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Vector Bundles on Products of Projective Spaces and Hyperquadrics

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Abstract. *Here we consider the space* $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s} \times \mathcal{Q}_{m_1} \times \cdots \times \mathcal{Q}_{m_q}$. *We introduce a notion of Castelnuovo-Mumford regularity in order to prove two splitting criteria for vector bundles with arbitrary rank.*

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1. Introduction

A well known result by Horrocks (see [8]) characterizes the vector bundles without intermediate cohomology on a projective space as direct sum of line bundles. This criterion fails on more general varieties. In fact there exist non-split vector bundles on X without intermediate cohomology. These bundles are called ACM bundles.

On a quadric hypersurface $\mathcal{Q}_n$ there is a theorem that classifies all the ACM bundles (see [11]) as direct sums of line bundles and spinor bundles (up to a twist — for generalities about spinor bundles see [14]).

Ottaviani has generalized Horrocks criterion to quadrics and Grassmanniann giving cohomological splitting conditions for vector bundles (see [13] and [15]).

The starting point of this note is [5] where Laura Costa and Rosa Maria Miró-Roig give a new proof of Horrocks and Ottaviani's criteria by using different techniques. Beilinson's Theorem was stated in 1978 and since then it has become a major tool in classifying vector bundles over projective spaces. Beilinson's spectral sequence was generalized by Kapranov (see [9] and [10]) to hyperquadrics and Grassmannians and by Costa and Miró-Roig (see [5]) to any smooth projective variety of dimension n with an n-block collection.

We specialize on a product X of finitely many projective spaces and smooth quadric hypersurfaces. In [2] and [1] we introduced a notion of Castelnuovo-Mumford regularity on quadric hypersurfaces and multiprojective spaces. We will give a suitable definition of regularity on such a product X in order to prove splitting criteria for vector bundle with arbitrary rank. Let E be a vector bundle on X. We will give two criteria which says when E is (up to a twist) a direct sum of $\mathcal{O}$ or the tensor product of pull-backs of spinor bundles on the quadric factors of X (see Theorems 2.14 and 2.15).

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2. Regularity on $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s} \times \mathcal{Q}_{m_1} \times \cdots \times \mathcal{Q}_{m_q}$

Let us consider a smooth quadric hypersurface $\mathcal{Q}_n$ in $\mathbf{P}^{n+1}$. We use the unified notation Σ_* meaning that for even n both the spinor bundles Σ_1 and Σ_2 are considered, and for n odd, the spinor bundle Σ. In [2] we introduced the following definition of regularity on $\mathcal{Q}_n$ (cfr [2] Definition 2.1 and Proposition 2.4):

Definition 2.1. *A coherent sheaf F on $\mathcal{Q}_n$ $(n \geq 2)$ is said to be m-Qregular if*

$$H^i(F(m-i)) = 0 \quad \text{for } i = 1, \ldots, n-1,$$

$$H^{n-1}(F(m) \otimes \Sigma_*(-n+1)) = 0 \quad \text{and} \quad H^n(F(m-n+1)) = 0.$$

We will say Qregular instead of 0*-Qregular.*

In [1] we introduced the following definition of regularity on $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ (cfr [1] Definition 4.1):

Definition 2.2. *A coherent sheaf F on $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s}$ is said to be $(p_1, \ldots, p_s)$-regular if, for all $i > 0$,*

$$H^i(F(p_1, \ldots, p_s) \otimes \mathcal{O}(k_1, \ldots, k_s)) = 0$$

whenever $k_1 + \cdots + k_s = -i$ and $-n_j \leq k_j \leq 0$ for any $j = 1, \ldots, s$.

Now we want to introduce a notion of regularity on

$$\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s} \times \mathcal{Q}_{m_1} \times \cdots \times \mathcal{Q}_{m_q}.$$

We recall the definition of n-block collection:

Definition 2.3. *An exceptional collection $(F_0, F_1, \ldots, F_m)$ of objects of $\mathcal{D}$ (see [5] Definition 2.1.) is a block if $Ext^i_{\mathcal{D}}(F_j, F_k) = 0$ for any i and $j \neq k$.*

An n-block collection of type $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ of objects of $\mathcal{D}$ is an exceptional collection

$$(\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_m) = (E^0_1, \ldots, E^0_{\alpha_0}, E^1_1, \ldots, E^1_{\alpha_1}, \ldots, E^n_1, \ldots, E^n_{\alpha_n})$$

such that all the subcollections $\mathcal{E}_i = (E^i_1, \ldots, E^i_{\alpha_i})$ are blocks.

Example 2.4. $(\mathcal{O}_{\mathbf{P}^n}(-n), \mathcal{O}_{\mathbf{P}^n}(-n+1), \ldots, \mathcal{O}_{\mathbf{P}^n})$ *is an n-block collection of type* $(1, 1, \ldots, 1)$ *on $\mathbf{P}^n$ (see [5] Example* 2.3.(1)*).*

EXAMPLE 2.5. *Let us consider a smooth quadric hypersurface $\mathcal{Q}_n$ in $\mathbf{P}^{n+1}$.*

$$(\mathcal{E}_0, \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O}),$$

where $\mathcal{E}_0 = (\Sigma_*(-n))$, is an n-block collection of type $(1,1,\dots,1)$ if n is odd, and of type $(2,1,\dots,1)$ if n is even (see [5] Example 3.4.(2)).

Moreover we can have several n-block collections:

$$\sigma_j = (\mathcal{O}(j), \dots, \mathcal{O}(n-1), \mathcal{E}_{n-j}, \mathcal{O}(n+1), \dots, \mathcal{O}(n-j-1))$$

where $\mathcal{E}_{n-j} = (\Sigma_*(n-1))$ and $1 \le j \le n$ (see [6] Proposition 4.4).

We need the following notation:

Notation. Let X, Y be two smooth projective varieties of dimension n and m. Let $(\mathcal{G}_0, \dots, \mathcal{G}_n)$, $\mathcal{G}_i = (G_0^i, \dots, G_{\alpha_i}^i)$ be a n-block collection for X and $(\mathcal{E}_0, \dots, \mathcal{E}_m)$, $\mathcal{E}_j = (E_0^j, \dots, E_{\beta_j}^j)$ a m-block collection for Y (see [5]).

We denote by $\mathcal{G}_i \boxtimes \mathcal{E}_j$ the set of all the bundles $G_k^i \boxtimes E_m^j$ on $X \times Y$ such that $G_k^i \in \mathcal{G}_i$ and $E_m^j \in \mathcal{E}_j$.

For any $0 \le k \le n+m$, we define $\mathcal{F}_k = \mathcal{G}_i \boxtimes \mathcal{E}_j$ where $i + j = k$.

Let us consider first $X = \mathbf{P}^n \times \mathcal{Q}_m$.

DEFINITION 2.6. *On $\mathbf{P}^n$ we consider the n-block collection:*

$$(\mathcal{E}_0, \dots \mathcal{E}_n) = (\mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O})$$

and on $\mathcal{Q}_m$ we consider the m-block collection:

$$(\mathcal{G}_0, \dots \mathcal{G}_m) = (\mathcal{O}(-m+1), \mathcal{G}_1, \dots, \mathcal{O})$$

where $\mathcal{G}_1 = (\Sigma_*(-m+1))$.

A coherent sheaf F on X is said to be (p,p')-regular if, for all $i > 0$,

$$H^i(F(p,p') \otimes \mathcal{E}_{n-j} \boxtimes \mathcal{G}_{m-k}) = 0$$

whenever $j + k = i$, $-n \le -j \le 0$ and $-m \le -k \le 0$.

REMARK 2.7. *If $m = 2$ Definition 2.6 coincides with Definition 2.2 on $\mathbf{P}^n \times \mathbf{P}^1 \times \mathbf{P}^1$. In fact the 2-block collection on $\mathcal{Q}_2$ is*

$$(\mathcal{O}(-1), \{\Sigma_1(-1), \Sigma_2(-1)\}, \mathcal{O}) = (\mathcal{O}(-1,-1), \{\mathcal{O}(-1,0), \mathcal{O}(0,-1)\}, \mathcal{O}).$$

In particular when $n = 0$, F is regular if

$$H^2(F(-1,-1)) = H^1(F(0,-1)) = H^1(F(-1,0)) = 0.$$

This definition is not equivalent to the definition of Qregularity on $\mathcal{Q}_2$ but it is a good definition of regularity. In fact, let F be a regular coherent sheaf. Since $H^1(F(-1,0)) = 0$ from the exact sequence

$$0 \to \mathcal{O}(-1,0) \to \mathcal{O}^2 \to \mathcal{O}(1,0) \to 0,$$

tensored by F we see that $H^0(F(1,0))$ is spanned by

$$H^0(F) \otimes H^0(\mathcal{O}(1,0)).$$

Moreover if we tensor the above sequence by $F(-1,-1)$, we have $H^2(F(-2,-1)) = 0$. From the sequences

$$0 \to F(-2,0) \to F^2(-1,0) \to F \to 0$$

and

$$0 \to F(-1,-1) \to F^2(0,-1) \to F(1,-1) \to 0,$$

we see that $H^1(F) = H^1(F(1,-1)) = 0$ and then $F(1,0)$ is regular.

Remark 2.8. *If $m = 0$ we can identify X with $\mathbf{P}^n$ and the sheaf $F(k,k')$ with $F(k)$. Under this identification F is (p,p')-regular in the sense of Definition 2.6, if and only if F is p-regular in the sense of Castelnuovo-Mumford.*

In fact, let $i > 0$, $H^i(F(p,p') \otimes \mathcal{E}_{n-j} \boxtimes \mathcal{G}_{m-k}) = H^i(F(p-j)) = 0$ whenever $j+k = i$, $-n \leq -j \leq 0$ and $-m \leq -k \leq 0$ if and only if $H^i(F(p-j)) = 0$ whenever $-i \leq -j \leq 0$ if and only if $H^i(F(p-i)) = 0$.

Lemma 2.9. (1) *Let H be a generic hyperplane of $\mathbf{P}^n$. If F is a regular coherent sheaf on $X = \mathbf{P}^n \times \mathcal{Q}_m$, then $F_{|L_1}$ is regular on $L_1 = H \times \mathcal{Q}_m$.*

(2) *Let H' be a generic hyperplane of $\mathcal{Q}_m$. If F is a regular coherent sheaf on $X = \mathbf{P}^n \times \mathcal{Q}_m$, then $F_{|L_2}$ is regular on $L_2 = \mathbf{P}^n \times H'$.*

Proof. (1) We follow the proof of [7] Lemma 2.6. We get this exact cohomology sequence:

$$H^i(F(-j,0) \otimes \mathcal{O} \boxtimes \mathcal{G}_{m-k}) \to H^i(F_{|L_1}(-j,0) \otimes \mathcal{O} \boxtimes \mathcal{G}_{m-k}) \to \\ \to H^{i+1}(F(-j-1,0) \otimes \mathcal{O} \boxtimes \mathcal{G}_{m-k}).$$

If $j+k = i$, $-n \leq -j \leq 0$ and $-m \leq -k \leq 0$, we have also $-n-1 \leq -j-1 \leq 0$, so the first and the third groups vanish by hypothesis. Then also the middle group vanishes and $F_{|L_1}$ is regular.

(2) We have to deal also with the spinor bundles. First assume m even, say $m = 2l$. We have $\Sigma_{1|\mathcal{Q}_{m-1}} \cong \Sigma_{2|\mathcal{Q}_{m-1}} \cong \Sigma$. Let $k = m-1$ and $j = m-1-i$ $(i \geq m-i)$. Let us consider the exact sequences

$$0 \to \mathcal{O}(-j) \boxtimes \Sigma_1(-m) \to \mathcal{O}(-j) \boxtimes \mathcal{O}(-m+1)^{2^l} \to \\ \to \mathcal{O}(-j) \boxtimes \Sigma_2(-m+1) \to 0$$

tensored by F.

Since $H^i(F \otimes \mathcal{O}(-j) \boxtimes \Sigma_2(-m+1)) = H^i(F \otimes \mathcal{E}_{n-j} \boxtimes \mathcal{G}_1) = 0$ and $H^{i+1}(F(-j,-m+1)) = H^{i+1}(F \otimes \mathcal{E}_{n-j} \boxtimes \mathcal{G}_0) = 0$, we also have $H^{i+1}(F \otimes \mathcal{O}(-j) \boxtimes \Sigma_1(-m)) = 0$.

From the exact sequences

$$0 \to \mathcal{O}(-j) \boxtimes \Sigma_1(-m+1) \to \mathcal{O}(-j) \boxtimes \Sigma_1(-m+2) \to \\ \to \mathcal{O}(-j) \boxtimes \Sigma_{1|\mathcal{Q}_{m-1}}(-m+2) \to 0$$

tensored by F, we get

$$H^i(F(-j,0) \boxtimes \Sigma_1(-m+1)) \to H^i(F(-j,0) \boxtimes \Sigma_{1|\mathcal{Q}_{m-1}}(-m+1)) \to \\ \to H^{i+1}(F(-j,0) \boxtimes \Sigma_1(-m))$$

If $i \geq m-1$ and $j = m-1-i$, the first and the third groups vanish by hypothesis. Then also the middle group vanishes. In the same way we can show that also $H^i(F(-j,0) \boxtimes \Sigma_{2|\mathcal{Q}_{m-1}}(-m+1)) = 0$.

Assume now m odd, say $m = 2l+1$. We have $\Sigma_{|\mathcal{Q}_{m-1}} \cong \Sigma_1 \oplus \Sigma_2$. We can consider the exact sequences

$$0 \to \mathcal{O}(-j) \boxtimes \Sigma(-m) \to \mathcal{O}(-j) \boxtimes \mathcal{O}(-m+1)^{2^{l+1}} \to \\ \to \mathcal{O}(-j) \boxtimes \Sigma(-m+1) \to 0$$

tensored by F. Then we argue as above.

All the others vanishing in Definition 2.6 can be proved as in (1) and we can conclude that $F_{|L_2}$ is regular. $\square$

PROPOSITION 2.10. *Let F be a regular coherent sheaf on $X = \mathbf{P}^n \times \mathcal{Q}_m$ then*

1. *$F(p,p')$ is regular for $p, p' \geq 0$.*
2. *$H^0(F(k,k'))$ is spanned by*

$$H^0(F(k-1,k')) \otimes H^0(\mathcal{O}(1,0))$$

if $k-1, k' \geq 0$; and it is spanned by

$$H^0(F(k,k'-1)) \otimes H^0(\mathcal{O}(0,1))$$

if $k, k'-1 \geq 0$ and $m > 2$.

Proof. (1) We want to prove part (1) by induction. Let F be a regular coherent sheaf, we want show that also $F(1,0)$ is regular. We follow the proof of [7] Proposition 2.7.

Consider the exact cohomology sequence:

$$H^i(F(-j,0)\otimes\mathcal{O}\boxtimes\mathcal{G}_{m-k})\to H^i(F(-j+1,0)\otimes\mathcal{O}\boxtimes\mathcal{G}_{m-k})\to$$
$$\to H^i(F_{|L_1}(-j+1,0)\otimes\mathcal{O}\boxtimes\mathcal{G}_{m-k})$$

If $j+k=i$, $-n\le -j\le 0$ and $-m\le -k\le 0$, the first group vanishes because F is regular and the third group vanishes by the inductive hypothesis. Then also the middle group vanishes. A symmetric argument shows the vanishing for $F(0,1)$. We only have to check the vanishing involving the spinor bundles. We have the sequences

$$H^i(F(-j,0)\boxtimes\Sigma_*(-m+1))\to H^i(F(-j,1)\boxtimes\Sigma_*(-m+1))\to$$
$$\to H^i(F(-j,1)\boxtimes\Sigma_{*|\mathcal{Q}_{m-1}}(-m+1))$$

If $k=m-1$ and $j=m-1-i$ ($i\ge m-i$), the first group vanishes because F is regular and the third group vanishes by the inductive hypothesis. Then also the middle group vanishes.

(2) We will follow the proof of [7] Proposition 2.8.

We consider the following diagram:

$$\begin{array}{ccc} H^0(F(k-1,k'))\otimes H^0(\mathcal{O}(1,0)) & \xrightarrow{\mu} & H^0(F(k,k')) \\ \downarrow\sigma & & \downarrow\nu \\ H^0(F_{|L_1}(k-1,k'))\otimes H^0(\mathcal{O}_{L_1}(1,0)) & \xrightarrow{\tau} & H^0(F_{|L_1}(k,k')) \end{array}$$

Note that σ is surjective if $k-1,k'\ge 0$ because $H^1(F(k-2,k'))=0$ by regularity.

Moreover also τ is surjective by (2) for $F_{|L_1}$.

Since both σ and τ are surjective, we can see as in [12] page 100 that μ is also surjective.

In order to prove that $H^0(F(k,k'))$ is spanned by $H^0(F(k,k'-1))\otimes H^0(\mathcal{O}(0,1))$ if $k,k'-1\ge 0$, we can use a symmetric argument since for $m>2$ the spinor bundles are not involved in the proof. □

Remark 2.11. *If F is a regular coherent sheaf on $X=\mathbf{P}^n\times\mathcal{Q}_m$ ($m>2$) then it is globally generated.*

In fact by the above proposition we have the following surjections:

$$H^0(F)\otimes H^0(\mathcal{O}(1,0))\otimes H^0(\mathcal{O}(0,1))\to$$
$$\to H^0(F(1,0))\otimes H^0(\mathcal{O}(0,1))\to H^0(F(1,1)),$$

and so the map

$$H^0(F)\otimes H^0(\mathcal{O}(1,1))\to H^0(F(1,1))$$

is a surjection.

Moreover we can consider a sufficiently large twist l such that $F(l,l)$ is globally generated. The commutativity of the diagram

$$\begin{array}{ccc} H^0(F)\otimes H^0(\mathcal{O}(l,l))\otimes\mathcal{O} & \to & H^0(F(l,l))\otimes\mathcal{O} \\ \downarrow & & \downarrow \\ H^0(F)\otimes\mathcal{O}(l,l) & \to & F(l,l) \end{array}$$

yields the surjectivity of $H^0(F)\otimes\mathcal{O}(l,l)\to F(l,l)$, which implies that F is generated by its sections.

If $m=2$, then F is globally generated by Remark 2.7 and [1] Remark 2.6.

Now we generalize Definition 2.6:

Definition 2.12. *Let us consider $X=\mathbf{P}^{n_1}\times\cdots\times\mathbf{P}^{n_s}\times\mathcal{Q}_{m_1}\times\cdots\times\mathcal{Q}_{m_q}$.*

On $\mathbf{P}^{n_j}$ (where $j=1,\dots,s$) we consider the n_j-block collections:

$$(\mathcal{E}^j_0,\dots\mathcal{E}^j_n)=(\mathcal{O}(-n_j),\mathcal{O}(-n_j+1),\dots,\mathcal{O})$$

and on $\mathcal{Q}_{m_l}$ (where $l=1,\dots,q$) we consider the m_q-block collections:

$$(\mathcal{G}^l_0,\dots\mathcal{G}^l_m)=(\mathcal{O}(-m_l+1),\mathcal{G}^l_1,\dots,\mathcal{O})$$

where $\mathcal{G}^l_1=(\Sigma_*(-m_l+1))$.

A coherent sheaf F on X is said to be $(p_1,\dots,p_{s+q})$-regular if, for all $i>0$,

$$H^i(F(p_1,\dots,p_{s+q})\otimes\mathcal{E}^1_{n_1-k_1}\boxtimes\cdots\boxtimes\mathcal{E}^s_{n_s-k_s}\boxtimes\mathcal{G}^1_{m_1-h_1}\boxtimes\cdots\boxtimes\mathcal{G}^q_{m_q-h_q})=0$$

whenever $k_1+\cdots+k_s+h_1+\cdots+h_q=i$, $-n_j\le -k_j\le 0$ for any $j=1,\dots,s$ and $-m_l\le -h_l\le 0$ for any $l=1,\dots,q$.

Remark 2.13. *As above can be proved (by using exactly the same arguments) that, if F is regular then is globally generated and $F(k_1,\dots,k_{s+q})$ is regular when $k_1,\dots,k_{s+q}\ge 0$.*

We use our notion of regularity in order to proving some splitting criterion on $X=\mathbf{P}^{n_1}\times\cdots\times\mathbf{P}^{n_s}\times\mathcal{Q}_{m_1}\times\cdots\times\mathcal{Q}_{m_q}$.

Theorem 2.14. *Let E be a rank r vector bundle on $X=\mathbf{P}^{n_1}\times\cdots\times\mathbf{P}^{n_s}\times\mathcal{Q}_{m_1}\times\cdots\times\mathcal{Q}_{m_q}$ $(m_1,\dots,m_q>2)$.*

Set $d=n_1+\cdots+n_s+m_1+\cdots+m_q$.

Then the following conditions are equivalent:

1. *for any $i=1,\dots,d-1$ and for any integer t,*

   $$H^i(E(t,\dots,t)\otimes\mathcal{E}^1_{n_1-k_1}\boxtimes\cdots\boxtimes\mathcal{E}^s_{n_s-k_s}\boxtimes\mathcal{G}^1_{m_1-h_1}\boxtimes\cdots\boxtimes\mathcal{G}^q_{m_q-h_q}))$$

   vanishes whenever $k_1+\cdots+k_s+h_1+\cdots+h_q=i$, $-n_j\le -k_j\le 0$ for any $j=1,\dots,s$ and $-m_l\le -h_l\le 0$ for any $l=1,\dots,q$.

2. *There are r integer $t_1, \dots, t_r$ such that $E \cong \bigoplus_{i=1}^{r} \mathcal{O}(t_i, \dots, t_i)$.*

Proof. (1) $\Rightarrow$ (2). Let us assume that t is an integer such that $E(t, \dots, t)$ is regular but $E(t-1, \dots, t-1)$ is not.

By the definition of regularity and (1) we can say that $E(t-1, \dots, t-1)$ is not regular if and only if

$$H^d(E(t-1, \dots, t-1) \otimes \mathcal{O}(-n_1, \dots, -n_s, -m_1+1, \dots, -m_q+1)) \neq 0.$$

By Serre duality we have that $H^0(E^\vee(-t, \dots, -t)) \neq 0$.

Now since $E(t, \dots, t)$ is globally generated by Remark 2.11 and $H^0(E^\vee(-t, \dots, -t)) \neq 0$ we can conclude that $\mathcal{O}$ is a direct summand of $E(t, \dots, t)$.

By iterating these arguments we get (2).

(2) $\Rightarrow$ (1). By Künneth formula for any $i = 1, \dots, m+n-1$ and for any integer t,

$$H^i(\mathcal{O}(t, \dots, t) \otimes \mathcal{E}^1_{n_1-k_1} \boxtimes \cdots \boxtimes \mathcal{E}^s_{n_s-k_s} \boxtimes \mathcal{G}^1_{m_1-h_1} \boxtimes \cdots \boxtimes \mathcal{G}^q_{m_q-h_q})) = 0$$

whenever $k_1 + \cdots + k_s + h_1 + \cdots + h_q = i$, $-n_j \leq -k_j \leq 0$ for any $j = 1, \dots, s$ and $-m_l \leq -h_l \leq 0$ for any $l = 1, \dots, q$.

Then $\mathcal{O}$ satisfies all the conditions in (1). $\square$

Theorem 2.15. *Let E be a rank r vector bundle on $X = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s} \times \mathcal{Q}_{m_1} \times \cdots \times \mathcal{Q}_{m_q}$ $(m_1, \dots, m_q > 2)$.*

Set $d = n_1 + \cdots + n_s + m_1 + \cdots + m_q$.

Then the following conditions are equivalent:

1. *for any $i = 1, \dots, d-1$ and for any integer t,*

$$H^i(E(t, \dots, t) \otimes \mathcal{E}^1_{n_1-k_1} \boxtimes \cdots \boxtimes \mathcal{E}^s_{n_s-k_s} \boxtimes \mathcal{G}^1_{m_1-h_1} \boxtimes \cdots \boxtimes \mathcal{G}^q_{m_q-h_q}))$$

vanishes whenever $k_1 + \cdots + k_s + h_1 + \cdots + h_q \leq i$, $-n_j \leq -k_j \leq 0$ for any $j = 1, \dots, s$ and $-m_l \leq -h_l \leq 0$ for any $l = 1, \dots, q$ except when $k_1 = n_1, \dots, k_s = n_s$ and $h_l = m_l - 1$ for any $l = 1, \dots, q$.

Moreover

$$H^{m_1-1}(E(t, \dots, t) \otimes \mathcal{O} \boxtimes \cdots \boxtimes \mathcal{O} \boxtimes \mathcal{O}(-m_1+1) \boxtimes \cdots \boxtimes \mathcal{O}) = \dots$$
$$\cdots = H^{m_q-1}(E(t, \dots, t) \otimes \mathcal{O} \boxtimes \cdots \boxtimes \mathcal{O} \boxtimes \mathcal{O} \boxtimes \cdots \boxtimes \mathcal{O}(-m_q+1)) = 0.$$

2. *E is a direct sum of bundles $\mathcal{O}$ and $\mathcal{O}(0, \dots, 0) \boxtimes \Sigma_* \boxtimes \cdots \boxtimes \Sigma_*$ with some twist.*

Proof. (1) $\Rightarrow$ (2). First we see the proof when $X = \mathbf{P}^n \times \mathcal{Q}_m$.

In this case the condition (1) is the following:

for any $i = 1, \ldots, m+n-1$ and for any integer t,

$$H^i(E(t,t) \otimes \mathcal{O}(j,k)) = 0$$

whenever $j+k = -i$, $-n \le k \le 0$ and $-m \le j \le 0$ $(j \ne -m+1)$.

Moreover $H^{k+m-1}(E(t,t) \otimes \mathcal{O}(k) \boxtimes \Sigma_*(-m+1)) = 0$ for $-n \le k < 0$ and $H^{m-1}(E(t,t) \otimes \mathcal{O} \boxtimes \mathcal{O}(-m+1)) = 0$.

Let us assume that t is an integer such that $E(t,t)$ is regular but $E(t-1,t-1)$ is not.

By the definition of regularity and (1) we can say that $E(t-1,t-1)$ is not regular if and only if one of the following conditions is satisfied:

i $H^d(E(t-1,t-1) \otimes \mathcal{O}(-n,-m+1)) \ne 0$.

ii $H^{n+m-1}(E(t-1,t-1) \otimes \mathcal{O}(-n) \boxtimes \Sigma_*(-m+1)) \ne 0$.

Let us consider one by one the conditions:

(i) Let $H^d(E(t-1,t-1) \otimes \mathcal{O}(-n,-m+1)) \ne 0$, we can conclude that $\mathcal{O}(t,t)$ is a direct summand as in the above theorem.

(ii) Let $H^{n+m-1}(E(t,t) \otimes \mathcal{O}(-n-1) \boxtimes \Sigma_*(-m)) \ne 0$.

Let us consider the following exact sequences tensored by $E(t,t)$:

$$0 \to \mathcal{O}(-n-1) \boxtimes \Sigma_*(-m) \to \mathcal{O}(-n) \boxtimes \Sigma_*(-m) \to \ldots$$
$$\cdots \to \mathcal{O}(1) \boxtimes \Sigma_*(-m) \to \mathcal{O} \boxtimes \Sigma_*(-m) \to 0,$$

by using the vanishing conditions in (1) we can see that there is a surjection from

$$H^{m-1}(E(t,t) \otimes \mathcal{O} \boxtimes \Sigma_*(-m))$$

to

$$H^{n+m-1}(E(t,t) \otimes \mathcal{O}(-n-1) \boxtimes \Sigma_*(-m)).$$

Let us consider now the following exact sequence tensored by $E(t,t)$:

$$0 \to \mathcal{O} \boxtimes \Sigma_*(-m) \to \mathcal{O} \boxtimes \mathcal{O}^{2^{([\frac{m+1}{2}])}}(-m+1) \to \ldots$$
$$\cdots \to \mathcal{O} \boxtimes \mathcal{O}^{2^{(\frac{m+1}{2})}}(-2) \to \mathcal{O} \boxtimes \Sigma_*(-1) \to 0.$$

By using the vanishing conditions in (1) as above (but here we need also the condition $H^{m-1}(E(t,t) \otimes \mathcal{O} \boxtimes \mathcal{O}(-m+1)) = 0$) we can see that there is a surjection from

$$H^0(E(t,t) \otimes \mathcal{O} \boxtimes \Sigma_*(-1))$$

to

$$H^{m-1}(E(t,t)\otimes\mathcal{O}\boxtimes\Sigma_*(-m))$$

and we can conclude that

$$H^0(E(t,t)\otimes\mathcal{O}\boxtimes\Sigma_*(-1))\neq 0.$$

This means that there exists a non zero map

$$g: E(t,t)\to\mathcal{O}\boxtimes\Sigma_*.$$

On the other hand

$$H^{n+m-1}(E(t,t)\otimes\mathcal{O}(-n-1)\boxtimes\Sigma_*(-m))\cong$$
$$\cong H^1(E^\vee(-t,-t)\otimes\mathcal{O}\boxtimes\Sigma_*(-1)).$$

Let us consider the following exact sequences tensored by $E^\vee(-t,-t)$:

$$0\to\mathcal{O}\boxtimes\Sigma_*(-1)\to\mathcal{O}\boxtimes\mathcal{O}^{2^{([\frac{m+1}{2}])}}\to\mathcal{O}\boxtimes\Sigma_*\to 0.$$

Since

$$H^1(E^\vee(-t,-t))\cong H^{n+m-1}(E(t-n-1,t-m))=0$$

we can conclude that

$$H^0(E^\vee(-t,-t)\otimes\mathcal{O}\boxtimes\Sigma_*)\neq 0.$$

This means that there exists a non zero map

$$f:\mathcal{O}\boxtimes\Sigma_*\to E(t,t).$$

Then, by arguing as in [1] Theorem 1.2, we see that the composition of the maps f and g is not zero so must be the identity and we have that $\mathcal{O}\boxtimes\Sigma_*$ is a direct summand of $E(t,t)$.

On $X=\mathbf{P}^{n_1}\times\cdots\times\mathbf{P}^{n_s}\times\mathcal{Q}_{m_1}\times\cdots\times\mathcal{Q}_{m_q}$ $(m_1,\dots,m_q>2)$, Let us assume that t is an integer such that $E(t,\dots,t)$ is regular but $E(t-1,\dots,t-1)$ is not.

By the definition of regularity and (1) we can say that $E(t-1,\dots,t-1)$ is not regular if and only if one of the following conditions is satisfied:

(i) $H^d(E(t-1,\dots,t-1)\otimes\mathcal{O}(-n_1,\dots,-n_s,-m_1+1,\dots,-m_q+1))\neq 0.$

(ii) $H^{n_1+\cdots+n_s+m_1-1+\cdots+m_q-1}(E(t-1,\dots,t-1)\otimes\mathcal{O}(-n_1,\dots,-n_s)\boxtimes\Sigma_*(-m_1+1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q+1))\neq 0.$

Let us consider one by one the conditions:

(i) Let $H^d(E(t-1,\dots,t-1)\otimes\mathcal{O}(-n_1,\dots,-n_s,-m_1+1,\dots,-m_q+1))\neq 0$, we can conclude that $\mathcal{O}(t,\dots,t)$ is a direct summand as in the above theorem.

(ii) Let $H^{n_1+\cdots+n_s+m_1-1+\cdots+m_q-1}(E(t,\dots,t)\otimes\mathcal{O}(-n_1-1,\dots,-n_s-1)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q))\neq 0$.

Let us consider the following exact sequences tensored by $E(t,\dots,t)$:

$$0\to\mathcal{O}(-n_1-1,\dots,-n_s-1)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q)\to\dots$$
$$\cdots\to\mathcal{O}(0,-n_2-1,\dots,-n_s-1)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q)\to 0,$$

$$0\to\mathcal{O}(0,-n_2-1,\dots,-n_s-1)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q)\to\dots$$
$$\cdots\to\mathcal{O}(0,0,-n_3-1,\dots,-n_s-1)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q)\to 0,$$

$$\dots$$

$$0\to\mathcal{O}(0,\dots,0,-n_s-1)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q)\to\dots$$
$$\cdots\to\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q)\to 0.$$

Since all the bundles in the above sequences are

$$\mathcal{E}^1_{n_1-k_1}\boxtimes\cdots\boxtimes\mathcal{E}^s_{n_s-k_s}\boxtimes\mathcal{G}^1_{m_1-h_1}\boxtimes\cdots\boxtimes\mathcal{G}^q_{m_q-h_q}$$

with decreasing indexes, by using the vanishing conditions in (1) we can see that there is a surjection from

$$H^{m_1-1+\cdots+m_q-1}(E(t,\dots,t)\otimes\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q))$$

to

$$H^{n_1+\cdots+n_s+m_1-1+\cdots+m_q-1}(E(t,\dots,t)\otimes$$
$$\otimes\,\mathcal{O}(-n_1-1,\dots,-n_s-1)\boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q)).$$

Let us consider now the following exact sequences on $\mathcal{Q}_{m_1}\times\cdots\times\mathcal{Q}_{m_q}$ for any integer p:

$$0\to\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(p-1)\to\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\mathcal{O}(p)^{2^{([\frac{m_q+1}{2}])}}\to$$
$$\to\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(p)\to 0.$$

We get the long exact sequence

$$0 \to \Sigma_*(-m_1) \boxtimes \cdots \boxtimes \Sigma_*(-m_q) \to \Sigma_*(-m_1) \boxtimes \dots$$
$$\cdots \boxtimes \mathcal{O}(-m_q+1)^{2^{([\frac{m_q+1}{2}])}} \to \cdots \to \boxtimes \Sigma_*(-m_1) \boxtimes \cdots \boxtimes \Sigma_*(-1) \to 0.$$

In the same way we can get

$$0 \to \Sigma_*(-m_1) \boxtimes \cdots \boxtimes \Sigma_*(-m_{q-1}) \boxtimes \Sigma_*(-1) \to \Sigma_*(-m_1) \boxtimes \dots$$
$$\cdots \boxtimes \mathcal{O}(-m_{q-1}+1)^{2^{([\frac{m_{q-1}+1}{2}])}} \boxtimes \Sigma_*(-1) \to \dots$$
$$\cdots \to \boxtimes \Sigma_*(-m_1) \boxtimes \cdots \boxtimes \Sigma_*(-1) \boxtimes \Sigma_*(-1) \to 0,$$

$$\dots$$

$$0 \to \Sigma_*(-m_1) \boxtimes \Sigma_*(-1) \boxtimes \cdots \boxtimes \Sigma_*(-1) \to$$
$$\to \mathcal{O}(-m_1+1)^{2^{([\frac{m_1+1}{2}])}} \boxtimes \Sigma_*(-1) \boxtimes \cdots \boxtimes \Sigma_*(-1) \to \dots$$
$$\cdots \to \Sigma_*(-1) \boxtimes \cdots \boxtimes \Sigma_*(-1) \to 0.$$

Then on $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_s} \times \mathcal{Q}_{m_1} \times \cdots \times \mathcal{Q}_{m_q}$ we can obtain the following exact sequence tensored by $E(t, \dots, t)$:

$$0 \to \mathcal{O}(0, \dots, 0) \boxtimes \Sigma_*(-m_1) \boxtimes \cdots \boxtimes \Sigma_*(-m_q) \to \dots$$
$$\cdots \to \mathcal{O}(0, \dots, 0) \boxtimes \Sigma_*(-1) \boxtimes \cdots \boxtimes \Sigma_*(-1) \to 0.$$

By using the vanishing conditions in (1) as above we can see that there is a surjection from

$$H^0(E(t, \dots, t) \otimes \mathcal{O}(0, \dots, 0) \boxtimes \Sigma_*(-1) \boxtimes \cdots \boxtimes \Sigma_*(-1))$$

to

$$H^{m_1-1+\cdots+m_q-1}(E(t, \dots, t) \otimes \mathcal{O}(0, \dots, 0) \boxtimes \Sigma_*(-m_1) \boxtimes \cdots \boxtimes \Sigma_*(-m_q))$$

and we can conclude that

$$H^0(E(t, \dots, t) \otimes \mathcal{O}(0, \dots, 0) \boxtimes \Sigma_*(-1) \boxtimes \cdots \boxtimes \Sigma_*(-1)) \neq 0.$$

This means that there exists a non zero map

$$g : E(t, \dots, t) \to \mathcal{O}(0, \dots, 0) \boxtimes \Sigma_* \boxtimes \cdots \boxtimes \Sigma_*.$$

On the other hand

$$H^{n_1+\cdots+n_s+m_1-1+\cdots+m_q-1}(E(t,\dots,t)\otimes\mathcal{O}(-n_1-1,\dots,-n_s-1)\boxtimes \\ \boxtimes\Sigma_*(-m_1)\boxtimes\cdots\boxtimes\Sigma_*(-m_q))\cong \\ \cong H^q(E^\vee(-t,\dots,-t)\otimes\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*(-1)\boxtimes\cdots\boxtimes\Sigma_*(-1)).$$

Let us consider the following exact sequences tensored by $E^\vee(-t,\dots,-t)$:

$$0\to\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*(-1)\boxtimes\cdots\boxtimes\Sigma_*(-1)\to\dots \\ \cdots\to\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*\boxtimes\cdots\boxtimes\Sigma_*\to 0.$$

By using the Serre duality and the vanishing conditions in (1) we can conclude that

$$H^0(E^\vee(-t,\dots,-t)\otimes\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*\boxtimes\cdots\boxtimes\Sigma_*)\neq 0.$$

This means that there exists a non zero map

$$f:\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*\boxtimes\cdots\boxtimes\Sigma_*\to E(t,\dots,t).$$

Then, by arguing as in [1] Theorem 1.2, we see that the composition of the maps f and g is not zero so must be the identity and we have that $\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*\boxtimes\cdots\boxtimes\Sigma_*$ is a direct summand of $E(t,\dots,t)$.

By iterating these arguments we get (2).

(2) $\Rightarrow$ (1). We argue as in Theorem 2.14. Since $H^i(\mathcal{Q}_n,\Sigma_*(e))\neq 0$ if and only if $i=0$ and $e\geq 0$ or $i=n$ and $e\leq -n-1$, we have that $\mathcal{O}(0,\dots,0)\boxtimes\Sigma_*\boxtimes\cdots\boxtimes\Sigma_*$ and $\mathcal{O}$ satisfy all the conditions in (1). □

References

[1] E. Ballico and F. Malaspina, *Regularity and cohomological splitting conditions for vector bundles on multiprojectives spaces*, preprint (2008).

[2] E. Ballico and F. Malaspina, *Qregularity and an extension of the Evans-Griffiths criterion to vector bundles on quadrics*, J. Pure Appl. Algebra **213** (2009), 194–202.

[3] A.A. Beĭlinson, *Coherent sheaves on* $\mathbf{P}^n$ *and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), 68–69.

[4] L. Costa and R.M. Miró-Roig, *Tilting sheaves on toric varieties*, Math. Z. **248** (2004), 849–865.

[5] L. Costa and R.M. Miró-Roig, *Cohomological characterization of vector bundles on multiprojective spaces*, J. Algebra **294** (2005), 73–96.

[6] L. Costa and R.M. Miró-Roig, *Monads and regularity of vector bundles on projective varieties*, Michigan Math. J. **55** (2007), 417–436.

[7] J.W. Hoffman and H.H. Wang, *Castelnuovo-Mumford regularity in biprojective spaces*, Adv. Geom. **4** (2004), 513–536.

[8] G. Horrocks, *Vector bundles on the punctured spectrum of a ring*, Proc. London Math. Soc. **14** (1964), 689–713.
[9] M.M. Kapranov, *Derived category of coherent sheaves on Grassmann manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), 192–202.
[10] M.M. Kapranov, *On the derived category of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), 479–508.
[11] H. Knörrer, *Cohen-Macaulay modules of hypersurface singularities I*, Invent. Math. **88** (1987), 153–164.
[12] D. Mumford, *Lectures on curves on an algebraic surface*, Princeton University Press, Princeton (1966).
[13] G. Ottaviani, *Critères de scindage pour les fibrés vectoriels sur les grassmanniennes et les quadriques*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 257–260.
[14] G. Ottaviani, *Spinor bundles on quadrics*, Trans. Amer. Math. Soc. **307** (1988), 301–316.
[15] G. Ottaviani, *Some extension of Horrocks criterion to vector bundles on Grassmannians and quadrics*, Ann. Mat. Pura Appl. **155** (1989), 317–341.

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