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Some Open Problems Concerning Expansive Systems[1]

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Abstract. *In this paper we present some open problems in expansive systems. In order to make them accessible we make first a brief survey about expansive systems. Besides the references we also present a list of recommended articles and books about this subject. In the last section we present the open problems.*

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1. Introduction

A discrete invertible (the case we shall mainly refer to) **expansive system** is a dynamical system such that every point of the underlying space has a distinctive behavior. A homeomorphism f from the compact metric space M onto M is expansive if there exists $\alpha > 0$, (called expansivity constant of f) such that if $x, y \in M$ and $dist(f^n(x), f^n(y)) \leq \alpha$ for every $n \in Z$ then, $x = y$. Thus, if $x \neq y$, then for some n, $dist(f^n(x), f^n(y)) > \alpha$.

Expansive systems are then wholly sensitive to initial conditions and therefore, in this sense, chaotic".

Assume the dynamics of f is observed with a precision that permits to distinguish points at a distance larger than α, meanwhile, points at a distance less than $\delta > 0$, $\delta << \alpha$, are not distinguished. Then, a δ-small neighbourhood of, say, $x \in M$ with infinite points, will be seen -at present- as only one point. However, for some $n \in Z$, the n-iterate through f of this point, will show many of them, since points at a distance larger than α are distinguished by the observer (see [2]).

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[1] A version of this paper is also available as a Scholarpedia article. A list of open problems has been included in the present version.

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Since M is compact, on account of the expansiveness of f, it is not difficult to show that given a δ like in the preceding paragraph, $\delta < \alpha/2$, there is a C^0-neighbourhood N of f, such that if $g \in N$, and

$$dist(g^n(x), g^n(y)) \leq \alpha \text{ for every } n \in Z,$$

then $dist(g^n(x), g^n(y)) \leq \delta$ for all $n \in Z$. Therefore the relation

$$\mathcal{R}_\delta = \{(x, y) \in M \times M : dist(g^n(x), g^n(y)) \leq \delta, \;\; n \in Z\}$$

is an equivalence relation on M,. The canonical projection $\pi : M \to M/\mathcal{R}_\delta$ is closed and consequently $M/\mathcal{R}_\delta$ is a Hausdorff compact topological space and therefore the $M/\mathcal{R}_\delta$ is a compact metrizable space, and $g^* : M/\mathcal{R}_\delta \to M/\mathcal{R}_\delta$ defined by $g^*(\pi(x)) = \pi(g(x))$, is an expansive homeomorphism of $M/\mathcal{R}_\delta$.

Again, an observer that can not distinguish points at a distance less than δ will see the motion as taking place in $M/\mathcal{R}_\delta$ (instead of M) under the action of g^*.

Clearly, homeomorphisms conjugate to an expansive one are also expansive.

2. Examples

2.1. The Shift

Consider

$$2^Z = \{(a_n) : a_n = 0 \text{ or } 1, n \in Z\}$$

and the distance:

$$dist((a_n), (b_n)) = \sum_{-\infty}^{\infty} |a_n - b_n| 2^{-|n|}$$

With this metric, which induces the product topology, 2^Z is compact. Let $\sigma : 2^Z \to 2^Z$ be defined by $\sigma(a_n) = (b_n)$, where $b_n = a_{n-1}$. σ is the usual shift homeomorphism. If $(a_n) \neq (b_n)$ then, for some $K \in Z, a_K \neq b_K$, and, therefore

$$dist(\sigma^K((a_n)), \sigma^K((b_n))) \geq 1,$$

showing that the shift is an expansive homeomorphism.

2.2. The Denjoy Map

Take a rotation of S^1 by an angle $2\pi\alpha$, where α is irrational, and replace the points of a dense orbit, say $\{x_n, n \in Z\}$, with arcs of diameter decreasing with $|n|$ in order to get a new space also homeomorphic to S^1. The Denjoy map, f, may be defined by assigning to each point not on the added arcs,

the former image under the rotation, and mapping (length) linearly the arc replacing x_n onto the one replacing $x_{n+1}, n \in Z$. It is easy to see that this map is a homeomorphism of S^1, and that the set D of points not lying in the interior of the added arcs is compact and invariant under the Denjoy map (in fact this set is homeomorphic to the Cantor set). A non-trivial arc whose end points lie on this set contains some of the added arcs, and, consequently, some iterate of this arc will include the one replacing x_0 of diameter, say d. Thus, d will be an expansivity constant for the restriction of f to D.

2.3. Anosov and Quasi-Anosov Diffeomorphisms

Let f be a diffeomorphism of a compact, Riemannian, smooth manifold M onto itself; f is Anosov if there exists $L > 0, 0 < \lambda < 1$, and continuous non-trivial Tf invariant sub-bundles S, U of TM, such that

$$S \oplus U = TM,$$

$$\|Tf^n(s)\| \leq L\lambda^n \text{ for } s \in S, n \geq 0, \text{ and}$$

$$\|Tf^{-n}(u)\| \leq L\lambda^n \text{ and } u \in U, n \geq 0.$$

If A is a compact f-invariant subset of M and the above decomposition holds on A, A is called a hyperbolic set. The restriction $f|A$, of f to a hyperbolic set A is also expansive. Anosov diffeomorphisms may also be characterized in a different way (see [15]). Let $B : TM \to R$ be a continuous quadratic form, i.e, $B_x = B|T_xM$ is a quadratic form on the vector space T_xM that depends continuously on $x \in M$. A diffeomorphism $f : M \to M$ is quasi-Anosov if there exists such a B with the property $B_{f(x)}((Tf)_x(v)) - B_x(v) > 0$, for every $x \in M$, and each $v \in T_xM, \|v\| \neq 0$. A diffeomorphism f is Anosov if and only if it is quasi-Anosov and B_x is non-degenerate for all $x \in M$. There are quasi-Anosov diffeomorphisms that fail to be Anosov (see [8, 30], the examples in this paper have a strange attractor and a strange repeller [21] and the motion of most points evolves to the attractor and comes from the repeller). This characterization of quasi-Anosov (Anosov) diffeomorphisms permits to conclude the existence of a C^1 neighbourhood of f such that any finite composition of diffeomorphisms in that neighbourhood is also quasi-Anosov (Anosov). We shall see below that Anosov and quasi-Anosov diffeomorphisms (and hyperbolic sets) are expansive.

2.4. Pseudo-Anosov Homeomorphisms

Let f be a homeomorphism of an oriented compact surface M of genus larger than 1 onto itself. The map f is pseudo-Anosov if there exist two f-invariant, transversal foliations with singularities (see figure 1) W^S, W^U, and also two transversal measures μ_S, μ_U (defined on the space of (stable, unstable) leaves of W^S, respectively W^U) and $\lambda > 1$ such that $f^*(\mu_U) = \lambda\mu_U$ and

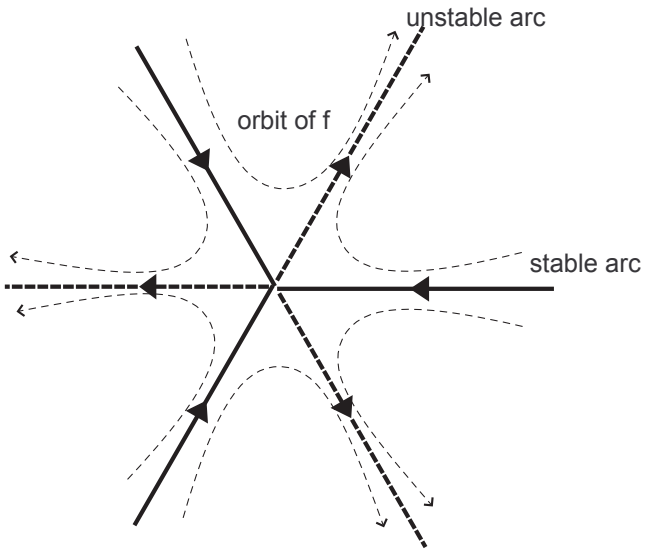

Figure 1: A singular point x of a pseudo-Anosov map f and nearby orbits.

$f^*(\mu_S) = \lambda^{-1}\mu_S$. The existence and expansivity of these homeomorphisms is proved in [32, 7].

2.5. Another Example

Let $f : T^2 \to T^2$ be defined by

$$f(x,y) = \left(2x + y - \frac{1}{2}\pi c \sin(2\pi x),\ x + y - \frac{1}{2}\pi c \sin(2\pi x)\right). \tag{1}$$

For $0 \leq c < 1$, f is Anosov (for $c = 0$ f is linear), but for $c = 1$, f is expansive but is neither Anosov nor quasi-Anosov since Tf_0 has no non-trivial invariant subspaces.

3. General Properties

Question: *Why not to define expansivity only for the future?*

Answer:

THEOREM 3.1. *[33]. Let M be a compact metric space and $f : M \to M$ be an homeomorphism such that there is $\alpha > 0$ with the property that for*

$x, y \in M, x \neq y$, $dist(f^n(x), f^n(y)) > \alpha$ for some $n \geq 0$. Then, M is finite.

3.1. Stable (Unstable) Sets

Let $f : M \to M$ be a homeomorphism; for $x \in M$, the stable set of x is

$$W^S(x) = \{y \in M : dist(f^n(x), f^n(y)) \to 0 \text{ if } n \to +\infty\},$$

and the unstable set is

$$W^U(x) = \{y \in M : dist(f^n(x), f^n(y)) \to 0 \text{ if } n \to -\infty\}.$$

The local stable (unstable) sets of x, are defined as follows: given $\varepsilon > 0$,

$$W^S_\varepsilon(x) = \{y \in M : dist(f^n(x), f^n(y)) \leq \varepsilon, n \geq 0\}$$

$$W^U_\varepsilon(x) = \{y \in M : dist(f^n(x), f^n(y)) \leq \varepsilon, n \leq 0\}$$

Let now f be expansive. May the stable set contain a neighbourhood of x for every $\varepsilon > 0$? In other words : may x be Lyapunov stable in the future? The answer is yes; it is easy to find a shift invariant subset of 2^Z for which 0 is Lyapunov stable in the future. Nevertheless,

THEOREM 3.2. *[17] If M is locally connected there are no stable points (either in the future or in the past).*

COROLLARY 3.3. *If M is locally connected, for every $\varepsilon > 0$, there is $r > 0$, such that for every $x \in M$, $W^S_\varepsilon(x)$ and $W^U_\varepsilon(x)$ contain a compact connected set of diameter larger than r.*

(Compare with the Denjoy map $f|D$; for points not lying on the added arcs the local stable (unstable) sets are trivial.)

APPLICATION 1. *There are no expansive homeomorphisms of S^1.*

Proof. Assume by contradiction that there exist an expansive homeomorphism on S^1. Then by the previous Corollary there are non-trivial stable open sets (a connected set of S^1 contains an open arc) and every point of it is a stable point, in contradiction with the above Theorem. □

3.2. Expansiveness and Lyapunov Functions.

THEOREM 3.4. *[15]. Let f be a homeomorphism of M, then f is expansive if and only if there exist a neighbourhood N of the diagonal in $M \times M$ and a real continuous function V (Lyapunov) defined on N, vanishing on the diagonal and such that for $(x, y) \in N, x \neq y$, $V(f(x), f(y)) - V(x, y) > 0$.*

In order to proof expansivity for Anosov and quasi-Anosov diffeomorphisms, the quadratic form B, mentioned in the section Anosov and quasi-Anosov diffeomorphisms, can be used to construct a Lyapunov function. In fact, for y close to x, the Lyapunov function is $V(x,y) = B_x(u)$, where $\exp_x(u) = y$. The expansivity of pseudo-Anosov maps may be shown also using Lyapunov functions [17]. For the examples in (1), choose

$$V(x,y) = V((x_1,x_2),(y_1,y_2)) = (y_1 - y_2)((x_1 - y_1) - (x_2 - y_2)).$$

4. On Surfaces.

THEOREM 4.1 (Classification Theorem). *([13, 18]). Let f be an expansive homeomorphism of a compact connected oriented boundaryless surface M. Then,*

- *S^2 does not support such a homeomorphism,*
- *if $M = T^2$, f is conjugate to an Anosov diffeomorphism*
- *if the genus of M is larger than 1, then f is conjugate to a pseudo-Anosov homeomorphism.*

(T^2 is the unique surface that supports Anosov diffeomorphisms.)

Those properties are consequences of the description of the local stable (unstable) sets of f.

Usually, the study of local stable (unstable) sets are made on the basis of strong assumptions on the dynamics of Tf, as for Anosov diffeomorphisms, hyperbolic sets, etc. In our case, even for expansive diffeomorphisms, we only have the dialogue between the topology of M and the dynamics of f. Nevertheless, after showing the local connectedness of the connected component containing x of $W^S_\varepsilon(x)(W^U_\varepsilon(x))$ the following theorem is proved.

THEOREM 4.2. *For $x \in M$, $W^S_\varepsilon(x)(W^U_\varepsilon(x))$ is the union of a finite number r of arcs, ($r \geq 2$) that meet only at x. Stable (unstable) sectors (the sets limited by two consecutive stable (unstable) arcs) are separated by unstable (resp. stable) arcs. If at $x \in M$, $r \geq 3$, x is called a singular point; the set of singular points is finite.*

When $r = 2$, as it is always the case for Anosov diffeomorphisms, x has a neighbourhood N such that if y and z belong to N, $W^S_\varepsilon(y) \cap W^U_\varepsilon(z)$ is not void. This is not the case for singular points (see figure 2).

Now a very brief mention of some steps of the proof of the Classification Theorem is given. For $r \geq 2$, if y and z lie in a sector then $W^S_\varepsilon(y)$ and $W^U_\varepsilon(z)$ meet only once. The set of these intersections includes, by the Theorem

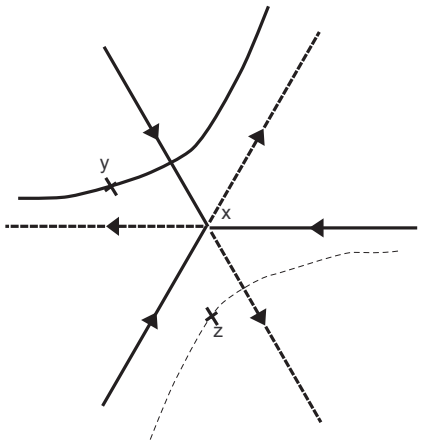

Figure 2: At the singular point x the stable manifold through y does not intersect the unstable one through z.

of invariance of domain, a neighbourhood of x in the sector (local product structure). This implies that singular points can not accumulate and then, their number is finite. Let now M^* be the universal cover of M. It is not difficult to show that the lifting to M^* of a stable or an unstable set is closed and that the union of the lifting of a stable arc and an unstable one can not be homeomorphic to S^1.

If S^2 supports an expansive homeomorphism, and $W^S(x)$ does not contain singular points, it is homeomorphic to S^1, and this in turn implies the existence of stable points; a contradiction.

That expansive homeomorphisms f of surfaces of genus ≥ 1 are conjugate to Anosov or to pseudo-Anosov maps follows from the following two Lemmas.

LEMMA 4.3. *An expansive homeomorphism f on a surface M of genus ≥ 1 is isotopic to an Anosov (if $M = T^2$) or to a pseudo-Anosov map (genus ≥ 1).*

Proof. It follows from [18] on account of Thurston's results [32]. □

DEFINITION 4.4. *Let f, g be homeomorphisms of the compact metric space M; f is semi-conjugate to g if there exists $h : M \to M$ continuous and surjective, such that $h \circ f = g \circ h$.*

LEMMA 4.5. *If the expansive homeomorphism f of the surface M is isotopic to an Anosov diffeomorphism, or to a pseudo-Anosov homeomorphism g, then f is semi-conjugate to g*

Proof. See [8, 18]. □

In both cases, $h^* : M^* \to M^*$, a lifting of the semi-conjugacy h is a proper map, and this fact is an essential tool to prove that the semi-conjugacy is, actually, a conjugacy.

5. Higher Dimension

Consider now expansive homeomorphisms f defined on compact boundaryless manifolds M of dimension larger than 2. In the case of surfaces, it follows from the Classification theorem that periodic points are dense on the surface, and, moreover, that on an open and dense set, $r = 2$. Thus for points x in that set, $W^S_\varepsilon(x)$ includes a topological 1-dimensional manifold and $W^U_\varepsilon(x)$ another such manifold, topologically transversal to the first one at x. The results concerning $dim M \geq 3$ assume the existence of a dense set of periodic points p such that $W^S_\varepsilon(p)$ contains a topological manifold of dimension d, $1 \leq d < \dim M$, and $W^U_\varepsilon(p)$ a manifold of complementary dimension, topologically transversal to $W^S_\varepsilon(p)$ at p. Points x with such a behaviour of $W^S_\varepsilon(x)$ and $W^U_\varepsilon(x)$ are called topologically hyperbolic. (This is the case for Anosov diffeomorphisms at every $x \in M$).

THEOREM 5.1. *([1, 34, 35]). Let f be an expansive homeomorphism of M with a dense set of topologically hyperbolic periodic points. Then there is an open and dense set with local product structure. Furthermore if* $\dim M \geq 3$, *and for some topologically hyperbolic periodic point p, either $W^S_\varepsilon(p)$ or $W^U_\varepsilon(p)$ is one-dimensional, M is a torus and f is conjugate to a linear Anosov diffeomorphism.*

Therefore, in this case, in contrast with what happens for surfaces, there are no singularities. This is, essentially, a consequence of the fact that if $\dim M \geq 3$, say, $W^S_\varepsilon(p)$ separates small balls centered at p, meanwhile $W^U_\varepsilon(p)$ does not. Of course, if we do not assume that one of this dimensions is one, the result is false: take the product of two pseudo-Anosov maps.

6. C^0-Perturbations of Expansive Systems

Let f be a homeomorphism of a compact metric space M onto itself.

6.1. Persistence

f is persistent if for any $\varepsilon > 0$ there exists a C^0-neighbourhood N of f such that for $g \in N$ and $x \in M$, there exists $y \in M$ with the following property: $dist(f^n(x), g^n(y)) \leq \varepsilon$, $n \in Z$

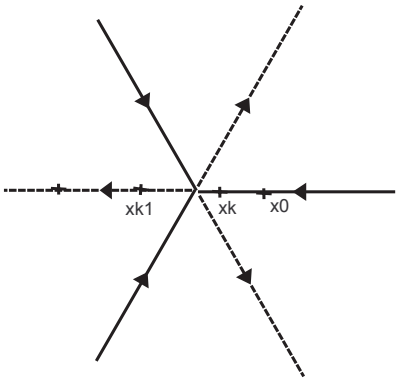

Figure 3: The pseudo orbit consisting of the f past of x_k and the f-future of x_{k+1} (x_{k+1} very close to x_k) is not shadowed by an f-orbit (see fig. 1)

6.2. Topological Stability

f is topologically stable if for $\varepsilon > 0$, there exists,N, a C^0-neighbourhood of f, such that any $g \in N$ is semi-conjugate to f (see 4)) and $dist(x, h(x)) < \varepsilon$.

6.3. Shadowing Property

A δ pseudo-orbit for f is a sequence $\{x_n : n \in Z\}$ such that $dist(f(x_n), x_{n+1}) < \delta$, $n \in Z$. Such a pseudo-orbit is ε shadowed if there is $y \in M$ such that $dist(x_n, f^n(y)) \leq \varepsilon, n \in Z$.

Clearly b) implies a) since the semi-conjugacy h is surjective, but a) does not imply b). All three properties are invariant under conjugacy. Anosov diffeomorphisms satisfy b) ([36]) and, since because of the classification theorem, every expansive homeomorphism of T^2 is conjugate to an Anosov, then all expansive homeomorphisms of T^2 satisfy b). A pseudo-Anosov homeomorphism f satisfies a) (see [10]) but not b); because, according to [37], for expansive systems b) is equivalent to c) and figure 3 shows an f pseudo-orbit shadowed by no f-trajectory; thus f does not satisfy c).

The quasi-Anosov diffeomorphisms are not even persistent. However each semi-trajectory is persistent : given $x \in M$, and $\varepsilon > 0$ there is, N_x, a C^0-neighbourhood of f such that for any $g \in N_x$, there is $y \in M$, with the property $dist(f^n(x), g^n(y)) \leq \varepsilon$, $n \geq 0$.

This is the f persistence of x in the future. We define similarly persistence in the past. A point x could be f persistent in the future and in the past without being persistent on both sides. This is the case of many points in a quasi-Anosov diffeomorphism. An open question is: are all the semi-trajectories of an expansive system persistent?

7. Links With the Tangent Map

Let M be a compact boundaryless smooth manifold, and let E be the set of all expansive diffeomorphisms of M.

THEOREM 7.1. *[21] The C^1-interior of E is the set of quasi-Anosov diffeomorphisms of M.*

On surfaces, quasi-Anosov diffeomorphisms are Anosov, and since in case M has genus larger than 1, M does not support Anosov diffeomorphisms, the interior mentioned in the theorem is, in this case, void. Thus, there are expansive diffeomorphisms which are not approximated by Anosov. Consider now the case $M = T^2$, where we do have Anosov diffeomorphisms. Since every expansive homeomorphism $f$is conjugate to a linear Anosov diffeomorphism l; $f = hlh^{-1}$ and according to [23] h may be C^0-approximated by a diffeomorphism g it follows easily, as glg^{-1} is Anosov, that f has arbitrarily C^0-close Anosov diffeomorphisms. However, it is not known, whether the C^1-closure of the C^1-interior of the expansive diffeomorphisms of T^2 includes all the expansive diffeomorphisms of the 2-torus. In other words: Is every expansive diffeomorphism the C^1-limit of Anosov diffeomorphisms? On the other hand, according to the results in [14], it is possible to conclude that such an expansive diffeomorphism has a dense set of periodic hyperbolic points.

8. Expansive Flows

We consider flows with no equilibrium points. Such a flow $\varphi_t : M \to M, t \in R$, is expansive if there exist $\alpha, \sigma > 0$, such that if $x, y \in M$, and

$$dist(\varphi_t(x), \varphi_{\tau(t)}(y)) \leq \alpha$$

for every $t \in R$, then $y = \varphi_{t_0}(x)$ for some $t_0, 0 \leq |t_0| \leq \sigma$. Here $\tau : R \to R$ is a re-parametrization of the flow through y, i.e, a surjective homeomorphism with $\tau(0) = 0$. This definition is somewhat more complicated than the one for discrete expansive systems as a consequence of the fact that we ask for geometric (instead of kinematic) separation. Important examples of expansive flows are geodesic flows on compact smooth Riemannian manifolds of negative curvature. A short list of papers concerning expansive flows are [3, 4, 5, 22, 16, 25, 26, 29, 30].

9. Non-Invertible Expansive Maps

This section refers to continuous maps f of a compact metric space M to itself that are not necessarily one-to-one. For those maps, a natural analogue to the notion of expansiveness is "positive expansiveness".

A map f is positively expansive if $dist(f^n(x), f^n(y)) \leq \alpha$; $n \geq 0$, implies $x = y$. A simple example of such a map is $f : S^1 \to S^1$;, $f(z) = z^n$, $n > 1$, where S^1 is the set of complex numbers z of modulus 1.

As in the preceding section we mention a short list of papers concerning, mainly, positively expansive maps [6, 11, 12, 24, 27, 28, 31].

10. Some Open Problems

As mentioned before, by the results of Mañe, the C^1-interior of Expansive Systems in the two-Torus consists of the Anosov diffeomorphisms.

Open Problem 10.1: Does the C^1-closure of the set of Anosov diffeomorphisms contains all expansive diffeomorphisms of the two-Torus? If the genus of the surface is larger than 1 this is not true, even for the C^0- closure; however, in the two-Torus this is true as mentioned before.

Open Problem 10.2: For an Expansive Homeomorphism is every semi-trajectory peristent in the future (in the past)? This is the case of all systems treated previously. (See [19]).

Open Problem 10.3: For $c > 1$ the map given in (1), has positive Lyapunov exponents on a set of positive measure? (See [20]).

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