \varphi_1 \text{ or } \mathfrak{M}, s \models \varphi_2$$
$$\mathfrak{M}, s \models \langle\gamma\rangle\varphi \Leftrightarrow s \in \widehat{E}_\gamma([\![\phi]\!]_{\mathfrak{M}}).$$

Define for a state $s \in S$ its *theory in* $\mathfrak{M}$ by

$$Th_{\mathfrak{M}}(s) := \{\varphi \mid \mathfrak{M}, s \models \varphi\},$$

so a state's theory is just the set of all formulas which are valid in this state.

3. Game Models, Coalgebraically

Given two game models, we relate them through a morphism; this is defined here, and we show that morphisms transport the validity of formulas. This observation will be important for the discussion to follow. Having morphisms available, we may use them for comparing the expressivity of models, and one way of doing so is to ask whether they are bisimilar.

This is traditionally described through a relation on the state spaces of the models under consideration, but morphisms permit a concise formulation through a span; this is what Aczels' Theorem says. We formulate this observation for the present scenario, before we show that game models may be looked at as coalgebras. This requires, however, to identify a suitable endofunctor $\mathfrak{F}$ on the category of sets with maps as morphisms. This functor is used for modelling the dynamics of a coalgebra, in this case for a game model. Recall that an $\mathfrak{F}$-*coalgebra* (A, α) is a set A together with a morphism $\alpha : A \to \mathfrak{F}(A)$, its dynamics.

Let $\mathfrak{M} = \big(S, (E_g)_{g\in\Gamma_0}, V\big)$ and $\mathfrak{M}' = \big(S', (E'_g)_{g\in\Gamma_0}, V'\big)$ be game models, then $f : \mathfrak{M} \to \mathfrak{M}'$ is called a *game morphism* iff $f : S \to S'$ is a map such that

1. $V(p) = f^{-1}[V'(p)]$ for all $p \in \Phi_0$,
2. $\{f^{-1}[X'] \mid X' \in E'_g(f(s))\} = E_g(s)$ for all states $s \in S$ and all atomic games $g \in \Gamma_0$.

Thus $s \in V(p)$ iff $f(s) \in V'(p)$ for all $p \in \Phi_0$ and all states $s \in S$. If upon playing atomic game $g \in \Gamma_0$, Angel has a strategy in model $\mathfrak{M}'$ for achieving X' in state $f(s)$, then it[1] has a strategy for achieving $f^{-1}[X']$ in model $\mathfrak{M}$

[1][8] address Angel and Demon as female. This may be politically correct, the present author has — in view of *Matthew 22:30* — some doubts, however, whether it is theologically correct.

in state s. And vice versa: if $X \in E_g(s)$ can be achieved by Angel by playing game $g \in \Gamma_0$ in state s, then there exists $X' \in E'_g(f(s))$ which Angel can achieve in $\mathfrak{M}'$ in state $f(s)$ such that $X = f^{-1}[X']$.

Morphisms respect and reflect the validity of formulas:

LEMMA 3.1. *Let* $\mathfrak{M}_1 = \big(S_1, (E_{1,g})_{g\in\Gamma_0}, V_1\big)$ *and* $\mathfrak{M}_2 = \big(S_2, (E_{2,g})_{g\in\Gamma_0}, V_2\big)$ *be game models,* $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ *a morphism. Then*

$$\mathfrak{M}_1, s \models \varphi \Leftrightarrow \mathfrak{M}_2, f(s) \models \varphi$$

holds for each state s *of* $\mathfrak{M}_1$ *and each formula* φ*.*

Proof. The proof proceeds by induction on the structure of φ. The induction starts for $\varphi = p \in \Phi_0$, for which the assertion follows immediately from the definition of a morphism. In the induction step, one shows first by induction on program γ that $f^{-1}\big[\widehat{E}_{2,\gamma}(X_2)\big] = \widehat{E}_{1,\gamma}(f^{-1}[X_2])$ for each $X_2 \subseteq S_2$ (because the test operator $\varphi?$ is involved, this is part of the induction). The crucial step is for a formula like $\langle\gamma\rangle\varphi$ with γ a game. Under the assumption that the assertion holds for formula φ, one obtains

$$\begin{aligned}\mathfrak{M}_2, f(s) \models \langle\gamma\rangle\varphi &\Leftrightarrow s \in f^{-1}\big[\widehat{E}_{2,\gamma}([\![\phi]\!]_{\mathfrak{M}_2})\big]\\ &\Leftrightarrow s \in \widehat{E}_{1,\gamma}\big(f^{-1}\big[[\![\phi]\!]_{\mathfrak{M}_2}\big]\big)\\ &\Leftrightarrow s \in \widehat{E}_{1,\gamma}([\![\phi]\!]_{\mathfrak{M}_1})\\ &\Leftrightarrow \mathfrak{M}_1, s \models \langle\gamma\rangle\varphi.\end{aligned}$$

This establishes the induction step. □

The decisive identity is apparently $f^{-1}\big[\widehat{E}_{2,\gamma}(X_2)\big] = \widehat{E}_{1,\gamma}(f^{-1}[X_2])$ for each γ. This can be established for $\widehat{E}_{\gamma^*}$, as defined above; it is at present not quite clear whether this identity holds if $\widehat{E}_{\gamma^*}$ is defined as a smallest fixed point.

Game models can be compared to each other through bisimulations which relate a step in one model to a step in the other one; we adopt [8, Definition 5]:

DEFINITION 3.2. *Let* $\mathfrak{M}_1 = \big(S_1, (E_{1,g})_{g\in\Gamma_0}, V_1\big)$ *and* $\mathfrak{M}_2 = \big(S_2, (E_{2,g})_{g\in\Gamma_0}, V_2\big)$ *be game models.* $B \subseteq S_1 \times S_2$ *is called a* bisimulation of $\mathfrak{M}_1$ and $\mathfrak{M}_2$*iff for each* $\langle s_1, s_2\rangle \in B$ *and each* $g \in \Gamma_0$

1. $s_1 \in V_1(p)$ *iff* $s_2 \in V_2(p)$ *for each atomic statement* $p \in \Phi_0$*.*
2. *For all* $X_1 \in E_{1,g}(s_1)$ *there exists* $X_2 \in E_{2,g}(s_2)$ *such that for each* $x'_2 \in X_2$ *there exists* $x'_1 \in X_1$ *with* $\langle x'_1, x'_2\rangle \in B$*.*
3. *For all* $X_2 \in E_{2,g}(s_2)$ *there exists* $X_1 \in E_{1,g}(s_1)$ *such that for each* $x'_1 \in X_1$ *there exists* $x'_2 \in X_2$ *with* $\langle x'_1, x'_2\rangle \in B$*.*

This is the relational version, based on Milner's [1] definition. Bisimulations are defined in terms of a relation between the state spaces of the game models. Handling relations may sometimes be a bit awkward. We show now that two game models are bisimilar provided we can find a span of morphisms between them.

PROPOSITION 3.3. *Let* $\mathfrak{M}_1 = (S_1, (E_{1,g})_{g\in\Gamma_0}, V_1)$ *and* $\mathfrak{M}_2 = (S_2, (E_{2,g})_{g\in\Gamma_0}, V_2)$ *be game models. Then the following statements are equivalent*

a. $B \subseteq S_1 \times S_2$ *is a bisimulation of* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$.

b. There exists a game model $\mathfrak{B} = \big(B, (F_g)_{g\in\Gamma_0}, W\big)$ *with state space* B *so that the projections* $\pi_1 : B \to S_1, \pi_2 : B \to S_2$ *are morphisms*

$$\mathfrak{M}_1 \xleftarrow{\pi_1} \big(B, \big(F_g\big)_{g\in\Gamma_0}, W\big) \xrightarrow{\pi_2} \mathfrak{M}_2.$$

Proof. "$a \Rightarrow b$": Define

$$F_g(s_1, s_2) := \{D \subseteq B \mid \pi_1[D] \in E_{1,g}(s_1) \text{ and } \pi_2[D] \in E_{2,g}(s_2)\},$$

$$W(p) := \big(V_1(p) \times V_2(p)\big) \cap B$$

for $g \in \Gamma_0, \langle s_1, s_2\rangle \in B$ and $p \in \Phi_0$. Because both $E_{1,g}(s_1)$ and $E_{2,g}(s_2)$ are upper closed, so is $F_g(s_1, s_2) \subseteq \mathcal{P}(B)$. Because B is a bisimulation,

$$\pi_1^{-1}[V_1(p)] = W(p) = \pi_2^{-1}[V_2(p)]$$

holds for each $p \in \Phi_0$.

Now fix $\langle s_1, s_2\rangle \in B$. Let $X_1 \in E_{1,g}(s_1)$, then $X_1 = \pi_1\left[\pi_1^{-1}[X_1]\right]$, so it suffices to show that $\pi_2\left[\pi_1^{-1}[X_1]\right] \in E_{2,g}(s_2)$. Given X_1 there exists $X_2 \in E_{2,g}$ with the additional property that for each $x_2 \in X_2$ there exists $x_1 \in X_1$ such that $\langle x_1, x_2\rangle \in B$.

We claim $X_2 = \pi_2[(X_1 \times X_2) \cap B]$. In fact, if $t \in \pi_2[(X_1 \times X_2) \cap B]$, then $t = x_2 \in X_2$ for some $\langle x_1, x_2\rangle \in B$, hence $\pi_2[(X_1 \times X_2) \cap B] \subseteq X_2$. Now let $x_2 \in X_2$, then there exists $x_1 \in X_1$ with $\langle x_1, x_2\rangle \in B$, thus $x_2 \in \pi_2[(X_1 \times X_2) \cap B]$, accounting for the other inclusion. But this implies $X_2 \subseteq \pi_2\left[\pi_1^{-1}[X_1]\right]$, hence $\pi_2\left[\pi_1^{-1}[X_1]\right] \in E_{2,g}(s_2)$. This shows $E_{1,g}(s_1) \subseteq \{X_1 \subseteq S_1 \mid \pi_1^{-1}[X_1] \in F_g(s_1, s_2)\}$.

Assume for the converse inclusion that $\pi_1^{-1}[X_1] \in F_g(s_1, s_2)$, for some $X_1 \subseteq S_1$. Then $\pi_1\left[\pi_1^{-1}[X_1]\right] \subseteq X_1$. Since $\pi_1\left[\pi_1^{-1}[X_1]\right] \in E_{1,g}(s_1)$ by construction of F_g, we conclude $X_1 \in E_{1,g}(s_1)$. Swapping the models one shows in the same way that $E_{2,g}(s_2) = \{X_2 \subseteq S_2 \mid \pi_2^{-1}[X_2] \in F_g(s_1, s_2)\}$ holds.

Summarizing, this means that

$$\pi_1 : \big(B, \big(F_g\big)_{g\in\Gamma_0}, W\big) \to \mathfrak{M}_1$$

$$\pi_2 : \big(B, \big(F_g\big)_{g\in\Gamma_0}, W\big) \to \mathfrak{M}_2$$

are morphisms.

"$b \Rightarrow a$": Now assume that the projections are game morphisms. Let $\langle s_1, s_2\rangle \in B$, and assume that $s_1 \in V_1(p)$ for some $p \in \Phi_0$. Because $W(p) = \big(V_1(p)\times X_2\big)\cap B = \big(X_1\times V_2(p)\big)\cap B$, it follows that $W(p) = \big(V_1(p)\times V_2(p)\big)\cap B$, hence we conclude $s_2 \in V_2(p)$; similarly, we obtain $s_1 \in V_1(p)$ from $s_2 \in V_2(p)$. Let $X_1 \in E_{1,g}(s_1)$, then $Y_1 := \pi_1^{-1}[X_1] \in F_g(s_1, s_2)$, thus $X_2 := \pi_2[Y_1] \in E_{2,g}(s_2)$. Hence we find $\langle x_1, t\rangle \in Y_1$ with $x_2 = \pi_2(x_1, t)$ for $x_2 \in X_2$. But $x_1 \in X_1$ and $t = x_2$, so that $\langle x_1, x_2\rangle \in \pi_1^{-1}[X_1] = (X_1 \times S_2) \cap B$, consequently, $\langle x_1, x_2\rangle \in B$. The third property of a bisimulation is proved exactly in the same way. Hence B is a bisimulation for $\mathfrak{M}_1$ and $\mathfrak{M}_2$. □

Proposition 3.3 is a version of Aczel's Theorem [9, Example 2.5] which permits the interpretation of bisimulations in a coalgebraic setting through a span of morphisms. This theorem says that for a set based functor $\mathfrak{F}$ the relation $B \subseteq A \times A'$ is a bisimulation of two $\mathfrak{F}$-coalgebras (A, α) and (A', α') iff there exists a coalgebra structure on B such that the projections are morphisms, see [9] for a general discussion.

Returning to bisimulations, $B \subseteq A \times A'$ is a bisimulation between the coalgebras iff there exists a morphism $\beta : B \to \mathfrak{F}(B)$ so that the projections are morphisms, i.e.,

$$(A, \alpha) \xleftarrow{\ \pi\ } (B, \beta) \xrightarrow{\ \pi'\ } (A', \alpha').$$

This expands to a commutative diagram

$$\begin{array}{ccccc} A & \xleftarrow{\ \pi\ } & B & \xrightarrow{\ \pi'\ } & A' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha'} \\ \mathfrak{F}(A) & \xleftarrow[\mathfrak{F}(\pi)]{} & \mathfrak{F}(B) & \xrightarrow[\mathfrak{F}(\pi')]{} & \mathfrak{F}(A') \end{array}$$

The transformation from a relational setting into a coalgebraic one may be performed in the present context as well, provided we interpret game models coalgebraically. In fact, game models may be perceived as coalgebras with valuations for atomic statements attached. The corresponding construction will be carried out now. Let

$$\mathcal{U}(X) := \{\mathfrak{x} \subseteq \mathcal{P}(X) \mid \mathfrak{x} \text{ is an upper set}\}$$

be the set of all upper subsets of $\mathcal{P}(X)$. Thus if $\mathfrak{x} \in \mathcal{U}(X)$, then $A \in \mathfrak{x}$ and $A \subseteq A'$ together imply $A' \in \mathfrak{x}$. Define for $f : X \to Y$ and for $\mathfrak{x} \in \mathcal{U}(X)$

$$\mathcal{U}(f)(\mathfrak{x}) := \{B \in \mathcal{P}(Y) \mid f^{-1}[B] \in \mathfrak{x}\},$$

thus $\mathcal{U}(f) : \mathcal{U}(X) \to \mathcal{U}(Y)$. It is easy to verify that $\mathcal{U}$ is an endofunctor on the category of sets with maps as morphisms. A game model $\mathfrak{M} = \big(S, (E_g)_{g \in \Gamma_0}, V\big)$ is perceived as an $\mathcal{U}^{\Gamma_0}$-coalgebra (S, ϱ) together with a valuation $V : \Phi_0 \to \mathcal{P}(S)$, where the dynamics $\varrho : S \to \mathcal{U}^{\Gamma_0}(S)$ is defined through $\varrho(g) := E_g$ for each $g \in \Gamma_0$, operations on $\mathcal{U}^{\Gamma_0}$ being carried out componentwise.

Model morphisms fit into this scenario as well: model morphism $f : \mathfrak{M} \to \mathfrak{M}'$ is a coalgebra morphism $f : (S, \varrho) \to (S', \varrho')$ such that $f^{-1}[V'(p)] = V(p)$ for each $p \in \Phi_0$. Recall that f as a coalgebra morphism renders this diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & S' \\ {\scriptstyle \varrho}\big\downarrow & & \big\downarrow{\scriptstyle \varrho'} \\ \mathcal{U}^{\Gamma_0}(S) & \xrightarrow[\mathcal{U}^{\Gamma_0}(f)]{} & \mathcal{U}^{\Gamma_0}(S') \end{array}$$

Thus we have $E'_g \circ f = \mathcal{U}(f) \circ E_g$ for each atomic game $g \in \Gamma_0$.

We are poised now to reformulate Proposition 3.3 in terms of coalgebras.

PROPOSITION 3.4. *Let $\mathfrak{M}_1 = (S_1, \varrho_1, V_1)$ and $\mathfrak{M}_2 = (S_2, \varrho_2, V_2)$ be game models. Then the following statements are equivalent*

a. $B \subseteq S_1 \times S_2$ is a bisimulation of $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

b. There exists a game model $\mathfrak{B} = (B, \beta, W)$ so that the projections $\pi_1 : B \to S_1, \pi_2 : B \to S_2$ are morphisms. Hence this diagram commutes:

$$\begin{array}{ccccc} S_1 & \xleftarrow{\quad \pi_1 \quad} & B & \xrightarrow{\quad \pi_2 \quad} & S_2 \\ {\scriptstyle \varrho_1}\big\downarrow & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \varrho_2} \\ \mathcal{U}^{\Gamma_0}(S_1) & \xleftarrow[\mathcal{U}^{\Gamma_0}(\pi_1)]{} & \mathcal{U}^{\Gamma_0}(B) & \xrightarrow[\mathcal{U}^{\Gamma_0}(\pi_2)]{} & \mathcal{U}^{\Gamma_0}(S_2) \end{array}$$

Proof. Proposition 3.3. □

Pausing, one might ask in view of Aczel's theorem whether the argumentation proposed here is circular: game models are coalgebras, bisimulations in coalgebras may be expressed through cospans of projections from the bisimilarity relation. This argument suggests that we do not need Proposition 3.3. But this argument fails to address the crucial point: Definition 3.2 defines bisimilarity in terms of game models, and Proposition 3.3 shows that bisimilarity of game models and bisimilarity for coalgebras manufactured from game models are the same. Hence we need Proposition 3.3 for bringing game bisimulation into the coalgebraic context.

4. Congruences

Now that the coalgebraic scenario is available, we use it for defining congruences and discussing two examples. We finally state and prove an isomorphism theorem which is similar to the classical one of group theory.

If τ is an equivalence relation on set S, the equivalence class for $s \in S$ is denoted by $[s]_\tau$, the factor map is η_τ, and the factor space is as usual denoted by S/τ. Call a set $A \subseteq S$ *τ-invariant* iff A is the union of τ-classes, or, equivalently, iff $s \in A$ and $s \; \tau \; s'$ together imply $s' \in A$. This in turn is equivalent to $A = \eta_\tau^{-1}[\eta_\tau[A]]$. Denote by $A^\tau := \bigcup\{[a]_\tau \mid a \in A\}$ the smallest τ-invariant subset of S which contains A.

Definition 4.1. *Let $\mathfrak{M} = (S, \varrho, V)$ be a game model, then an equivalence relation τ on the state space S is called a* congruence for $\mathfrak{M}$ *iff*

1. *There exists $\varrho_\tau : S/\tau \to \mathcal{U}^{\Gamma_0}(S/\tau)$ such that this diagram commutes*

$$\begin{array}{ccc} S & \xrightarrow{\;\eta_\tau\;} & S/\tau \\ {\scriptstyle\varrho}\downarrow & & \downarrow{\scriptstyle\varrho_\tau} \\ \mathcal{U}^{\Gamma_0}(S) & \xrightarrow[\mathcal{U}^{\Gamma_0}(\eta_\tau)]{} & \mathcal{U}^{\Gamma_0}(S/\tau) \end{array}$$

2. *$V(p)$ is τ-invariant for each atomic program $p \in \Phi_0$.*

Denote by $\mathfrak{M}/\tau := \left(S/\tau, \varrho_\tau, V_\tau\right)$ the factor model*, where $V_\tau : p \mapsto \eta_\tau[V(p)]$.*

Because η_τ is onto, the map ϱ_τ is uniquely determined, if it exists. Consequently, $\eta_\tau : \mathfrak{M} \to \mathfrak{M}/\tau$ is a game morphism. Expanding this definition, we see that for a congruence τ and every atomic program g there exists a map

$$E_{\tau,g} : S/\tau \to \mathcal{U}(S/\tau)$$

such that

$$E_{\tau,g}([s]_\tau) = \{\eta_\tau[B] \mid B \in E_g(s)\}$$

for all states s. In fact, $E_{\tau,g}([s]_\tau) = \{A \subseteq S/\tau \mid \eta_\tau^{-1}[A] \in E_g(s)\}$ follows from the definition of a morphism, so that $E_{\tau,g}([s]_\tau) \subseteq \{\eta_\tau[B] \mid B \in E_g(s)\}$. Conversely, if $B \in E_g(s)$, then $B \subseteq \eta_\tau^{-1}[\eta_\tau[B]]$, so that the other inclusion follows from the fact that $E_g(s)$ is upper closed.

This characterization yields — together with the τ-invariance of the valuations for the primitive programs — a concise description of a congruence. We obtain from Proposition 3.4

Corollary 4.2. *Let* τ *be a congruence for game model* $\mathfrak{M}$*, then we have*

$$\mathfrak{M}, s \models \varphi \Leftrightarrow \mathfrak{M}/\tau, [s]_\tau \models \varphi.$$

for each formula φ *and each state* s *of* $\mathfrak{M}$*.* $\dashv$

Game logic induces an equivalence relation $\sim$ on S: Two states are related iff they have the same theories, so iff they cannot be separated by a formula. Formally,

$$s \sim s' \text{ iff } \left[\mathfrak{M}, s \models \varphi \Leftrightarrow \mathfrak{M}, s' \models \varphi\right] \text{ for each formula } \varphi.$$

This is in fact a congruence. One first notes that $V(p)$ is $\sim$-invariant for each primitive proposition $p \in \Phi_0$. Now define for the atomic game $g \in \Gamma_0$ and $B \subseteq S/\sim$

$$B \in E_{\sim,g}([s]_\sim) \text{ iff } \{s' \in S \mid \exists\, [s'']_\sim \in B : Th_{\mathfrak{M}}(s') = Th_{\mathfrak{M}}(s'')\} \in E_g(s),$$

then the set $E_{\sim,g}([s]_\sim)$ is upper closed, since $E_g(s)$ is, and if $B \in E_{\sim,g}([s]_\sim)$, then $B = \eta_\sim\left[\bigcup B\right]$ and $\bigcup B \in E_g(s)$. Moreover, if $A \in E_g(s)$, then $\eta_\sim[A] \in E_{\sim,g}([s]_\sim)$. In fact, A may assumed to be $\sim$-invariant (otherwise we switch to $A^\sim \in E_g(s)$), so that we obtain $\{s' \in S \mid \exists\, [s'']_\sim \in \eta_\sim[A] : Th_{\mathfrak{M}}(s') = Th_{\mathfrak{M}}(s'')\} = A$. We have shown

Proposition 4.3. *The equivalence relation* $\sim$ *induced by the logic is a congruence on* $\mathfrak{M}$*.* $\dashv$

In fact, this congruence is the largest one on a game model.

Corollary 4.4. *Let* τ *be a congruence on game model* $\mathfrak{M}$*. Then* $\tau \subseteq \sim$*.*

Proof. Let $s\ \tau\ s'$, then $\mathfrak{M}, s \models \varphi \Leftrightarrow \mathfrak{M}, s' \models \varphi$ for each formula φ by Corollary 4.2. Hence $s \sim s'$. □

Let $f : \mathfrak{M} \to \mathfrak{M}'$ be a game morphism for the game models $\mathfrak{M} = (S, \varrho, V)$ and $\mathfrak{M}' = (S', \varrho', V')$, and define

$$\mathsf{ker}\,(f) := \{\langle s_1, s_2\rangle \mid f(s_1) = f(s_2)\},$$

the kernel of f. We claim that this is a congruence on $\mathfrak{M}$.

One first notes that each $V(p)$ is $\mathsf{ker}\,(f)$-invariant by the definition of a morphism, and that

$$E_{\mathsf{ker}(f),g}([s]_{\mathsf{ker}(f)}) := \{A^{\mathsf{ker}(f)}/\mathsf{ker}\,(f) \mid A \in E_g(s)\}$$

is upper closed with $B \in E_{\mathsf{ker}(f),g}([s]_{\mathsf{ker}(f)})$ iff $\eta^{-1}_{\mathsf{ker}(f)}[B] \in E_g(s)$. Consequently, this defines a game model $\mathfrak{M}/\mathsf{ker}\,(f)$ with $\eta_{\mathsf{ker}(f)} : \mathfrak{M} \to \mathfrak{M}/\mathsf{ker}\,(f)$ as a morphism, so that we have established

Proposition 4.5. *Given a game model $\mathfrak{M} = (S, \varrho, V)$ and an equivalence relation τ on S, these properties are equivalent.*

1. *τ is a congruence on $\mathfrak{M}$.*
2. *There exists a model $\mathfrak{M}'$ and a game morphism $f : \mathfrak{M} \to \mathfrak{M}'$ such that $\tau = \mathsf{ker}\,(f)$.* ⊣

We finally establish an isomorphism akin to the second isomorphism theorem of group theory [13, par. 50].

Proposition 4.6. *Let $\mathfrak{M} = (S, \varrho, V)$ be a game model with a congruence τ. Given a congruence σ on $\mathfrak{M}/\tau$, there exists a congruence ϑ on $\mathfrak{M}$ such that $\mathfrak{M}/\vartheta$ is isomorphic to $(\mathfrak{M}/\tau)/\sigma$.*

Proof. Define $s\ \vartheta\ s'$ iff $[s]_\tau\ \sigma\ [s']_\tau$ for $s, s' \in S$, then ϑ is an equivalence relation on S such that

$$\Psi : \begin{cases} S/\vartheta & \to (S/\tau)/\sigma \\ [s]_\vartheta & \mapsto [[s]_\tau]_\sigma \end{cases}$$

is a bijection. Given $X \subseteq S/\vartheta$, it is noted that $\eta_\sigma\left[\eta_\tau\left[\eta_\vartheta^{-1}[X]\right]\right] \in E_{\sigma,\tau,g}\big([[s]_\tau]_\sigma\big)$ iff $\eta_\vartheta^{-1}[X] \in E_g(s)$. Define

$$\begin{aligned} E_{\vartheta,g}([s]_\vartheta) &:= \{X \subseteq S/\vartheta \mid \eta_\vartheta^{-1}[X] \in E_g(s)\}, \\ \varrho_\vartheta(g) &:= \big(t \mapsto E_{\vartheta,g}(t)\big), \\ V_\vartheta(p) &:= \eta_\vartheta^{-1}[V(p)]\ \big(= \eta_\sigma^{-1}[\eta_\tau V(p)]\big). \end{aligned}$$

This yields the game model $\mathfrak{M}/\vartheta = (S/\theta, \varrho_\vartheta, V_\vartheta)$ such that $\Psi : \mathfrak{M}/\vartheta \to (\mathfrak{M}/\tau)/\sigma$ is a game morphism. Since a bijective game morphism is a game isomorphism, the assertion follows. □

The proof suggests a counterpart to the correspondence theorem of universal algebra [4, Theorem 6.20].

Corollary 4.7. *Given a congruence τ on a game model $\mathfrak{M} = (S, \varrho, V)$, there exists a bijection between the congruences on $\mathfrak{M}/\tau$ and the set of all congruences on $\mathfrak{M}$ which contain τ.*

Proof. Let σ be a congruence on $\mathfrak{M}$ with $\tau \subseteq \sigma$, and define the equivalence relation θ on S/τ through $[s]_\tau\ \theta\ [s']_\tau$ iff $s\ \sigma\ s'$. Thus $\eta_\theta = \mathsf{ker}\,(f)$ with $f : [s]_\tau \mapsto [s]_\sigma$; the latter map is well defined since $\tau \subseteq \sigma$. f is actually a morphism $\mathfrak{M}/\tau \to \mathfrak{M}/\sigma$. In fact, consider this diagram:

$$\begin{array}{ccccc} S & \xrightarrow{\ \eta_\tau\ } & S/\tau & \xrightarrow{\ f\ } & S/\sigma \\ \downarrow{\scriptstyle \varrho} & & \downarrow{\scriptstyle \varrho_\tau} & & \downarrow{\scriptstyle \varrho_\sigma} \\ \mathcal{U}^{\Gamma_0}(S) & \xrightarrow[\mathcal{U}^{\Gamma_0}(\eta_\tau)]{} & \mathcal{U}^{\Gamma_0}(S/\tau) & \xrightarrow[\mathcal{U}^{\Gamma_0}(f)]{} & \mathcal{U}^{\Gamma_0}(S/\sigma) \end{array}$$

The outer diagram and the leftmost one commute. Thus, since η_τ is onto, the rightmost diagram commutes as well. Because f is a morphism, its kernel θ is a congruence on $\mathfrak{M}/\tau$ by Proposition 4.5. The rest of the claim is straightforward. □

5. Conclusion

We propose a coalgebraic interpretation of game logic, making the results of coalgebraic logic available. We study some properties of a coalgebraic interpretation, showing among others that Aczel's Theorem on the characterization of bisimilar models through spans of morphisms is valid in the present scenario. We investigate also congruences as those equivalences on the state space which preserve the structure of the model.

The logic contains iteration, which is commonly interpreted through a smallest fixed point (in this fashion opening an avenue to μ-calculus). We propose using a computationally easier feasible fixed point construction which probably does not always lead to the smallest one. The reason for doing so is the compatibility of the iteration construct with morphisms, a price to be paid for the coalgebraic interpretation. We know that transporting fixed points through maps is not a straightforward business [10, Appendix A.3], hence the relationship of fixed points and morphisms should be investigated further. Given that a coalgebraic approach is available, other interpretations may be interesting as well: the functor governing the coalgebra is a model parameter, which can be varied. One might think of employing M. Giry's subprobability functor, which then leads to stochastic game models and their various properties, parametrized through the underlying measurable structures [11, 5].

The next step is the formulation of a coalgebraic logic along these lines. Syntactically, this is done by replacing all modal operators by predicate liftings. A lifting is a natural transformation which involves the functor governing the coalgebra. As in modal logic, we say for each lifting how it is interpreted, using the corresponding transformation. Thus a Kripke model for a coalgebraic logic contains a set of rules for the interpretation of all its liftings. But this is the crux of the matter. We argue in [5] that the correspondence between modal operators and interpretation rules is somewhat incomplete due to the fact that the operators' semantics cannot be matched directly by the facilities provided by the family of interpreting relations, so that some artifacts have to be introduced. This applies *a fortiori* to the general coalgebraic approach. We have a coalgebraic representation of Kripke models for game logic, but we still have to find ways of expressing coalgebraic logics along these lines.

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Received September 12, 2010
Revised October 30, 2010