A Deformed Bargmann Transform by an $SU(2)$ Matrix Parameter

ALLAL GHANMI AND ZOUHAİR MOUAYN

ABSTRACT. The Laguerre 2D polynomials depending on an arbitrary matrix $Q$ in $SU(2)$ as a fixed parameter are used to construct a set of coherent states. The corresponding coherent state transforms constitute a deformation by matrix $Q$ of a generalized Bargmann transform.

Keywords: Laguerre 2D Polynomials, Coherent States Transform, Deformed Bargmann Transform.

MS Classification 2000: 33C45, 81R30, 44A05

1. Introduction

The Bargmann transform, originally introduced by V. Bargmann [1], is a windowed Fourier transform [5]. It is closely connected to the Heisenberg group and has many applications in quantum optics as well as in signal processing and harmonic analysis on phase space [3]. This transform defined through

$$\mathfrak{B}_0[f](z) := \int_{\mathbb{R}} f(x) e^{-x^2 + 2xz - \frac{1}{2}z^2} dx, \quad z \in \mathbb{C},$$

maps isometrically the space $L^2(\mathbb{R}, dx)$ of square integrable functions on the real line onto the Bargmann-Fock space $\mathcal{F}(\mathbb{C})$ of entire complex-valued functions which are $e^{-|z|^2} d\mu$-square integrable, $d\mu$ being the Lebesgue measure on $\mathbb{C}$.

In [2] H-Y. Chen and J. Fan have constructed an integral transform, called there generalized Bargmann transform, by

$$\mathfrak{B}[\varphi](z, w) := \int_{\mathbb{C}} \exp \left( -zw + w\xi + z\bar{\xi} - \frac{1}{2} |\xi|^2 \right) \overline{\varphi(\xi)} d\mu(\xi) \quad (1)$$

as a transform of two-mode Fock space represented by a two-variable complex Laguerre polynomials, which naturally accompanies Einstein-Podolsky-Rosen entangled states of continuous variables.

Our aim here is to construct a kind of deformation $\mathfrak{B}^Q$ of (1) by means of an arbitrary parameter matrix $Q$ belonging to the special unitary group $SU(2)$. 
such that for $Q = I$, being the identity matrix, the kernel of $\mathfrak{B}^I$ coincides with that of (1). Indeed, we define:

$$\mathfrak{B}^Q [\varphi](\xi) := \int_C \exp \left( 3Q^t \Xi(\xi) - \frac{1}{2} 3\Lambda^t Q^t 3 - \frac{1}{2} |\xi|^2 \right) \overline{\varphi(\xi)} d\mu(\xi),$$

where $\varphi$ belongs to a suitable class of functions, $^t 3$ (resp. $^t \Xi(\xi)$) denotes the matrix transpose of $3 = (z, w) \in \mathbb{C}^2$ (resp. $\Xi(\xi) = (\xi, \xi^*)$) and $\Lambda := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This can be handled by adopting a coherent states method [7]. The physical meaning of the obtained deformed Bargmann transform (2) is encoded in the two-variable complex Laguerre polynomials depending on a matrix $Q$ as introduced by A. Wünsche [10], and occurring in the quantum mechanics of a degenerate $2D$ harmonic oscillator.

The paper is organized as follows. In Section 2, we shall recall some needed facts on the Laguerre $2D$ polynomials. Section 3 deals with a formalism of generalized coherent states. This formalism is applied in Section 4 so as to define a matrix parameter family of generalized coherent states and to discuss the corresponding coherent state transforms.

2. The Laguerre $2D$ Polynomials

The Laguerre $2D$ polynomials $L_{m,n}^Q(\xi, \xi^*)$ defined in [10] are polynomials of the pair complex conjugated variables $(\xi, \xi^*)$, which depend on an arbitrary fixed $2D$ matrix $Q$ as parameter. In fact, we have

$$L_{m,n}^Q(\xi, \xi^*) = \exp \left( - \frac{\partial^2}{\partial \xi \partial \xi^*} \right) (\xi')^m (\xi^*)^n; \quad m, n = 0, 1, 2, ...$$

where for $Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}$ we have

$$\begin{pmatrix} \xi' \\ \xi^* \end{pmatrix} = Q \begin{pmatrix} \xi \\ \xi^* \end{pmatrix} = \begin{pmatrix} \alpha \xi + \beta \xi^* \\ \gamma \xi + \sigma \xi^* \end{pmatrix}.$$ 

In the special case of $Q$ being the identity matrix $I$, definition (3) provides explicitly

$$L_{m,n}^I(\xi, \xi^*) = (-1)^n n! \xi^{m-n} L_n^{(m-n)}(\xi \xi^*) = (-1)^m m! \xi^{*n-m} L_m^{(n-m)}(\xi^*),$$

where $L_m^{(\alpha)}(\cdot)$ denote the generalized Laguerre polynomials and $L_m^{(0)}(\cdot) = L_m(\cdot)$ are the ordinary Laguerre polynomials [4].
Note that for an arbitrary matrix $Q$ the polynomials $L^Q_{m,n}(\xi,\xi^*)$ are still connected to the polynomials $L_{m,n} := L^I_{m,n}$ through the relation [10, p. 670]:

\[
L^Q_{m,n}(\xi,\xi^*) = (\sqrt{\det Q})^{m+n} \sum_{j=0}^{m+n} \left( \frac{\beta}{\sqrt{\det Q}} \right)^{m-j} \left( \frac{\sigma}{\sqrt{\det Q}} \right)^{n-j} \times \mathcal{P}_{j(m-j,n-j)} \left( 1 + \frac{2\alpha\gamma}{\det Q} \right) L_{j,m+n-j}(\xi,\xi^*)
\]

where \( \mathcal{P}_j(\alpha,\beta) (\cdot) \) denotes the Jacobi polynomial [4]. It should be also noted that there is a relation between the two polynomials $L^Q_{m,n}(\cdot,\cdot)$ and $L^Q_{p,s}(\cdot,\cdot)$ in the degenerate case of vanishing determinant of $Q$ see [10, p. 671].

Beside the Laguerre 2D polynomials, Wünsche has introduced the Laguerre 2D functions as

\[
\mathcal{L}^Q_{m,n}(\xi,\xi^*) := e^{-\frac{1}{2}|\xi|^2} \frac{L^Q_{m,n}(\xi,\xi^*)}{\sqrt{\pi m! n!}}
\]

and has established for general 2D matrix $Q$ the following orthonormalization relations:

\[
\int_C i \frac{1}{2} (d\xi \wedge d\xi^*) \mathcal{L}^Q_{m,n}(\xi,\xi^*) \mathcal{L}^{(Q)^{-1}}_{k,l}(\zeta^*,\zeta) = \delta_{m,k} \delta_{n,l},
\]

where \( \frac{1}{2}(d\xi \wedge d\xi^*) = d\mu(\xi) \) is the area element of the plane. Here \( (Q)^{-1} \) denotes the transposed matrix of $Q$ and \( \delta_{m,k} \) the Kronecker symbol. In addition, we have the completeness relation:

\[
\sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \mathcal{L}^Q_{m,n}(\xi,\xi^*) \mathcal{L}^{(Q)^{-1}}_{m,n}(\zeta^*,\zeta) = \delta(\xi - \zeta, \xi^* - \zeta^*),
\]

where \( \delta(\xi,\xi^*) = \delta(\Re \xi) \delta(\Im \xi) \) denotes the two-dimensional delta function.

For our purpose, we fix $Q$ in the special unitary group $SU(2)$, i.e., so that its inverse $Q^{-1}$ be equal to the transpose of its conjugate. Thus, one can easily see from (5) and (6) that the Laguerre 2D polynomials satisfy the following property

\[
\int_C |L^Q_{m,n}(\xi,\xi^*)|^2 e^{-|\xi|^2} d\mu = \sqrt{\pi m! n!}
\]

which means that the function $\xi \mapsto L^Q_{m,n}(\xi,\xi^*)$ belongs to the Hilbert space $L^2(\mathbb{C}; e^{-|\xi|^2} d\mu)$ of complex-valued Gaussian square integrable functions on $\mathbb{C}$. Consequently, the Laguerre 2D functions are elements of the Hilbert space.
Indeed, these functions can be viewed as unitary transforms of the normalized Laguerre $2D$ polynomials as

$$L_m^Q(\xi, \xi^*) := T^{-1}[L_m^Q(\xi, \xi^*)]$$

where $T$ is the unitary map from $L^2(\mathbb{C}; d\mu)$ to $L^2(\mathbb{C}; e^{-|\xi|^2} d\mu)$ defined by

$$T[\phi](\zeta) := e^{\frac{1}{2}|\zeta|^2} \sqrt{\rho_k} \phi(\zeta), \quad \phi \in L^2(\mathbb{C}; d\mu),$$

called a ground state transformation. These precisions are just to make sense when talking about the closure in $L^2(\mathbb{C}; d\mu)$ of the vector space spanned by all linear combinations of the Laguerre $2D$ functions.

**Remark 2.1.** The involved polynomials $L_{m,n}^Q(\xi, \xi^*)$ in (4), corresponding to the special case of the identity matrix $Q = I$, play an important role when studying representations of quasi-probabilities in quantum optics [8, 9]. Indeed, for $Q = I$ the identity (4) can be used to describe the transition from linear polarization to circular polarization or for a beam splitter to the splitting of a beam into two partial beams of equal intensity [6].

### 3. Generalized Coherent States

In this section, we present a generalization of coherent states according to the procedure in [7]. For this, let $(X, \nu)$ be a measure space and $A \subset L^2(X, \nu)$ a closed subspace of infinite dimension. Let $\{f_k\}_{k=0}^\infty$ be a given orthogonal basis of $A$ satisfying

$$\omega(a) := \mathcal{K}(a, a) := \sum_{k=0}^\infty |f_k(a)|^2 < +\infty; \quad a \in X,$$

where $\rho_k := \|f_k\|_{L^2(X, \nu)}^2$ and

$$\mathcal{K}(a, b) := \sum_{k=0}^\infty \rho_k^{-1} f_k(a) \overline{f_k(b)}, \quad a, b \in X,$$

is the reproducing kernel of the Hilbert space $A$.

**Definition 3.1.** Let $\mathcal{H}$ be an infinite Hilbert space with an orthonormal basis $\{\psi_k\}_{k=0}^\infty$. The coherent states labeled by points $a \in X$ are defined as the ket-vectors $|\phi_a >$ of $\mathcal{H}$:

$$|\phi_a > := (\omega(a))^{-\frac{1}{2}} \sum_{k=0}^\infty \frac{f_k(a)}{\sqrt{\rho_k}} \psi_k.$$
Then, it is straightforward to show that $<\phi_a | \phi_a> = 1$.

**Definition 3.2.** The coherent state transform corresponding to the set of coherent states $|\phi_a>$ is the isometric mapping $W : \mathcal{H} \rightarrow \mathcal{A} \subset L^2(X, \nu)$ defined by

$$W[\psi](a) := (\omega(a))^\frac{1}{2} <\phi_a | \psi>$$

Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$<\phi | \psi>_{\mathcal{H}} = <W[\phi] | W[\psi]>_{L^2(X, \nu)} = \int_X d\nu(a) \omega(a) <\phi | \phi_a><\phi_a | \psi>.$$ 

Thereby, we have a resolution of the identity of $\mathcal{H}$ which can be expressed in Dirac’s bra-ket notation as

$$1_{\mathcal{H}} = \int_X d\nu(a) \omega(a) |\phi_a><\phi_a|,$$

where $\omega(a)$ appears as a weight function. The notation $|\phi_a><\phi_a|$ means the rank one operator.

**Remark 3.3.** Note that the formula (11) can be considered as a generalization of the series expansion of the canonical coherent states:

$$|\phi_z> := e^{-\frac{1}{2} |z|^2} \sum_{k=0}^{+\infty} \frac{z^k}{\sqrt{k!}} \psi_k, z \in \mathbb{C},$$

where $\{\psi_k\}_{k=0}^{+\infty}$ denotes an orthonormal basis of eigenstates of the quantum harmonic oscillator, consisting of Gaussian-Hermite functions in $L^2(\mathbb{R}, dx)$. In this case, the space $\mathcal{A}$ is nothing but the Bargmann-Fock space $\mathcal{F}(\mathbb{C})$ and $\omega(z) = \pi^{-1} e^{-|z|^2}, z \in \mathbb{C}$.

### 4. A Coherent State Transform Associated with Laguerre 2D Functions

We are now going to attach to Laguerre 2D polynomials with a fixed matrix parameter $Q \in SU(2)$ a set of coherent states by using the formalism described in Section 3. This can be handled by considering the following points:

- $(X, \nu) = (\mathbb{C}^2, e^{-|z|^2-|w|^2} d\mu), d\mu(z, w)$ is the Lebesgue measure on $\mathbb{C}^2$. 

\( A := \mathcal{F}(\mathbb{C}^2) \subset L^2(\mathbb{C}^2, e^{-|z|^2} \, d\mu) \) denotes the Bargmann-Fock space of entire functions \( \varphi : \mathbb{C}^2 \to \mathbb{C} \) with finite norm square
\[
||\varphi||^2 := \int_{\mathbb{C}^2} \varphi(z,w)\overline{\varphi(z,w)} e^{-|z|^2-|w|^2} \, d\mu(z,w) < +\infty. \tag{17}
\]

Its reproducing kernel is known to be given by
\[
\omega(z,w) = K((z_1,w_1), (z_2,w_2)) = \pi^{-2} e^{-|z_1|^2+|w_1|^2}. \tag{18}
\]

\( \{f_{m,n}\}_{m,n=0}^{+\infty} \) is an orthogonal basis of \( A \) given by
\[
f_{m,n}(z,w) := z^m w^n; \quad m, n = 0, 1, 2, \ldots \tag{19}
\]
whose the norm is given by \( \rho_{m,n} = ||f_{m,n}||^2 = \pi^m m! \).

\( Q \in SU(2) \) is a fixed matrix parameter and \( \mathcal{H}_Q(\mathbb{C}) \) denotes the Hilbert subspace of \( L^2(\mathbb{C}, \mu) \) obtained as the closure of vector space \( \text{span}(L^Q_{m,n}) \) spanned by all linear combinations of the Laguerre 2D functions \( L^Q_{m,n} \) in (5).

**Definition 4.1.** The vectors \( (\Phi_{3,Q}) \) of the Hilbert space \( \mathcal{H}_Q(\mathbb{C}) \) labelled by elements \( 3 = (z,w) \in \mathbb{C}^2 \) and defined formally through (13) by
\[
\Phi_{3,Q} \equiv |(z,w), Q> := (\omega(z,w))^{-\frac{1}{2}} \sum_{m,n=0}^{+\infty} \frac{f_{m,n}(z,w)}{\sqrt{\rho_{m,n}}} \mathcal{Q}^Q_{m,n}, \tag{20}
\]
are called generalized coherent states.

**Proposition 4.2.** The wave functions of the states in (20) admit the following closed form
\[
\Phi_{3,Q}(\xi) = e^{-\frac{1}{2}(|3|^2+|\xi|^2)} \exp\left(3Q^2 \Xi(\xi) - \frac{1}{2}3A^tQ^t3\right), \tag{21}
\]
where \( ^t3 \) (resp. \( ^t\Xi(\xi) \)) denotes the matrix transpose of \( 3 = (z,w) \) (resp. \( \Xi(\xi) = (\xi, \xi^*) \)), \( |3|^2 = |z|^2 + |w|^2 \) its square modulus and \( A \) denotes the first Pauli spin matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).
Proof. By definition (20), the associated wave functions read

\[ \Phi_{Z,Q}(\xi) := \langle \xi | z, w, Q \rangle = \langle \xi | (z, w) \rangle \]

\[ = (\omega(z, w))^{-1/2} \sum_{m,n=0}^{+\infty} \frac{f_{m,n}(z, w)}{\sqrt{p_{m,n}}} L_{m,n}^{Q}(\xi, \xi^*) \]

\[ = (\pi^{-1}e^{\frac{1}{2}|z|^2 + |w|^2})^{-1/2} \sum_{m,n=0}^{+\infty} \frac{z^m w^n}{\sqrt{\pi m! n!}} e^{-\frac{1}{2}|\xi|^2} \frac{L_{m,n}^{Q}(\xi, \xi^*)}{\sqrt{\pi m! n!}} \]

\[ = e^{-\frac{1}{2}(|z|^2 + |w|^2)} e^{-\frac{1}{2}|\xi|^2} \sum_{m,n=0}^{+\infty} \frac{z^m w^n}{m! n!} L_{m,n}^{Q}(\xi, \xi^*). \]

Now, making use of the generating function for the Laguerre 2D polynomials [10, p. 675]:

\[ \sum_{m,n=0}^{+\infty} \frac{z^m w^n}{m! n!} L_{m,n}^{Q}(\xi, \xi^*) = \exp \left[ (z, w) Q \left( \xi, \xi^* \right) - \frac{1}{2} (z, w) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) Q \left( \begin{array}{c} z \\ w \end{array} \right) \right], \]

one gets the announced result. \[\Box\]

The constructed generalized coherent states give rise to a transform according to Definition 3.2. Thus, the coherent state transform (CST) associated to \( \Phi_{3,Q}: Q \in SU(2) \), is the unitary map \( \mathcal{B}^{Q} \) from the Hilbert space \( \mathcal{H}_{Q}(\mathbb{C}) \subset L^2(\mathbb{C}, d\mu) \) into the Bargmann-Fock space \( \mathcal{F}(\mathbb{C}^2) \) defined by

\[ \mathcal{B}^{Q}[\varphi](3) := (\omega(3))^{1/2} \langle \Phi_{3,Q}, \varphi \rangle_{L^2(\mathbb{C}, d\mu)}, \quad \varphi \in \mathcal{H}_{Q}(\mathbb{C}). \quad (22) \]

Being motivated by this construction, we state the following definition

**Definition 4.3.** The coherent state transform \( \mathcal{B}^{Q} \) whose integral representation is given by

\[ \mathcal{B}^{Q}[\varphi](3) = \int_{\mathbb{C}} \exp \left( 3Q^{t} \Xi(\xi) - \frac{1}{2} 3A^{Q} 3^{t} Q^{t} - \frac{1}{2} |\xi|^2 \right) \varphi(\xi) d\mu(\xi) \quad (23) \]

will be called a deformed Bargmann transform by the SU(2) matrix parameter \( Q \).

**Remark 4.4.** For the particular case of \( Q = I \) being the identity matrix, the CST in (23) reduces further to

\[ \mathcal{B}^{I}[\varphi](z, w) = \int_{\mathbb{C}} \exp \left( -zw + w\xi^* + z\xi - \frac{1}{2} |\xi|^2 \right) \varphi(\xi) d\mu(\xi) \]

which has the same integral kernel as the transform considered in [2].
References


Authors’ addresses:

Allal Ghanmi
Department of Mathematics, Faculty of Sciences
P.O. Box 1014 Mohammed V University
Agdal, 10 000 Rabat-Morocco
E-mail: allalghanmi@gmail.com

Zouhaïr Mouayn
Department of Mathematics, Faculty of Sciences and Technics (M’Ghila)
Sultan Moulay Slimane University
BP. 523 Béni Mellal-Morocco
E-mail: mouayn@fstbm.ac.ma

Received September 9, 2009
Revised February 16, 2010