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Variational Theory for Liouville Equations with Singularities

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Abstract. *In this note we consider a singular Liouville equation on compact surfaces, arising from the study of Chern-Simons vortices. Using improved versions of the Moser-Trudinger inequality and a min-max scheme, we prove existence of solutions in cases with lack of coercivity. Full details and further references can be found in the forthcoming paper [17].*

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1. Introduction

On a compact orientable surface (Σ, g) we consider the following problem

$$-\Delta_g u = \rho \left(\frac{h(x)e^{2u}}{\int_\Sigma h(x)e^{2u} dV_g} - 1 \right) - 2\pi\alpha \left(\delta_p - 1\right), \tag{1}$$

where ρ is a positive parameter, $h : \Sigma \to \mathbb{R}$ a smooth positive function, $\alpha \geq 0$ and $p \in \Sigma$. This equation has interest in physical models like the abelian Chern-Simons-Higgs theory and the Electroweak theory, see [11], [13], [14], [15]. We also refer to [20], [22] and the references therein for a more recent and complete description of the subject.

Most of the existing literature concerns asymptotic analysis or compactness of solutions, while relatively few results are available on existence. In [6], [12] some perturbative results are given, providing solutions of multi-bump type for special values of the parameter ρ. Our aim is to describe a global variational theory for the equation, which relies on improved Moser-Trudinger inequalities and min-max methods.

With a change of variables of the form

$$u \mapsto u + \alpha \log dist(x,p) \qquad \text{near } p \tag{2}$$

one can ransform (1) into the problem

$$-\Delta_g u + h_\rho = \rho \frac{\tilde{h}(x) e^{2u}}{\int_\Sigma \tilde{h}(x) e^{2u} dV_g} \qquad \text{on } \Sigma, \tag{3}$$

where

$$\tilde{h} > 0 \text{ on } \Sigma \setminus \{p\}; \qquad \tilde{h}(x) \simeq dist(x,p)^{2\alpha} \quad \text{near } p, \tag{4}$$

and where h_ρ satisfies

$$\int_\Sigma h_\rho dV_g = \rho, \tag{5}$$

which is a necessary condition for the solvability of (3), as one can see using integration by parts.

Equation (3) has a variational structure and the corresponding Euler-Lagrange functional is the following

$$I_\rho(u) = \int_\Sigma |\nabla_g u|^2 dV_g + 2 \int_\Sigma h_\rho u dV_g - \rho \log \int_\Sigma \tilde{h}(x) e^{2u} dV_g; \qquad u \in H^1(\Sigma). \tag{6}$$

The logarithmic term can be controlled by Moser-Trudinger type inequalities. In presence of a singular term $\tilde{h}$ as in (4), the value of the best constant was found in [21], see also [4], and the result under interest is reported in Proposition 2.1. From that result it follows that I_ρ is bounded from below if $\rho < 4\pi$, and that 4π is a threshold value, in the sense that for larger values of ρ the functional does not have a finite lower bound. However one can still hope to find critical points of saddle type using min-max schemes: using variational methods we prove the following results (which are particular cases of the results obtained in [17]).

THEOREM 1.1. *Suppose $\alpha \in (0,1]$ and that $\rho \in (4\pi, 4\pi(1+\alpha))$. Then if $\Sigma \not\simeq S^2$ problem* (1) *is solvable.*

THEOREM 1.2. *Suppose $\alpha \in (0,1)$ and that $\rho \in (4\pi(1+\alpha), 8\pi)$. Then* (1) *is always solvable.*

REMARK 1.3. *The assumption that $\Sigma \not\simeq S^2$ in Theorem 1.1 is necessary, since in [2] it is shown (via a Pohozaev identity) that on the standard sphere (S^2, g_0)* (1) *has no solution for $\rho \in (4\pi, 4\pi(1+\alpha))$, $\alpha > 0$. An explanation of the role of this condition in our proof is given below.*

One main tool for applying variational arguments in Liouville type equations is some kind of improvement of the Moser-Trudinger inequality. A classical example is a result by J.Moser, [18], in which he considered the case of the standard sphere. Assuming that u is an even function on S^2, he showed that the constant in (10) can be taken to be $\frac{1}{8\pi}$ and he found applications in prescribing even functions as Gauss curvatures on S^2. A more general improvement was obtained by T.Aubin in [1], still in the case of the standard sphere. He considered *balanced* metrics, namely those for which the conformal factor e^{2u} satisfies

$$\int_{S^2} x_i e^{2u} dV_{g_0} = 0; \qquad i = 1,2,3, \tag{7}$$

where the functions x_i are the restrictions of the Euclidean coordinates to S^2, embedded canonically into $\mathbb{R}^3$. In this case, Aubin showed that in the Moser-Trudinger inequality one can take any constant which is larger than $\frac{1}{8\pi}$. Applications were found in [4] where rather general conditions were given for prescribing the Gauss curvature on the sphere. An even more general improvement, which includes Aubin's one as a particular case, was obtained by W.Chen and C.Li in [5], where it was shown that if the conformal volume *spreads* into two distinct regions (separated by a fixed positive distance) of any given surface, then the best constant in the embedding reduces by nearly a factor two, see Proposition 2.2 below. Some applications were found in [7] to produce solutions of the regular Liouville equation, see also [8], [9] and [16] for further progress on this direction.

We will describe next a new improvement of the Moser-Trudinger inequality obtained in [17], which applies to the study of (1). To explain the spirit of the improvement we consider Proposition 2.1: if α is negative the best constant in front of the Dirichlet energy is larger than $\frac{1}{4\pi}$, but if α is positive (as in our case) the best constant is simply $\frac{1}{4\pi}$, as for the standard Moser-Trudinger inequality.

One can easily see this testing the inequality on a *standard bubble*, namely a function of the form

$$\varphi_{\lambda,x}(y) = \log \frac{\lambda}{1 + \lambda^2 dist(x,y)^2}, \tag{8}$$

with center point x different from p. This function realizes the best constant in the regular case (in the limit $\lambda \to \infty$), and for the above choice of x there is basically no effect from the vanishing of $\tilde{h}$ somewhere on Σ. On the other hand, in [10] it was shown that for any $\alpha > -1$ there exists C_α such that

$$\log \int_B |x|^{2\alpha} e^{2(u-\overline{u})} dV_g \leq \frac{1}{4(1+\alpha)\pi} \int_B |\nabla_g u|^2 dx + C_\alpha; \qquad u \in H_r^1(B).$$

In the latter formula B stands for the unit disk of $\mathbb{R}^2$ and H_r^1 denotes the space of radial functions of class H^1 in B.

The improvement in [17] basically substitutes the symmetry requirement with a condition on the *center of mass* of the function $\tilde{h}e^{2u}$, which holds for a family of functions in $H^1(\Sigma)$ with codimension 2. A more precise statement is given in Section 2, but we can roughly state that if the center of mass of $\tilde{h}e^{2u}$ coincides with the singularity p then the constant in (10) can be taken to be $\frac{1}{4\pi(1+\alpha)}$. The proof of this result, which is not reported here for reasons of brevity, requires different arguments from Aubin's and Chen-Li's. The main new feature of our improvement is that it is scaling invariant, and in a sense what matters to get a better constant is the closeness to p of the center of mass to $\tilde{h}e^{2u}$, compared to its *scale* of concentration.

To prove existence we look at the structure of very low sublevels of I_ρ. From Proposition 2.2 one deduces that if $\rho < 8\pi$ and if $I_\rho(u)$ is sufficiently negative, then $\tilde{h}e^{2u}$ should be concentrated near at most one point of Σ. Under the assumptions of Theorem 1.1, namely for $\rho < 4\pi(1+\alpha)$, we learn from the improved inequality that if $I_\rho(u)$ is negative then $\tilde{h}e^{2u}$ should not be too concentrated near p. It is then natural to define a projection $\Psi : \{I_\rho \leq -L\} \to \Sigma \setminus \{p\}$ which associates to u the center of mass of $\tilde{h}e^{2u}$.

Viceversa, it is possible to construct a map from $\Sigma \setminus \{p\}$ into arbitrarily low sublevels of I_ρ using the above functions $\varphi_{\lambda,x}$, namely $I_\rho(\varphi_{\lambda,x}) \to -\infty$ uniformly for x in any compact set of $\Sigma \setminus \{p\}$. Furthermore, one has that $x \mapsto \varphi_{\lambda,x} \mapsto \Psi(\varphi_{\lambda,x})$ is a map homotopic to the identity. Using this fact and the non contractibility of $\Sigma \setminus \{p\}$ (for $\Sigma \not\equiv S^2$) we use a min-max scheme which provides existence of solutions. The details are described in Section 3.

In the case of Theorem 1.2, for $\rho > 4\pi(1+\alpha)$, the improved inequality does not give any new information, and the map Ψ is now with values in Σ, including possibly also p. On the other hand, one can use test functions like

$$\varphi_{\alpha,\lambda,x}(y) = \log \frac{\lambda^{\alpha+1}}{\left(1 + (\lambda dist(x,y))^{2(1+\alpha)}\right)}, \tag{9}$$

to repeat the above argument substituting $\Sigma \setminus \{p\}$ with the whole surface Σ, which is always non contractible.

In Section 2 we collect some preliminary results on the Moser-Trudinger inequality and a known improvement, together with some compactness results. In Section 3 we then illustrate the new improved inequality from [17] and the min-max scheme which is used to prove existence.

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2. Preliminary Facts

In this section we fix the notation and recall some useful known results, concerning Moser-Trudinger type inequalities and compactness results.

We write $dist(x,y)$ to denote the distance between two points $x, y \in \Sigma$. Moreover, the symbol $B_r(p)$ denotes the open metric ball of radius r and center p. $H^1(\Sigma)$ is the Sobolev space of the functions on Σ which are in $L^2(\Sigma)$ together with their first derivatives. The symbol $\|\cdot\|$ will denote the norm of $H^1(\Sigma)$. If $u \in H^1(\Sigma)$, $\overline{u} = \frac{1}{|\Sigma|}\int_\Sigma u dV_g$ stands for the average of u.

Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc.. Also constants with this kind of subscripts are allowed to vary.

We begin by recalling the Moser-Trudinger inequality, and some if its variants. The Moser-Trudinger inequality asserts that, on a compact surface Σ

$$\log \int_\Sigma e^{2(u-\overline{u})} dV_g \leq \frac{1}{4\pi}\int_\Sigma |\nabla u|^2 + C_g \qquad \text{for all } u \in H^1(\Sigma).$$

In presence of a singular term, the value of the best constant was derived by M.Troyanov, who obtained the following result.

PROPOSITION 2.1. *([21]) Let $\alpha > -1$, and let $\tilde{h} : \Sigma \to \mathbb{R}$ be as in* (4)*. Then one has the inequality*

$$\log \int_\Sigma \tilde{h}(x) e^{2(u-\overline{u})} dV_g \leq \frac{1}{4\pi \min\{1, 1+\alpha\}}\int_\Sigma |\nabla u|^2 + C_{\tilde{h},g} \tag{10}$$

for all $u \in H^1(\Sigma)$.

As one can see, if $\alpha < 0$ the constant in front of the Dirichlet energy is worse than the standard one $\frac{1}{4\pi}$. However, if $\alpha > 0$ the inequality does not improve in general. Below, we will show that this happens indeed for suitable functions u.

The next result, proved in [5], states that if the function $\tilde{h}e^{2u}$ is *spread* into two distinct regions of Σ, then the constant in the Moser-Trudinger inequality can be basically divided by two. The presence of the singular function $\tilde{h}$ only requires obvious modifications compared to the regular case.

PROPOSITION 2.2. *([5]) Let Σ be a compact surface, and $\tilde{h}$ as in* (4)*. Let Ω_1, Ω_2 be subsets of Σ with $dist(\Omega_1, \Omega_2) \geq \delta_0$ for some $\delta_0 > 0$, and fix $\gamma_0 \in \left(0, \frac{1}{2}\right)$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \delta_0, \gamma_0)$ such that*

$$\log \int_\Sigma \tilde{h}(x) e^{2(u-\overline{u})} dV_g \leq C + \frac{1}{8\pi - \varepsilon}\int_\Sigma |\nabla_g u|^2 dV_g$$

for all functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{\Omega_i} \tilde{h}(x) e^{2u} dV_g}{\int_\Sigma \tilde{h}(x) e^{2u} dV_g} \geq \gamma_0, \qquad i = 1, 2. \tag{11}$$

A useful corollary of this result in the following one: for the proof see [5] or [7].

COROLLARY 2.3. *Suppose $\rho < 8\pi$. Then, given any $\varepsilon, r > 0$ there exists $L = L(\varepsilon, r) > 0$ such that*

$$I_\rho(u) \leq -L \qquad \Rightarrow \qquad \frac{\displaystyle\int_{B_x(r)} \tilde{h}(x) e^{2u} \, dV_g}{\displaystyle\int_\Sigma \tilde{h}(x) e^{2u} \, dV_g} > 1 - \varepsilon \quad \text{for some } x \in \Sigma.$$

We conclude this section by mentioning a compactness result obtained in [3], proved though blow-up analysis of solutions.

THEOREM 2.4. *([3]) Let Σ be a compact surface, and let u_i solve (3) with $\rho = \rho_i$, $\rho_i \to \overline{\rho}$, with $\alpha > 0$ and $p \in \Sigma$. Suppose that $\int_\Omega e^{2u_i} \leq C_1$ for some fixed $C_1 > 0$. Then along a subsequence u_{i_k} one of the following alternative holds:*

(i) *u_{i_k} is uniformly bounded from above on Σ;*

(ii) $\max_\Sigma \left(u_{i_k} - \frac{1}{2} \log \int_\Sigma \tilde{h} e^{2u_{i_k}} \right) \to +\infty$ *and there exists a finite blow-up set $S = \{q_1, \dots, q_l\} \in \Sigma$ such that*

(a) for any $s \in \{1, \dots, l\}$ there exist $x_k^s \to q_s$ such that $u_{i_k}(x_k^s) \to +\infty$ and $u_{i_k} \to -\infty$ uniformly on the compact sets of $\Sigma \setminus S$,

(b) $\rho_{i_k} \frac{\tilde{h} e^{2u_{i_k}}}{\int_\Sigma \tilde{h} e^{2u_{i_k}} dV_g} \rightharpoonup \sum_{s=1}^l \beta_s \delta_{q_s}$ in the sense of measures, with $\beta_s = 4\pi$ for $q_s \neq \{p\}$, or $\beta_s = 4\pi(1+\alpha)$ if $q_s = p$. In particular one has that

$$\overline{\rho} = 4\pi n + 4\pi(1 + \alpha),$$

for some $n \in \mathbb{N} \cup \{0\}$, and $\overline{\rho} > 0$.

From the above result we obtain immediately the following corollary.

COROLLARY 2.5. *Suppose we are in the above situation, and that*

$$\rho \in (4\pi, 8\pi), \qquad \rho \neq 4\pi(1 + \alpha).$$

Then the solutions of (3) stay uniformly bounded in $C^2(\Sigma)$.

3. Proof of the Theorems

We begin by considering some test functions on which the functional attains arbitrarily negative values.

PROPOSITION 3.1. *Let $\varphi_{\lambda,x}$ be as in* (8)*. Then for $\rho > 4\pi$ one has*

$$I_\rho(\varphi_{\lambda,x}) \to -\infty$$

uniformly for x in any compact set of $\Sigma \setminus \{p\}$.

Proof. First of all we notice that, working in geodesic coordinates at x, the following estimates can be easily derived

$$\int_\Sigma e^{2\varphi_{\lambda,x}} dV_g \geq C^{-1} \text{ uniformly for } x \in \Sigma.$$

Therefore, if x belongs to a compact set K, by (4) we also have that

$$\log \int_\Sigma \tilde{h} e^{2\varphi_{\lambda,x}} dV_g \geq -C_K \text{ uniformly for } x \in \Sigma.$$

Fixing a small number δ, we notice that

$$\varphi_{\lambda,x}(y) \leq \log \lambda \quad \text{ in } B_\delta(x),$$

and that

$$|\varphi_{\lambda,x}(y) + \log \lambda| \leq C_\delta \text{ in } \Sigma \setminus B_\delta(x).$$

From the latter two formulas we deduce

$$\fint_\Sigma \varphi_{\lambda,x} dV_g = -(1 + O(\delta)) \log \lambda + O_\delta(1) \quad \text{ as } \lambda \to +\infty.$$

It remains to estimate the Dirichlet energy: by elementary calculations one finds the estimate

$$|\nabla \varphi_{\lambda,x}(y)| \leq \min \left\{ C\lambda, \frac{2}{dist(x,y)} \right\}$$

for some large constant C. Dividing the Dirichlet integral into the ball $B_{\frac{1}{\lambda}}(x)$ and its complement one then finds

$$\int_\Sigma |\nabla \varphi_{\lambda,x}|^2 dV_g \leq 8\pi \log \lambda - (1 + O(\delta)) \log \lambda + O_\delta(1) \quad \text{ as } \lambda \to +\infty.$$

From the last formulas it follows that

$$I_\rho(\varphi_{\lambda,x}) \leq (8\pi - 2\rho + O(\delta)) \log \lambda + O_\delta(1) + O_K(1) \quad \text{ as } \lambda \to +\infty,$$

which concludes the proof. $\square$

We next state, without proof, the following result, yielding continuous (and non trivial if $\Sigma \not\simeq S^2$) maps from low sublevels of I_ρ into the pointed surface $\Sigma \setminus \{p\}$.

Proposition 3.2. *Suppose $\rho \in (4\pi, 4\pi(1+\alpha))$. Then there exists $L > 0$ and a map $\Psi : \{I_\rho \leq -L\} \to \Sigma \setminus \{p\}$ with the following property. If $\varphi_{\lambda,x}$ is as in (8), then one has $\Psi(\varphi_{\lambda,x}) \to x$ as $\lambda \to +\infty$ uniformly on the compact sets of $\Sigma \setminus \{p\}$.*

We notice that for Ψ to be well defined on the sublevel $\{I_\rho \leq -L\}$ we are using Proposition 3.1. The fact that Ψ has values into $\Sigma \setminus \{p\}$ is one of the main novelties in [17], and relies on a new improved Moser-Trudinger inequality. The way to prove it is to construct a sort of barycentric map β when $\tilde{h}e^{2u}$ is mostly concentrated near p. To do this, one first finds a *scale of concentration* for any point x finding an annulus (with a fixed ratio of the radii) for which the integrals of $\tilde{h}e^{2u}$ on the components of its complement coincide. We then look at the points where these integrals are maximal possible. If some of these are sufficiently close to p (at a proper scale), then the improvement holds true.

We next define the min-max scheme which is needed to prove Theorem 1.1. Consider the compact set $\Sigma_\tau := \Sigma \setminus B_\tau(p)$, and fix L so large that the map Ψ in Proposition 3.2 is well defined on $\{I_\rho \leq -L\}$. We then define the set

$$\Lambda_\lambda = \{\varphi_{\lambda,x} \; : \; x \in \Sigma_\tau\}.$$

Next, we consider the topological cone over Σ_τ

$$\hat{\Sigma}_\tau = (\Sigma_\tau \times [0,1]) / (\Sigma_\tau \times \{1\}),$$

where we are identifying all points in $\Sigma_\tau \times \{1\}$. Finally, we introduce the family of continuous maps

$$\mathcal{H}_{\lambda,\rho} = \left\{ \mathfrak{h} : \hat{\Sigma}_\tau \to H^1(\Sigma) \; : \; \mathfrak{h}(y) = \varphi_{\lambda,x} \text{ for every } x \in \Sigma_\tau \right\},$$

and the number

$$\overline{\mathcal{H}}_{\lambda,\rho} = \inf_{\mathfrak{h} \in \mathcal{H}_\lambda} \sup_{z \in \hat{\Sigma}_\tau} I_\rho(\mathfrak{h}(z)).$$

We have the following result.

Proposition 3.3. *Under the assumptions of Theorem 1.1, if λ is sufficiently large the number $\overline{\mathcal{H}}_{\lambda,\rho}$ is finite.*

Proof. If L is as in Proposition 3.2 and if λ is so large that $\sup_{x \in \Sigma_\tau} I_\rho(\varphi_{\lambda,x}) \leq -2L$, we show that $\overline{\mathcal{H}}_{\lambda,\rho} > -\frac{3}{2}L$. In fact, suppose by contradiction that there exists a map $\mathfrak{h}_0$ such that

$$\mathfrak{h}_0 \in \mathcal{H}_{\lambda,\rho} \qquad \text{and} \qquad \sup_{z \in \hat{\Sigma}_\tau} I_\rho(\mathfrak{h}_0(z)) \leq -\frac{3}{2}L. \tag{12}$$

Then Proposition 3.2 applies and gives a continuous map $F_{\lambda,\rho} : \hat{\Sigma}_\tau \to \Sigma_\tau$ defined as the composition

$$F_{\lambda,\rho} := T_\tau \circ \Psi \circ \mathfrak{h}_0,$$

where T_τ is a retraction of $\Sigma \setminus \{p\}$ to Σ_τ.

Since $\mathfrak{h}_0 \in \mathcal{H}_{\lambda,\rho}$, and hence it coincides with $\varphi_{\lambda,\cdot}$ on $\partial\hat{\Sigma}_\tau \simeq \Sigma_\tau$, by Proposition 3.2 we deduce that

$$F_{\lambda,\rho} \text{ is homotopic to } Id|_{\Sigma_\tau}. \tag{13}$$

On the other hand, if we let ϖ run between 1 and 0, and we consider the maps $F_{\lambda,\rho}|_{\cdot,t=\varpi} : \Sigma_\tau \to \Sigma_\tau$, we obtain a homotopy between the identity on Σ_τ and a constant map. Since by our assumptions Σ_τ is non contractible, we obtain a contradiction. This proves the desired statement. □

Proof of Theorem 1.1. To check that $\overline{\mathcal{H}}_{\lambda,\rho}$ is a critical level, one can use a monotonicity method introduced by Struwe, see [19], and which has been used extensively in the study of Liouville type equations. We consider a sequence $\rho_n \to \rho$ and the corresponding functionals I_{ρ_n}. All the above estimates and results can be worked out for I_{ρ_n} as well with minor changes.

We then define the min-max value $\tilde{\mathcal{H}}_{\lambda,\rho} := \frac{\overline{\mathcal{H}}_{\lambda,\rho}}{\rho}$, which corresponds to the functional $\frac{I_\rho}{\rho}$. It is immediate to see that

$$\rho \mapsto \tilde{\mathcal{H}}_{\lambda,\rho} \qquad \text{is monotone.}$$

This implies that the map $\rho \mapsto \overline{\mathcal{H}}_{\lambda,\rho}$ is almost everywhere differentiable. Reasoning as in [7], from the differentiability one finds that there exists a subsequence of $(\rho_n)_n$ such that I_{ρ_n} has a solution u_n at level $\overline{\mathcal{H}}_{\lambda,\rho_n}$. Then, applying Theorem 2.4 and passing to a further subsequence, we obtain that u_n converges to a critical point u of I_ρ at level $\overline{\mathcal{H}}_{\lambda,\rho}$. □

Proof of Theorem 1.2. In the existence argument for the previous theorem we substitute the set Σ_τ with the whole surface Σ. The main change which is needed is the counterpart of Proposition 3.1. If one chooses the test function in (9) then the following estimate for the gradient holds true

$$|\nabla\varphi_{\alpha,\lambda,x}(y)| \leq \min\left\{C\lambda, \frac{2(1+\alpha)}{dist(x,y)}\right\}$$

for some large constant C. Using estimates similar to the above ones one gets

$$I_\rho(\nabla\varphi_{\alpha,\lambda,x}) \leq (8\pi(1+\alpha)^2 - 2(1+\alpha)\rho + O(\delta))\log\lambda + O_\delta(1) \quad \text{ as } \lambda \to +\infty.$$

If $\rho > 4\pi(1+\alpha)$ then $I_\rho(\nabla\varphi_{\alpha,\lambda,x}) \to -\infty$ as $\lambda \to +\infty$ uniformly for $x \in \Sigma$. Then the above argument goes though with minor modifications. □

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