Rend. Istit. Mat. Univ. Trieste
Vol. XXXIX, 421–444 (2007)

Countability Properties of the Pseudocompact-Open Topology on $C(X)$: A Comparative Study

S. Kundu and Pratibha Garg [(*)]

Dedicated to Professor Robert A. McCoy

Summary. - *The main goal of this paper is to study the countability properties, such as the countable chain condition, Lindeöf property and second countability of the pseudocompact-open topology on $C(X)$, the set of all continuous real-valued functions on a Tychonoff space X. But in order to make this study fruitful, these countability properties of the pseudocompact-open topology are compared with those of the point-open and compact-open topologies on $C(X)$.*

1. Introduction

The set $C(X)$ of all real-valued continuous functions as well as the set $C^*(X)$ of all bounded real-valued continuous functions on a Tychonoff space X has a number of natural topologies. Three commonly used among them are the point-open topology p, the compact-

[(*)] Authors' addresses: S. Kundu, Department Of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India; E-mail: skundu@maths.iitd.ernet.in
Pratibha Garg, Department Of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India; E-mail: pratibhagarg@gmail.com
Keywords: Function Space, Separability, $\aleph_0$-Bounded, the Countable Chain Condition, Lindelöf, $\aleph_0$-Space, Cosmic Space, Second Countable.
AMS Subject Classification: 54C35, 54D20, 54D99, 54E20.

open topology k and the topology of uniform convergence u. The compact-open topology and the topology of uniform convergence on $C(X)$ (or on $C^*(X)$) are equal if and only if X is compact. Because compactness is such a strong condition, there is a considerable gap between these two topologies. This gap has been especially felt in the topological measure theory; consequently in the last five decades, there have been quite a few topologies introduced that lie between k and u, such as the strict topology, the σ-compact-open topology, the topology of uniform convergence on σ-compact subsets and the topology of uniform convergence on bounded subsets. (see, for example, [2, 8, 10, 12, 16, 19, 20, 21, 30] and [31].

The pseudocompact-open topology ps is another natural and interesting locally convex topology on $C(X)$, from the viewpoint of both topology and measure theory, though it has not received much attention from the researchers until a formal study of this topology was done in [18]. The spaces $C(X)$, equipped with the point-open topology p, the compact-open topology k and the pseudocompact-open topology ps, are denoted by $C_p(X)$, $C_k(X)$ and $C_{ps}(X)$ respectively.

In [18], in addition to studying some basic properties of $C_{ps}(X)$ and comparing it with $C_k(X)$, the submetrizability and metrizability of $C_{ps}(X)$ have been studied. Also in [18], a characterization result on the separability of $C_{ps}(X)$ has been given, but no illustrative example on this result has been given there. An important family of properties, the completeness properties of $C_{ps}(X)$, has been studied in detail in [17]. But another important family of properties, the countability properties of $C_{ps}(X)$, (except separability), is yet to be studied. In this paper, we plan to do exactly that. More precisely, we would like to study the $\aleph_0$-boundedness, countable chain condition, Lindelöf property and second countability of $C_{ps}(X)$ in comparison with the corresponding properties of $C_p(X)$ and $C_k(X)$. *Here we would like to mention that these comparisons essentially turn this paper into a sort of an expository research article.* But we should keep in mind that while a compact subset in a Tychonoff space is C-embedded, a closed pseudocompact subset need not be so. Also a closed subset of a pseudocompact subset need not be pseudocompact. Often these two facts make the study of $C_{ps}(X)$ difficult.

In Section 2 of this paper, we compare the possibilities of $C_{ps}(X)$ and $C_k(X)$ having the properties of $\aleph_0$-bounded and countable chain condition. In order to have a better perspective of the comparison of these possibilities, we recall the result on the separability of $C_{ps}(X)$ from [18] with the addition of one more equivalent condition. In Section 3, we compare $C_{ps}(X)$, $C_k(X)$ and $C_p(X)$ in regard to the Lindelöf property. In section 4, first we prove some equivalent characterizations of the second countability of $C_p(X)$. Then we move to the second countability of $C_k(X)$ and $C_{ps}(X)$ in order to show that the situation for second countability of $C_{ps}(X)$ is similar to that of separability, that is, $C_{ps}(X)$ is second countable if and only if $C_k(X)$ is second countable. Here we also present four important, *but already known*, equivalent characterizations of the second countability of $C_k(X)$ in term of the topological properties of X.

Throughout this paper, all spaces are Tychonoff spaces and $\mathbb{R}$ denotes the space of real numbers with the usual topology. The constant zero function defined on X is denoted by 0, more precisely by 0_X. We call it the constant zero function in $C(X)$. The symbols ω_0 and ω_1 denote the first infinite and the first uncountable ordinal respectively. Given a set X, $card\,X$ denotes the cardinality of X. If X and Y are two spaces with the same underlying set, then we use $X = Y$, $X \leq Y$ and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X and that the topology on Y is strictly finer than the topology on X. So for any space X, $C_p(X) \leq C_k(X) \leq C_{ps}(X)$.

2. Separability, $\aleph_0$-boundedness and countable chain condition

Similar to the point-open and compact-open topologies, there are three ways to consider the pseudocompact-open topology ps on $C(X)$.

First, one can use as subbase the family $\{[A, V] : A$ is a pseudocompact subset of X and V is an open subset of $\mathbb{R}\}$ where $[A, V] = \{f \in C(X) : f(A) \subseteq V\}$. But one can also consider this topology as the topology of uniform convergence on the pseudocompact subsets of X, in which case the basic open sets will be of the form $\langle f, A, \epsilon\rangle = \{g \in C(X) : |g(x) - f(x)| < \epsilon$ for all $x \in A\}$, where $f \in C(X)$, A is a pseudocompact subset of X and ϵ is a positive real number.

The third way is to look at the pseudocompact-open topology as a locally convex topology on $C(X)$. For each pseudocompact subset A of X and $\epsilon > 0$, we define the seminorm p_A on $C(X)$ and $V_{A,\epsilon}$ as follow: $p_A(f) = \sup\{|f(x)| : x \in A\}$ and $V_{A,\epsilon} = \{f \in C(X) : p_A(f) < \epsilon\}$. Let $\mathcal{V} = \{V_{A,\epsilon} : A$ is a pseudocompact subset of X, $\epsilon > 0\}$. Then for each $f \in C(X)$, $f + \mathcal{V} = \{f + V : V \in \mathcal{V}\}$ forms a neighborhood base at f. This topology is locally convex since it is generated by a collections of seminorms and it is same as the pseudocompact-open topology ps on $C(X)$. It is also easy to see that this topology is Hausdorff.

In order to have a better perspective of $\aleph_0$-boundedness and countable chain condition, first we recall the result that $C_{ps}(X)$ is separable if and only if $C_k(X)$ is separable. More precise and detailed statement follows.

THEOREM 2.1. *For a space X, the following assertions are equivalent.*

(a) *$C_{ps}(X)$ is separable.*

(b) *$C_k(X)$ is separable.*

(c) *$C_p(X)$ is separable.*

(d) *X has a separable metrizable compression, that is, X has a weaker separable metrizable topology.*

(e) *X is submetrizable and has a dense subset of cardinality less than or equal to $2^{\aleph_0}$.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$. These are immediate.

$(c) \Rightarrow (d)$. This can be proved in a manner similar to Lemma 3 of [34].

$(d) \Rightarrow (e)$. Let $f : X \longrightarrow Y$ be a continuous bijection where Y is a separable metric space. So $card\, X = card\, Y$. But Y being separable, $card\, Y \leq 2^{\aleph_0}$. Hence $card\, X \leq 2^{\aleph_0}$.

$(e) \Rightarrow (a)$. Since X is submetrizable, every pseudocompact subset of X is compact and consequently, $C_{ps}(X) = C_k(X)$. But by the theorem in [33], $(e) \Rightarrow (b)$. $\square$

COROLLARY 2.2. *If $C_{ps}(X)$ is separable, then $C_{ps}(X) = C_k(X)$.*

REMARK 2.3. *In Theorem 5.8 of [18], the equivalence of the conditions (a), (b), (c) and (d) has been proved by using the Corollary 4.2.2 of [25]. But this corollary is essentially due to Warner [34].*

EXAMPLE 2.4. *Suppose X is pseudocompact, but not metrizable. Since a pseudocompact submetrizable space is metrizable, (see Lemma 4.3 in [19]), X cannot be submetrizable either. Hence $C_{ps}(X)$ cannot be separable. In particular for $X = [0, \omega_1)$ or $X = [0, \omega_1]$, neither $C_{ps}(X)$ nor $C_k(X)$ is separable. Note that for $X = [0, \omega_1]$, $C_p(X) < C_k(X) = C_{ps}(X)$, but for $X = [0, \omega_1)$, $C_p(X) < C_k(X) < C_{ps}(X)$.*

EXAMPLE 2.5. *Since $\mathbb{R}$ is a separable metric space, $C_{ps}(\mathbb{R})$ is separable. For this space, we have*

$$C_p(\mathbb{R}) < C_k(\mathbb{R}) = C_{ps}(\mathbb{R}).$$

EXAMPLE 2.6. *For every infinite subset $C \subseteq \mathbb{N}$, let x_C be a point in $cl_{\beta X} C \backslash \mathbb{N}$. Consider the space $X = \mathbb{N} \cup \{x_C : C$ is an infinite subset of $\mathbb{N}\}$ with the relative topology induced from $\beta\mathbb{N}$. So X is a Tychonoff space. This space is also pseudocompact. Every compact subset of X is finite. See [15]. Since X is pseudocompact but not compact, X is not submetrizable. Consequently $C_{ps}(X)$ is not separable. For this space X, we have*

$$C_p(X) = C_k(X) < C_{ps}(X).$$

EXAMPLE 2.7. *Consider the Fortissimo space F, (Example 25, page 53 in [32]), which is uncountable. This space is Lindelöf and every compact subset of F is finite. Consequently every pseudocompact subset of F is also finite. But F is not submetrizable, since there exists a non-G_δ-point in F. Hence $C_p(F)$ is not separable. For this space F, we have*

$$C_p(F) = C_k(F) = C_{ps}(F).$$

Example 2.8. *Suppose X is a countable discrete space. Then $C_{ps}(X)$ is separable. For this space X, we have*

$$C_p(X) = C_k(X) = C_{ps}(X).$$

Now we would like to study two closely related topological properties of $C_{ps}(X)$ and $C_k(X)$, each of which is weaker than separability. A space X is said to have the *countable chain condition* (called ccc in brief) if any family of pairwise disjoint nonempty open subsets of X is countable. The ccc is also known as the Souslin property. It is clear that every separable space has ccc. *Note that for every nonempty set X, by [9, Corollary 2.3.18], $\mathbb{R}^X$ has always ccc. But for any space X, $C_p(X)$ is dense in $\mathbb{R}^X$. Consequently for any space X, $C_p(X)$ has always ccc.*

The second property, weaker than separability, is known as being $\aleph_0$-bounded. The precise definition follows.

Definition 2.9. *Let G be a topological group (under addition). Then G is said to be $\aleph_0$-bounded provided that for each neighborhood U of the identity element in G, there exists a countable subset S of G such that $G = S + U = \{s + u : s \in S, u \in U\}$.*

Arhangel'skii studied $\aleph_0$-bounded topological groups in the Section 9 of [1] in a more general setting of τ-bounded topological groups. According to Arhangel'skii, the τ-bounded topological groups were first studied by Guran in [14]. We would like to state the following interesting and significant results on $\aleph_0$-bounded topological groups mentioned in [1].

(*a*) The product of any family of $\aleph_0$-bounded topological groups is $\aleph_0$-bounded.

(*b*) Any subgroup of an $\aleph_0$-bounded topological group is $\aleph_0$-bounded.

(*c*) A topological group having a dense $\aleph_0$-bounded topological subgroup is itself $\aleph_0$-bounded.

(*d*) The image of an $\aleph_0$-bounded topological group under a continuous homomorphism is $\aleph_0$-bounded.

(*e*) The class of $\aleph_0$-bounded groups contains all subgroups of compact Hausdorff groups.

(*f*) A topological group is $\aleph_0$-bounded if and only if it is topologically isomorphic to a subgroup of a topological group having ccc. Then obviously a topological group having ccc is itself $\aleph_0$-bounded.

(*g*) A Lindelöf topological group is $\aleph_0$-bounded.

(*h*) A metrizable $\aleph_0$-bounded group is separable.

In [25], an $\aleph_0$-bounded topological group has been called totally $\aleph_0$-bounded. Here we would like to put a note of caution. In [4], Arhangel'skii has used the term '$\aleph_0$-bounded' for an entirely different concept. In [4], a topological space has been called $\aleph_0$-bounded if the closure of every countable subset of X is compact.

The next result gives a necessary condition for $C_{ps}(X)$ to be $\aleph_0$-bounded.

THEOREM 2.10. *For a space X, assume that $C_{ps}(X)$ is $\aleph_0$-bounded. Then every C-embedded pseudocompact subset of X is metrizable and compact.*

Proof. Let A be a C-embedded pseudocompact subset of X. Now first we will show that $C_{ps}(A)$ is $\aleph_0$-bounded. Since A is pseudocompact, $C_{ps}(A)$ is metrizable. To avoid the confusion, for each $f \in C(A)$ and $\epsilon > 0$, let us denote $\langle f, A, \epsilon\rangle_A = \{g \in C(A) : |f(x) - g(x)| < \epsilon \ \forall \ x \in A\}$. Then for each f, the collection $\{\langle f, A, \epsilon\rangle_A : \epsilon > 0\}$ forms a neighborhood base of f in $C_{ps}(A)$. Let $\langle 0_A, A, \epsilon\rangle_A$ be a basic neighborhood of the zero function 0_A in $C_{ps}(A)$. Now $\langle 0_X, A, \epsilon\rangle$ is a basic neighborhood of the zero function 0_X in $C_{ps}(X)$. Since $C_{ps}(X)$ is $\aleph_0$-bounded, there exists a countable set B in $C_{ps}(X)$ such that $C_{ps}(X) = B + \langle 0_X, A, \epsilon\rangle$. Now let $B_A = \{f|_A : f \in B\}$. Let $f \in C(A)$. Since A is C-embedded in X, there exists a continuous extension f^* of f to X. Then $f^* = h + g$ where $h \in B$ and $g \in \langle 0_X, A, \epsilon\rangle$. Then $h|_A \in B_A$, $g|_A \in \langle 0_A, A, \epsilon\rangle$ and $f = g|_A + h|_A$. Thus $C_{ps}(A)$ is $\aleph$-bounded. Now, $\aleph_0$-bounded metrizable groups are separable. Therefore, $C_{ps}(A)$ is separable. By Theorem 2.1, A is submetrizable. But by Lemma 4.3 in [19], a pseudocompact submetrizable space is metrizable and compact. Thus A is metrizable and compact. □

Before stating the next corollary, we need to define a σ-functionally normal space. A space X is called *σ-functionally normal* if for any two disjoint closed sets A and B in X, there is a sequence (f_n) in $C(X)$ such that if $x \in A$ and $y \in B$, then there exists n such that $f_n(x) \neq f_n(y)$. Obviously a normal space is σ-functionally normal, but the converse need not be true. The Niemytzki plane L and the deleted Tychonoff plank T_∞ are σ-functionally normal, but not normal. Every closed pseudocompact subset in a σ-functionally normal space is C-embedded. For details on σ-functionally normal spaces, see [7].

Corollary 2.11. *If X is a σ-functionally normal space and $C_{ps}(X)$ is $\aleph_0$-bounded, then every pseudocompact subset of X is metrizable and compact.*

Corollary 2.12. *If $C_{ps}(X)$ is either Lindelöf or has ccc, then every C-embedded pseudocompact subset of X is metrizable and compact.*

Corollary 2.13. *Suppose that X is pseudocompact and $C_{ps}(X)$ is either Lindelöf or has ccc, then X is metrizable and compact.*

By using Theorem 2.1, an alternate proof of Corollary 2.13 can be given as follows. If X is pseudocompact, the pseudocompact-open topology ps on $C(X)$ is actually generated by the supremum metric on $C(X)$ and consequently $C_{ps}(X)$ is metrizable. So in addition, if $C_{ps}(X)$ is either Lindelöf or has ccc, then $C_{ps}(X)$ would be separable and consequently by Theorem 2.1, X would be submetrizable. But a pseudocompact submetrizable space is metrizable and compact.

Corollary 2.14. *If X is pseudocompact, then the following statements are equivalent.*

(a) *$C_{ps}(X)$ is separable.*

(b) *$C_{ps}(X)$ has ccc.*

(c) *X is metrizable.*

Proof. $(a) \Rightarrow (b)$. This is immediate.

$(b) \Rightarrow (c)$. By Corollary 2.13, X is metrizable.

$(c) \Rightarrow (a)$. If X is metrizable, then X, being pseudocompact, is also compact. Hence X is separable and consequently by Theorem 2.1, $C_{ps}(X)$ is separable. $\square$

Note that if $C_{ps}(X)$ has ccc, then $C_k(X)$ also has ccc. So we would like to find an example of a space X such that $C_k(X)$ has ccc, but $C_{ps}(X)$ does not have ccc. If there exists an infinite pseudocompact space X whose compact subsets are finite, then $C_p(X) = C_k(X) < C_{ps}(X)$. So $C_k(X)$ has ccc, but by Corollary 2.13, $C_{ps}(X)$ will not have ccc. But does there exist such a space X? The answer is affirmative and such a space has already been presented in Example 2.6.

In the Corollary 4.2.7 of [25], we have a stronger version of Theorem 2.10 for $C_k(X)$. This corollary says that $C_k(X)$ is $\aleph_0$-bounded if and only if every compact subset of X is metrizable. This corollary also helps us to have the following converse of Theorem 2.10.

Theorem 2.15. *If every pseudocompact subset of X is submetrizable, then $C_{ps}(X)$ is $\aleph_0$-bounded.*

Proof. Here each pseudocompact subset of X is actually metrizable and compact. Hence $C_{ps}(X) = C_k(X)$ and consequently $C_{ps}(X)$ is $\aleph_0$-bounded. $\square$

Corollary 2.16. *Assume that X is σ-functionally normal. Then $C_{ps}(X)$ is $\aleph_0$-bounded if and only if every pseudocompact subset of X is metrizable and compact.*

If X itself is assumed to be submetrizable, then we get a stronger conclusion that $C_{ps}(X)$ has ccc. This result can be proved in a manner similar to Proposition 7.1.3 in [28].

Theorem 2.17. *If X is submetrizable, then $C_{ps}(X)$ has ccc.*

Proof. Let $\{\langle f_\lambda, A_\lambda, \epsilon_\lambda\rangle : \lambda \in \Lambda\}$ be a family of pairwise disjoint (nonempty) basic open sets in $C_{ps}(X)$. Here each A_λ is pseudocompact and $\epsilon_\lambda > 0$. Since X is submetrizable, each A_λ is actually compact and metrizable, and consequently each A_λ is separable. For each $\lambda \in \Lambda$, let D_λ be a countable dense subset of A_λ.

If possible, suppose that Λ is uncountable. Then there exists an uncountable subset Λ_0 of Λ such that $card\,\Lambda_0 \leq 2^{\aleph_0}$. Let $Y = \cup\{A_\lambda : \lambda \in \Lambda_0\}$ and $D = \cup\{D_\lambda : \lambda \in \Lambda_0\}$. It is easy to see that D is dense in Y and $card\,D \leq 2^{\aleph_0}$. But Y is also submetrizable. Hence by Theorem 2.1, $C_{ps}(Y)$ is separable and consequently $C_{ps}(Y)$ has ccc.

Now for each $\lambda \in \Lambda_0$, let $W_\lambda = \langle f_\lambda|_Y, A_\lambda, \epsilon_\lambda\rangle$. Note that each W_λ is open in $C_{ps}(Y)$. Also we claim that $W_{\lambda_1} \cap W_{\lambda_2} = \emptyset$ whenever $\lambda_1, \lambda_2 \in \Lambda_0$ and $\lambda_1 \neq \lambda_2$. If possible, suppose that $g \in W_{\lambda_1} \cap W_{\lambda_2}$. Since $A_{\lambda_1} \cup A_{\lambda_2}$ is compact in X, there exists $h \in C(X)$ such that $h(x) = g(x)\ \forall\ x \in A_{\lambda_1} \cup A_{\lambda_2}$. But then $h \in \langle f_{\lambda_1}, A_{\lambda_1}, \epsilon_{\lambda_1}\rangle \cap \langle f_{\lambda_2}, A_{\lambda_2}, \epsilon_{\lambda_2}\rangle = \emptyset$. We arrive at a contradiction. Hence $\{W_\lambda : \lambda \in \Lambda_0\}$ is a family of pairwise disjoint nonempty open sets in $C_{ps}(Y)$. But since $C_{ps}(Y)$ has ccc, Λ_0 must be countable. We arrive at a contradiction. Hence Λ must be countable and consequently $C_{ps}(X)$ has ccc. $\square$

Corollary 2.18 (Proposition 7.1.3 of [28]). *If X is submetrizable, then $C_k(X)$ has ccc.*

The following counterexample shows that the converse of Theorem 2.17 need not be true.

Counterexample. Consider the Fortissimo space F mentioned in Example 2.7, For this space, we have $C_p(F) = C_k(F) = C_{ps}(F)$. We have already noted that for any space X, $C_p(X)$ has always ccc. So here for the Fortissimo space F, $C_{ps}(F) = C_p(F)$ has ccc. But F is not submetrizable, since there exists a non-G_δ-point in F.

3. Lindelöf Property

In this section, we study the situations when possibly $C_j(X)$ ($j = p,\ k,\ ps$) can be Lindelöf. Since $C_p(X) \leq C_k(X) \leq C_{ps}(X)$, any necessary condition for $C_p(X)$ to be Lindelöf also becomes necessary for $C_k(X)$ as well as for $C_{ps}(X)$ to be Lindelöf. Therefore it becomes expedient to search for criteria in term of topological properties of X so that $C_p(X)$ becomes Lindelöf. But here we should mention that though many well-known mathematicians have searched and studied several such criteria, no satisfactory intrinsic characterization of the space X, for which $C_p(X)$ is Lindelöf, is yet to emerge. We would also like to mention that while Arhangel'skii presented his works on the conditions for $C_p(X)$ to be Lindelöf in 5th Prague Topology Symposium held in 1981, (see [3], the Section 4 of Chapter I in [4], pp. 29-32 in [5] and Exercise 3 in page 68 of [25]), the paper [23] appeared in 1980. But an important necessary condition for $C_p(X)$ to be Lindelöf in term of the tightness of X was found earlier by Asanov in 1979, see [6]. Recall that a space X is said to have countable tightness if for each $x \in X$ and $A \subseteq X$ such that $x \in \overline{A}$, there exists a countable subset C of A such that $x \in \overline{C}$.

Theorem 3.1 (M. O. Asanov [6]). *If $C_p(X)$ is Lindelöf, then for every $n \in \mathbb{N}$, X^n has countable tightness.*

Proof. For the proof of a more general version of this result, see Theorem I.4.1, page 33 in [4]. □

Remark 3.2. *For $n = 1$, Theorem 3.1 was proved independently by McCoy in [23].*

Of-course, the condition in Theorem 3.1 is not sufficient for $C_p(X)$ to be Lindelöf. The following examples justify this assertion.

Example 3.3 (Example 3 in [23]). *Let X be the interval $[0, 1)$ with the Sorgenfrey topology. It has been shown in [27] that $C_p(X, [0, 1])$ is not normal. Here $C_p(X, [0, 1])$ is the space $C(X, [0, 1])$ equipped with the point-open topology and $C(X, [0, 1]) = \{f \in C(X) : f(X) \subseteq [0, 1]\}$. But $C_p(X, [0, 1])$ is closed in $C_p(X)$. Hence $C_p(X)$ cannot be normal either. Consequently $C_p(X)$ is not Lindelöf. Yet X is a Lindelöf space such that for every $n \in \mathbb{N}$, X^n has countable tightness. Note that for this space X, $C_p(X) < C_k(X) = C_{ps}(X)$.*

Example 3.4. *Let X be a discrete space. Then $C_p(X) = \mathbb{R}^X$. If X is countable, then $\mathbb{R}^X$ is second countable and consequently it is Lindelöf. Conversely, if we assume $\mathbb{R}^X$ to be Lindelöf, then X must be countable. In order to prove this, for each $x \in X$, let Z_x^+ be the discrete space of positive integers. Now the product space $\Pi\{Z_x^+ : x \in X\}$ is closed in $\mathbb{R}^X$. If $\mathbb{R}^X$ is Lindelöf, then $\mathbb{R}^X$ is normal and consequently $\Pi\{Z_x^+ : x \in X\}$ will also be normal. But* *it has been shown in [32, Example 103] that* $\Pi\{Z_x^+ : x \in X\}$ *is not normal. So if* X *is an uncountable discrete space, then* $C_p(X)$ *is not Lindelöf, though* X^n *obviously has countable tightness for each* $n \in \mathbb{N}$.

EXAMPLE 3.5. *The "double arrow" space* X *is first countable and compact, but it is not metrizable. In literature, this space is also called the "two arrows" space. For details on this space, see Exercise 3.10 C, page 212 in [9]. Also see page 30 in [5]. As argued in Example 1 of [29], it can be shown that* $C_p(X)$ *is not even normal. But since* X *is first countable,* X^n *has countable tightness for each* $n \in \mathbb{N}$. *Note that for this space* X, $C_p(X) < C_k(X) = C_{ps}(X)$.

On the other hand. by using the necessity of the countable tightness of X, often we can conclude that $C_p(X)$ is not Lindelöf. The following examples justify this observation.

EXAMPLE 3.6. *Let* X *be the ordinal space* $[0, \omega_1]$. *Since* X *does not have countable tightness,* $C_p(X)$ *is not Lindelöf. Note that for this space* X, $C_p(X) < C_k(X) = C_{ps}(X)$.

EXAMPLE 3.7. *The Fortssimo space* F *does not have countable tightness and consequently* $C_p(F)$ *is not Lindelöf. We have already noted in Example 2.7 that* $C_p(F) = C_k(F) = C_{ps}(F)$.

Now we are going to have eleven more necessary conditions for $C_p(X)$ to be Lindelöf. But ten of these conditions can actually be obtained as corollaries to Theorem 3.8. Most of these conditions first appeared in [3] and [4] or in [23].

THEOREM 3.8 (Proposition I.4.3 in [4]). *Suppose that* $C_p(X)$ *is Lindelöf and let* Y *be a* C*-embedded subset of* X. *Then* $C_p(Y)$ *is Lindelöf.*

Proof. Consider the inclusion map $i : Y \longrightarrow X$. Then since Y is C-embedded in X, the induced map $i^* : C_p(X) \longrightarrow C_p(Y)$, given by $i^*(f) = f \circ i$, is a continuous surjection. But a continuous image of a Lindelöf space is also Lindelöf. □

COROLLARY 3.9. *Suppose that* $C_p(X)$ *is Lindelöf and let* Y *be a discrete* C*-embedded subset of* X. *Then* Y *is countable.*

Proof. By Theorem 3.8, $C_p(Y)$ is Lindelöf. But since Y is discrete, $C_p(Y) = \mathbb{R}^Y$. But it has already shown in Example 3.4 that unless Y is countable, $\mathbb{R}^Y$ cannot be Lindelöf. □

COROLLARY 3.10. *Suppose that X is normal and $C_p(X)$ is Lindelöf. Then every point-finite family of open sets in X is countable.*

Proof. Let $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$ be a point-finite family of nonempty open sets in X. For each $\gamma \in \Gamma$, choose $x_\gamma \in V_\gamma$ and consider the set $Y = \{x_\gamma : \gamma \in \Gamma\}$. Note that since $\mathcal{V}$ is point-finite, Y is closed and discrete in X. Hence by Corollary 3.9, Y is countable. Now since Y is countable, again by using the point-finiteness of $\mathcal{V}$, it can be shown that Γ must be countable. □

COROLLARY 3.11 (Corollary 2 in [23]). *If X is metacompact and normal, and if $C_p(X)$ is Lindelöf, then X is Lindelöf.*

COROLLARY 3.12. *If X is paracompact and $C_p(X)$ is Lindelöf, then X is Lindelöf.*

REMARK 3.13. *We have already noted that for any space X, $C_p(X)$ has always ccc. Hence $C_p(X)$ is Lindelöf if and only if $C_p(X)$ is paracompact. So the Corollary 3.12 can be rephrased as follows : If both X and $C_p(X)$ are paracompact, then X is Lindelöf.*

COROLLARY 3.14. *If X is metrizable and $C_p(X)$ is Lindelöf, then X is Lindelöf.*

COROLLARY 3.15. *Suppose that X is normal and $C_p(X)$ is Lindelöf. Then every closed discrete subset of X is countable. Moreover, every pseudocompact subset of X is metrizable and compact. In particular, $C_{ps}(X) = C_k(X)$.*

Proof. Since X is normal, the first part of the corollary follows from Corollary 3.9, while the second part of the corollary follows from Corollary 2.12. □

COROLLARY 3.16. *If X is normal, then $C_{ps}(X)$ is Lindelöf if and only if $C_k(X)$ is Lindelöf. Moreover in this case, $C_{ps}(X) = C_k(X)$.*

Corollary 3.17 (Theorem 3 in [23]). *Suppose that X is normal and $C_p(X)$ is Lindelöf. Then every discrete family of closed sets in X is countable.*

Proof. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a discrete family of nonempty closed sets in X. Hence $A_{\lambda_1} \cap A_{\lambda_2} = \emptyset$ whenever $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 \neq \lambda_2$. For each $\lambda \in \Lambda$, choose $x_\lambda \in A_\lambda$ and consider the set $Y = \{x_\lambda : \lambda \in \Lambda\}$. Note that Y is closed and discrete in X. Hence by Corollary 3.9, Y is countable. But since $card\, Y = card\, \Lambda$, Λ is also countable. □

Corollary 3.18 (Proposition I.4.4 in [4]). *If $C_p(X)$ is Lindelöf, then every discrete family of open sets in X is countable.*

Proof. This corollary follows from Corollary 3.9. For a complete proof, see Proposition I.4.4 in [4]. □

Now we would like to see if the normality condition on X in Corollary 3.17 can be weakened. For this query, we have the following interesting result.

Theorem 3.19 (Theorem 5 in [23], Corollary I.4.13 in [4]). *Assume that $C_p(X)$ is Lindelöf and that the space X satisfies the condition (α): whenever A and B are countable subsets of X with $\overline{A} \cap \overline{B} = \emptyset$, then there is an f in $C(X)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then X is normal.*

Proof. For the proof, either see Theorem 5 in [23] or Corollary I.4.13 in [4]. □

The following example illustrates a nice application of Theorem 3.19.

Example 3.20 (Example 4 in [23]). *The Tychonoff plank T is defined to be $[0, \omega_1] \times [0, \omega_0]$, where both ordinals spaces $[0, \omega_1]$ and $[0, \omega_0]$ are given the interval topology. The subspace $T_\infty = T \setminus \{(\omega_1, \omega_0)\}$ is called the deleted Tychonoff plank. The space T_∞ is not normal, but it is pseudocompact. For more details on T_∞, see Example 87 in [32]. Now if A is a countable subset of T_∞, then it can be shown that A must actually be contained in some compact open subspace of T_∞. From this, it can easily be concluded that T_∞ must satisfy the condition (α) of Theorem 3.19. Hence $C_p(T_\infty)$ is not Lindelöf. Note that $C_p(T_\infty) < C_k(T_\infty) < C_{ps}(T_\infty)$.*

We do not know yet of any simple purely topological condition sufficient for $C_p(X)$ to be Lindelöf except one that X has a countable network. In order to look at this sufficient condition from a larger perspective, we need to define network and some related concepts.

DEFINITION 3.21. *A nonempty family $\mathcal{F}$ of nonempty subsets of a space X is called a network provided that for each $x \in X$ and each open neighborhood U of x, there exists $F \in \mathcal{F}$ such that $x \in F \subseteq U$. A space X is called a σ-space (cosmic space) if X has a σ-discrete (countable) network.*

A nonempty family $\mathcal{F}$ of nonempty subsets of a space X is called a k-network provided that for each compact subset K and each open set U in X such that $K \subseteq U$, there exists $F \in \mathcal{F}$ such that $K \subseteq F \subseteq U$. A space X is called an $\aleph_0$-space if it has a countable k-network.

REMARK 3.22. *Clearly an $\aleph_0$-space is a cosmic space and a cosmic space is a σ-space. In Theorem 4.11 of [11], it has been shown that a regular T_1-space has a σ-discrete network if and only if X has a σ-locally finite network. In [26], a network has been called a point-pseudobase, while a k-network has been called pseudobase. It has been shown in [26] that a space X is cosmic if and only if X is an image of a separable metric space under a continuous bijection. Now it follows that a cosmic space is submetrizable, hereditarily Lindelöf and hereditarily separable.*

THEOREM 3.23 (Proposition 10.5 in [26]). *For a space X, $C_p(X)$ is a cosmic space if and only if X is a cosmic space.*

Proof. See Proposition 10.5 in [26]. □

COROLLARY 3.24. *If X is cosmic, then $C_p(X)$ is Lindelöf.*

COROLLARY 3.25. *Suppose that X is a normal σ-space. Then $C_p(X)$ is cosmic if and only if $C_p(X)$ is Lindelöf.*

Proof. If $C_p(X)$ is Lindelöf, then by Corollary 3.17, every σ-discrete network is countable and consequently X is a cosmic space. Hence by Theorem 3.23, $C_p(X)$ is cosmic. □

THEOREM 3.26. *For a space X, the following statements are equivalent.*

(a) *X is an $\aleph_0$-space.*

(b) *$C_k(X)$ is an $\aleph_0$-space.*

(c) *$C_k(X)$ is cosmic.*

(d) *$C_{ps}(X)$ is cosmic.*

Proof. The equivalence of the conditions (a), (b) and (c) has been proved in Proposition 10.3 in [26]. (a) $\Rightarrow$ (d). If X is an $\aleph_0$-space, then X is submetrizable and consequently $C_{ps}(X) = C_k(X)$. Hence by (c), $C_{ps}(X)$ is cosmic.

(d) $\Rightarrow$ (c). It is immediate, since cosmicness is preserved by a weaker topology. □

COROLLARY 3.27. *If X is an $\aleph_0$-space, then $C_k(X)$ is Lindelöf. Moreover in this case, $C_k(X) = C_{ps}(X)$.*

COROLLARY 3.28. *If X is metrizable, then $C_k(X)$ is Lindelöf if and only if X is Lindelöf.*

Proof. If X is metrizable as well as Lindelöf, then X is second countable and consequently by Corollary 3.27, $C_k(X)$ is Lindelöf. Conversely, if $C_k(X)$ is Lindelöf, then by Corollary 3.14, X is Lindelöf. □

We end this section by having a few more examples in relation to the Lindelöf property of $C_j(X)$, ($j = p,\ k,\ ps$).

EXAMPLE 3.29. *Let X be a discrete space. Then $\mathbb{R}^X = C_p(X) = C_k(X) = C_{ps}(X)$. It has already been proved in Example 3.4 that $\mathbb{R}^X$ is Lindelöf if and only if X is countable.*

EXAMPLE 3.30. *From Corollary 3.28, it follows that for a separable metric space X, $C_{ps}(X)$ is Lindelöf. In particular, for $\mathbb{R}$, $C_{ps}(\mathbb{R})$ is Lindelöf and $C_p(\mathbb{R}) < C_k(\mathbb{R}) = C_{ps}(\mathbb{R})$.*

EXAMPLE 3.31. *Let X be the discrete irrational extension of $\mathbb{R}$, that is, in addition to the usual open sets, take the singleton irrational points to be open. This space has been discussed in detail in Example 71 of [32]. This space X is paracompact, but not Lindelöf. Hence by Corollary 3.12, $C_p(X)$ is not Lindelöf. For this space X, $C_p(X) < C_{ps}(X) = C_k(X)$.*

EXAMPLE 3.32. *Let $X = [0, \omega_1)$. Since X is countably compact, the pseudocompact-open topology on $C(X)$ is actually generated by the supremum metric on $C(X)$ and consequently $C_{ps}(X)$ is metrizable. If $C_{ps}(X)$ is Lindelöf, then it would be separable also. But in Example 2.4, it has been shown that $C_{ps}(X)$ is not separable. Hence $C_{ps}(X)$ is not Lindelöf. But by using Theorem 2 of [13], it has been shown in [23] that $C_k([0, \omega_1))$ is Lindelöf. For this space X, we have $C_p(X) < C_k(X) < C_{ps}(X)$*

4. Second Countability

In the last section of this paper, we study the second countability of $C_j(X)$, $j = p, k, ps$. We begin the study by first looking at the second countability of $C_p(X)$ from a larger perspective. Then we show that $C_k(X)$ is second countable if and only if $C_{ps}(X)$ is second countable. The second countability of $C_p(X)$ and $C_k(X)$ has been well-studied in the literature. While the second countability of $C_p(X)$ has been studied [3] and [4], the second countability $C_p(X)$ and $C_k(X)$ has been studied from a general perspective in [22, 24] and [25]). In particular, the second countability of $C_k(X)$ has been studied in [28]. For readers' convenience, we present two known results on the second countability of $C_k(X)$ with complete proofs, but obviously in the presence of $C_{ps}(X)$. For the first result in this section, we need the following definition.

DEFINITION 4.1. *A space X is a q-space if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of x such that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point.*

THEOREM 4.2. *For a space X, the following statements are equivalent.*

(*a*) *$C_p(X)$ is second countable.*

(*b*) *$C_p(X)$ is metrizable.*

(*c*) *$C_p(X)$ is first countable.*

(*d*) *$C_p(X)$ is a q-space.*

(*e*) *X is countable.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. These are all immediate.

$(d) \Rightarrow (e)$. Suppose that $C_p(X)$ is a q-space. Hence there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the constant zero function 0 in $C_p(X)$ such that if $f_n \in U_n$ for each $n \in \mathbb{N}$, then $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $C_p(X)$. Now for each n, there exist a finite subset F_n of X and $\epsilon_n > 0$ such that $\langle 0, F_n, \epsilon_n\rangle \subseteq U_n$. We claim that $X = \cup_{n=1}^{\infty} F_n$. If not, let $x_0 \in X \backslash \cup_{n=1}^{\infty} F_n$. Then for each $n \in \mathbb{N}$, there exists a continuous function $f_n : X \longrightarrow [0, 1]$ such that $f_n(x_0) = n$ and $f_n(x) = 0$ for all $x \in F_n$. It is clear that $f_n \in \langle 0, F_n, \epsilon_n\rangle$. But the sequence $\{f_n : n \in \mathbb{N}\}$ does not have a cluster point. If possible, suppose that this sequence has a cluster point f in $C_p(X)$. Then for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $f_{n_k} \in \langle f, \{x_0\}, 1\rangle$. So for all $k \in \mathbb{N}$, $f(x_0) > f_{n_k}(x_0) - 1 = n_k - 1 \geq k$. Since this is not possible, the sequence $\{f_n : n \in \mathbb{N}\}$ cannot have a cluster point in $C_p(X)$ and consequently $C_p(X)$ fails to be a q-space. Hence X must be countable.

$(e) \Rightarrow (a)$. If X is countable, then $\mathbb{R}^X$ is second countable and consequently $C_p(X)$ is also second countable. □

THEOREM 4.3. *For a space X, $C_{ps}(X)$ is second countable if and only if $C_k(X)$ is second countable. Moreover, in this case $C_{ps}(X) = C_k(X)$.*

Proof. If either $C_{ps}(X)$ or $C_k(X)$ is second countable, then it is separable and consequently by Theorem 2.1, X is submetrizable. Hence in this case, $C_{ps}(X) = C_k(X)$. □

For the next result on the second countability of $C_j(X)$, $j = k,\ ps$, we need the definition of π-base.

DEFINITION 4.4. *A family of nonempty open sets in a space X is called a π-base for X if every nonempty open set in X contains a member of this family.*

The routine proof of the following lemma is omitted.

LEMMA 4.5. *Let D be a dense subset of a space X. Then D has a countable π-base if and only if X has a countable π-base.*

Now Theorem 4.3 can be strengthened as follows.

THEOREM 4.6. *For a space X, the following statements are equivalent.*

(a) *$C_{ps}(X)$ contains a dense subspace which has a countable π-base.*

(b) *$C_{ps}(X)$ has a countable π-base.*

(c) *$C_{ps}(X)$ is second countable.*

(d) *$C_k(X)$ is second countable.*

(e) *X is hemicompact and submetrizable.*

Proof. By Lemma 4.5, (a) $\Leftrightarrow$ (b) and by Theorem 4.3, (c) $\Leftrightarrow$ (d). (b) $\Rightarrow$ (c). If $C_{ps}(X)$ has a countable π-base, then by Theorem 5.7 in [18], $C_{ps}(X)$ is metrizable. But it is easy to see that a space having a countable π-base is separable. Hence $C_{ps}(X)$ is second countable.

(d) $\Rightarrow$ (e). If $C_k(X)$ is second countable, then it is submetrizable as well as it is separable. Hence X is hemicompact and submetrizable.

(e) $\Rightarrow$ (d). If X is hemicompact, then $C_k(X)$ is metrizable. Note that X, being hemicompact, is Lindelöf. Since X is also submetrizable, X has a separable metrizable compression and consequently by Theorem 2.1, $C_k(X)$ is separable. Hence $C_k(X)$ is second countable. $\square$

In the next result, we would like to present a few more known equivalent characterizations of hemicompact submetrizable spaces.

THEOREM 4.7. *For a space X, the following statements are equivalent.*

(a) *X is hemicompact and submetrizable.*

(b) *X is a hemicompact $\aleph_0$-space.*

(c) *X is a hemicompact cosmic space.*

(d) *X is a hemicompact σ-space.*

Proof. $(a) \Rightarrow (b)$. Let $\{K_n : n \in \mathbb{N}\}$ be a countable family of compact subsets of X such that for each compact subset K of X, there exists n such that $K \subseteq K_n$. Note that $X = \bigcup_{n=1}^{\infty} K_n$. Since X is submetrizable, each K_n is metrizable and consequently each K_n is second countable. It is easy to show that a second countable space has a countable k-network, that is, a second countable space is an $\aleph_0$-space. Hence each K_n has a countable k-network $\mathcal{B}_n$. Now we claim that $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a k-network for X. Let $K \subseteq U$ where K is a compact subset of X and U is open in X. Then $K \subseteq K_n$ for some n and so $K \subseteq U \cap K_n$. Hence there exists $B \in \mathcal{B}_n$ such that $K \subseteq B \subseteq U \cap K_n$. So there exists $B \in \mathcal{B}$ such that $K \subseteq B \subseteq U$. Hence X is an $\aleph_0$-space.

$(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$. These are immediate.

$(d) \Rightarrow (a)$. If X is hemicompact, then it is Lindelöf. In addition, if X is a σ-space, then it becomes a paracompact σ-space. Hence by Theorem 4.4 in [11], X is a cosmic space and consequently X is submetrizable.

□

In the next result, we show that in presence of local compactness of X, the second countability of $C_{ps}(X)$ is equivalent to the second countability of X. The precise and detailed statement follows.

THEOREM 4.8. *For a locally compact space X, the following statements are equivalent.*

(*a*) *$C_{ps}(X)$ is second countable.*

(*b*) *$C_k(X)$ is second countable.*

(*c*) *X is hemicompact and submetrizable.*

(*d*) *X is Lindelöf and submetrizable.*

(*e*) *X is σ-compact and submetrizable.*

(*f*) *X is the union of a countable family of compact metrizable subsets of X.*

(*g*) *X is second countable.*

Proof. First we note that if X is locally compact, then X is hemicompact if and only if X is either Lindelöf or σ-compact; see 3.8.C(b), page 195 in [9]. Hence $(c) \Leftrightarrow (d) \Leftrightarrow (e)$.

Clearly $(e) \Rightarrow (f)$.

$(f) \Rightarrow (g)$. First we note that a compact metrizable space is second countable and consequently it is a cosmic space. Then as argued in the proof of $(a) \Rightarrow (b)$ in Theorem 4.7, it can be shown that if X is the union of a countable family of cosmic subspaces, then X is itself a cosmic space. Hence if (f) holds, then X becomes a cosmic space. But a cosmic space is submetrizable.

Now since X is locally compact, for each $x \in X$, there exists an open set V_x in X such that $x \in V_x$ and $\overline{V}_x$ is compact. Note that $\{V_x : x \in X\}$ is an open cover of X. But X, being σ-compact, is Lindelöf and consequently, there exists a countable subset $\{x_n : n \in \mathbb{N}\}$ of X such that $X = \bigcup\limits_{n=1}^{\infty} V_{x_n}$. But since X is submetrizable and each $\overline{V}_{x_n}$ is compact, each $\overline{V}_{x_n}$ is second countable. Consequently each V_{x_n} is also second countable and X becomes the union of a countable family of second countable open subsets of X. Hence X is second countable.

$(g) \Rightarrow (a)$. If X is second countable, then X is metrizable as well as X is Lindelöf. But since a locally compact Lindelöf space is hemicompact, by Theorem 4.6, it follows that $C_{ps}(X)$ is second countable. □

Corollary 4.9. *If X is locally compact and $C_p(X)$ is second countable, then $C_k(X)$ is also second countable.*

Proof. If $C_p(X)$ is second countable, by Theorem 4.2, X is countable. Hence, X being locally compact, X is hemicompact. Also since $C_p(X)$ is separable, X is submetrizable. Hence by Theorem 4.8, $C_k(X)$ is second countable. □

We end this paper with some examples in relation to the second countability of $C_j(X)$, $j = p,\ k,\ ps$.

Example 4.10. *If $X = \mathbb{Q}$, the space of rational numbers with the usual topology, then $C_p(\mathbb{Q}) < C_k(\mathbb{Q}) = C_{ps}(\mathbb{Q})$. Here $C_p(\mathbb{Q})$ is second countable, but since $\mathbb{Q}$ is not hemicompact, $C_k(\mathbb{Q})$ is not second countable.*

EXAMPLE 4.11. *If* $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ *with the usual topology, then* $C_p(X) < C_k(X) = C_{ps}(X)$*. Here both* $C_p(X)$ *and* $C_k(X)$ *are second countable.*

EXAMPLE 4.12. *If* X *is a discrete space, then* $\mathbb{R}^X = C_p(X) = C_k(X) = C_{ps}(X)$*. Then* $C_p(X)$ *is second countable if and only if* X *is countable.*

EXAMPLE 4.13. *If* $X = \mathbb{R}$*, then* $C_p(X) < C_k(X) = C_{ps}(X)$*. Here* $C_p(X)$ *is not second countable, while* $C_k(X)$ *is second countable.*

EXAMPLE 4.14. *For the Sorgenfrey line* $\mathbb{R}_l$*,* $C_p(\mathbb{R}_l) < C_k(\mathbb{R}_l) = C_{ps}(\mathbb{R}_l)$*. Since every compact subset in* $\mathbb{R}_l$ *is countable,* $\mathbb{R}_l$ *is not even* σ*-compact. Here neither* $C_p(\mathbb{R}_l)$ *nor* $C_k(\mathbb{R}_l)$ *is second countable.*

EXAMPLE 4.15. *If* $X = [0, \omega_1)$*, then* $C_p(X) < C_k(X) < C_{ps}(X)$*. Here none of* $C_j(X)$*,* $j = p,\ k,\ ps$*, is second countable.*

EXAMPLE 4.16. *For the space* X *mentioned in Example 2.6, none of* $C_j(X)$*,* $j = p,\ k,\ ps$*, is second countable.*

REFERENCES

[1] A.V. ARHANGEL'SKIĬ, *Classes of topological groups*, Russian Math. Surveys **36**, no. 3 (1981), 151–174.

[2] A.V. ARHANGEL'SKIĬ, *On linear homeomorphisms of function spaces*, Soviet Math. Dokl. **25** (1982), 852–855.

[3] A.V. ARHANGEL'SKIĬ, *On relationships between topological properties of* X *and* $C_p(X)$, General topology and its relations to modern analysis and algebra, V (Prague, 1981), Sigma Ser. Pure Math., vol. **3**, Heldermann, Berlin, (1983), 24–36.

[4] A.V. ARHANGEL'SKIĬ, *Topological function spaces*, Mathematics and its Applications, **78** (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, (1992). Translated from the Russian by R.A.M. Hoksbergen.

[5] A.V. ARKHANGEL'SKIĬ, C_p*-theory*, in: *Recent progress in general topology*, M. Hušek, Miroslav and J. van Mill eds., North-Holland Publishing Co. (1992), pp. 1–56.

[6] M.O. ASANOV, *On cardinal invariants of spaces of continuous functions*, Sovr. Topol. i Teor. Mnozhestv. Izhevsk **2** (1979), 8–12. In Russian.

[7] C.E. Aull, *Absolute C-embedding in functionally normal spaces and related spaces*, Colloq. Math. Soc. János Bolyai **23** (1978), 129–136.
[8] R.C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104.
[9] R. Engelking, *General topology*, revised and completed ed., Heldermann Verlag, Berlin (1989).
[10] R. Giles, *A generalization of the strict topology*, Trans. Amer. Math. Soc. **161** (1971), 467–474.
[11] G. Gruenhage, *Generalized metric spaces*, in *Handbook of set-theoretic topology*, K. Kunen and J.E. Vaughan eds., Elsevier Science Publishers B. V. (1984), 423–501.
[12] D. Gulick, *The σ-compact-open topology and its relatives*, Math. Scand. **30** (1972), 159–176.
[13] S.P. Gul'ko, *On properties of subsets of Σ-products*, Soviet Math. Dokl. **18**, no. 6 (1977), 1438–1442.
[14] I.L. Guran, *On embeddings of topological groups*, L'vorsk Univ., L'vov, (1981). Manuscript deposited at VINITI.
[15] J.E. Jayne, *Spaces of Baire functions. I*, Ann. Inst. Fourier (Grenoble) **24**, no. 4 (1974), 47–76.
[16] S. Kundu, *$C_b(X)$ revisited: induced map and submetrizability*, Quaest. Math. **27** (2004), 47–56.
[17] S. Kundu and Pratibha Garg, *Completeness properties of the pseudocompact-open topology on $C(X)$*, to appear in Math. Slovaca.
[18] S. Kundu and Pratibha Garg, *The pseudocompact-open topology on $C(X)$*, Topology Proc. **30**, no. 1 (2006), 279–299.
[19] S. Kundu and R.A. McCoy, *Topologies between compact and uniform convergence on function spaces*, Internat. J. Math. Math. Sci. **16** (1993), 101–109.
[20] S. Kundu, R.A. McCoy and A.B. Raha, *Topologies between compact and uniform convergence on function spaces. II*, Real Anal. Exchange **18** (1992/93), 176–189.
[21] S. Kundu and A.B. Raha, *The bounded-open topology and its relatives*, Rend. Istit. Mat. Univ. Trieste **27** (1995), 61–77.
[22] R.A. McCoy, *Countability properties of function spaces*, Rocky Mountain J. Math. **10**, no. 4 (1980), 717–730.
[23] R.A. McCoy, *Necessary conditions for function spaces to be Lindelöf*, Glasnik Matematički **15**, no. 35 (1980), 163–168.
[24] R.A. McCoy and I. Ntantu, *Countability properties of function spaces with set-open topologies*, Topology Proc. **10** (1985), 329–345.
[25] R.A. McCoy and I. Ntantu, *Topological properties of spaces of continuous functions*, Lecture Notes in Mathematics **1315**, Springer-Verlag, Berlin (1988).

[26] E. Michael, $\aleph_0$-*spaces*, J. Math. Mech. **15** (1966), 983–1002.

[27] E. Michael, $\aleph_0$-*spaces and a function space theorem of R. Pol*, Indiana Univ. Math. J. **26**, no. 2 (1977), 299–306.

[28] I. Ntantu, *The compact-open topology on $C(X)$*, PhD Thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, U.S.A. (1985).

[29] R. Pol, *Normality in function spaces*, Fund. Math. **84**, no. 2 (1974), 145–155.

[30] J. Schemets, *Espaces de fonctions continues*, Lecture Notes in Mathematics **519**, Springer-Verlag, New York, U.S.A. (1976).

[31] F.D. Sentilles, *Bounded continuous functions on a completely regular space*, Trans. Amer. Math. Soc. **168** (1972), 311–336.

[32] L.A. Steen and J.A. Seebach Jr., *Counterexamples in topology*, 2^{nd} ed., Springer-Verlag, New York, U.S.A. (1978).

[33] G. Vidossich, *Characterizing separability of function spaces*, Inv. Math. **10** (1970), 205–208.

[34] S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. **25** (1958), 265–282.

Received May 31, 2007.