Rend. Istit. Mat. Univ. Trieste
Vol. XXXIX, 407–420 (2007)

A Problem of Minimum Area for Bodies with Constrains on Width and Curvature

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Summary. - *In this paper we solve the problem of finding the convex plane body of minimum area with the hypothesis of having a limited radius of curvature, together with an assigned minimum width. This result allows us to give a new and mainly analytical proof of Pal's Theorem.*

1. Introduction

By a convex n-dimensional body K we mean a compact, convex subset of $\mathbb{R}^n$ with nonempty interior.

We consider the projections of K on a linear subspace of dimension r, with $0 < r < n$: if $r = n - 1$, what we get is the projection of K on a hyperplane, whose measure we call *brightness*; if $r = 1$, K is projected on a line and the measure of this projection is called *width*.

In other words, the width of K in a given direction is the distance between two supporting hyperplanes which are perpendicular to it. The reconstruction of geometric properties of convex bodies, such as volume, surface measure and shape, knowing informations about sections or projections, is the main topic of Geometric Tomography.

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Keywords: Pal's Theorem, Width, Triarc.
AMS Mathematics Subject Classification (2000): 52A40.

While the Blaschke-Lebesgue Theorem solves the problem of finding the convex body of minimum area having constant width ω, giving the Reuleaux triangle with width ω as solution, according to Pal's Theorem, the only convex body of minimum width ω_0 and minimum area is the equilateral triangle with height equal to ω_0. This result has been proved by Pal in 1921 (see [9]) and later by Yaglom and Boltjanski (see [12]) in a rather similar way; as far as we know, the only other proof which can be found in literature is given by Campi, Colesanti and Gronchi (see [3]).
Pal's problem arises from the so-called *Kakeya problem,* which has a curious background. In 1917, Besicovitch posed the following question:

If f is a Riemann-integrable function, defined on the plane, is it always possible to find an orthogonal coordinate system in which $F(y) = \int f(x, y)\,dx$ exists, as a Riemann integral, for every y, and $F(y)$ is also Riemann-integrable ?

Besicovitch realised that this is not true if one can find a compact plane set C, having Lebesgue measure equal to zero, containing a segment in every direction. In 1919, Besicovitch succeeded in constructing such a set, but, due to the Russian political instability at the beginning of the century, the result got attention only in 1928. In the meantime, Kakeya and Fujiwara in 1917 were investigating the problem of finding the convex body of minimum area in which a unit segment can be rotated.

They rightly conjectured that the equilateral triangle of height 1 was the solution and the result was proved, as we said, by Pal in 1920.

Later on, in 1928, Perron and Besicovitch proved that a small modification of the set he had found 9 years before, leads to a solution of Kakeya problem (without the convexity hypothesis of course) providing a set of arbitrarily small measure.

In this paper we present a new proof of Pal's Theorem, which is substantially different from the others that can be found in literature; the starting point is a new characterization, which may be of independent interest, for plane convex bodies of minimum area, minimum width ω_0 and carrying a suitable restriction on the radius of curvature. Such bodies are triarcs whose boundary is given by three circular arcs with maximum radius. With a suitable limit operation, from these bodies we get the equilateral triangle as solution to Pal's Theorem. The plan of the paper is the following. In the second section, we recall some definitions and results in convex geometry that we are going to use in the sequel.

In the third section, we study Pal's problem with restrictions on the curvature. In particular, we work in the class F_k of the convex bodies K having minimum width 1 and radius of curvature ρ, such that

$$\rho(z) + \rho(-z) \leq k \text{ a.e..}$$

This class is closed with respect to the Hausdorff metric and it can be proved that if $K \in F_k$, then K is $k-$convex, that is it can be constructed by intersection of circles of radius k (see Proposition 3.3).

According to Sholander-Chakerian's Theorem, the extremal body is a triarc; using the Steiner symmetrization and its properties, we are able to prove that the three basic points of our minimal triarc T are vertices of an equilateral triangle and finally that the boundary of T is given by three circles of radius k, each joining two of the three basic points.

In the final section, the previous results lead us to the conclusion that the equilateral triangle of height 1 is a body of minimum area among those having minimum width 1.

Our approach does not allow us to prove that such an optimal body is unique.

I wish to express my gratitude to Professor Stefano Campi and Professor Paolo Gronchi for their useful advices and hints.

2. Preliminaries

Given a planar convex body K, its *support function* H_K is defined by

$$H_K(\xi) = \sup\{\langle x, \xi \rangle;\ x \in K\}, \qquad \xi \in \mathbb{R}^2. \tag{1}$$

Let h_K be the restriction of H_K to $S^1 = \{x \in \mathbb{R}^2 : |\, x \,| = 1\}$, that is $x = (\cos\theta, \sin\theta)$.

The *width* of K in the direction corresponding to the angle θ, is given by

$$\omega_K(\theta) = h_K(\theta) + h_K(\theta + \pi), \tag{2}$$

and then it measures the distance between two distinct parallel supporting lines orthogonal to such a direction. $\omega_K(\theta)$ is called the *width function* of K and, from now on, we are going to use this notation for ω_K, meaning that we are working along the direction of the vector $(\cos\theta, \sin\theta)$.

The thickness ω_0 of a convex body K, that is $\omega_0 = \min\{\,\omega_K(\theta) : \theta \in \mathbb{R}\,\}$, satisfies the following property: if a chord of K has length equal to ω_0 and there exist two parallel supporting lines passing through its ends, then such lines are perpendicular to the chord. We say that K is a body of constant width ω, if $\omega_K(\theta) = \omega$ for all θ. If we are looking for the minimum area among such bodies, the answer is given by the following theorem:

Theorem 2.1 (Blaschke-Lebesgue). *Among all planar convex bodies of given constant width, the Reuleaux triangle is the unique body having the least area.*

We recall that the Reuleaux triangle of width ω is the plane convex figure obtained by intersecting three circles of radius ω centered at the vertices of an equilateral triangle of side ω.

Sholander and Chakerian gave a partial answer to the more general problem of finding the body of minimum area among those having the same width function:

Theorem 2.2 (Sholander-Chakerian). *A plane convex body K having assigned width function $\omega(\theta)$ and minimum area is a triarc.*

A triarc is a convex figure having 3 basic points for the supporting lines, that is, for every two parallel supporting lines at least one of them goes through one of the basic points.

Finally, we state:

Theorem 2.3 (Pal). *Among all plane convex bodies K having minimum width ω_0, the equilateral triangle of height ω_0 is the unique body having the least area.*

3. Pal's problem with restrictions on the curvature

Consider the class F_k of all convex bodies K with minimum width 1 and area measure σ_K such that

$$\sigma_K(B) + \sigma_K(-B) \le k\,\lambda(B), \tag{3}$$

for all Borel sets $B \subset S^1$, where λ stands for the Lebesgue measure and $k \geq 1$.

Actually, consider the body $D = K + (-K)$. Since $H_D = H_K + H_{-K} \geq 1$, the circle of radius 1 centered at the origin is contained in D. So the perimeter $\mathcal{L}$ of D is larger than 2π and, from $2\pi \leq \mathcal{L} = \sigma_D(S^1) \leq k\,2\pi$, we get $k \geq 1$.

The measure σ_K has the following meaning:

For each Borel subset $B \subset S^1$, define $g(K, B)$ to be the set of points in ∂K at which there is an outward unit normal vector in B. Now, the measure on S^1

$$\sigma_K(B) = \lambda(g(K, B)$$

is called the area measure of K. Note that $\sigma_K(S^1)$ is just the length of ∂K. If $K = \mathbb{B}^2$ (unit 2-dimensional ball), then $\sigma_K = \lambda$.

The measure σ_K is particularly simple when K is a polygon, since $\sigma_K(\cdot)$ is then the sum of point masses at the outward unit normal vectors to the facets of K, the weight of each being the area of the corresponding facet.

Since σ_K is a positive measure and satisfies condition (3), it follows that σ_K is absolutely continuous with respect to λ, and this, due to the Radon-Nikodym theorem, implies the existence of a function ρ such that, for all Borel sets B, we have $\sigma_K(B) = \int_B \rho\, d\lambda$.

This means that condition (3) can be rewritten as $\rho(z) + \rho(-z) \leq k$ for almost every $z \in S^1$.

From a geometrical point of view, the function ρ is the radius of curvature of K. If K and L are compact, nonempty subsets of $\mathbb{R}^2$, the *Hausdorff distance* of K from L is defined by

$$\delta(K, L) = \min\{\lambda \geq 0 : K \subset L + \lambda\mathbb{B}^2,\ L \subset K + \lambda\mathbb{B}^2\}, \quad (4)$$

or, equivalently,

$$\delta(K, L) = \max\left\{ \sup_{x \in K} \inf_{y \in L} |\, x - y \,|,\ \sup_{x \in L} \inf_{y \in K} |\, x - y \,| \right\}. \quad (5)$$

Moreover, the Hausdorff distance between two convex bodies K and L is equivalent to $|\, h_K - h_L \,|_\infty$.

Working with such a metric, if $\mathcal{K}^n$ is the set of all convex bodies in $\mathbb{R}^n$, one can prove that

Theorem 3.1. *A sequence $(K_j)_j$, with $K_j \in \mathcal{K}^n$ for all j, converges to K with respect to the Hausdorff metric if and only if the corresponding support functions h_{K_j} uniformly converge to h_K on $S^{n-1} = \{ x \in \mathbb{R}^n : | x | = 1 \}$ and if and only if the corresponding area measures σ_{K_j} weakly converge to σ_K.*

(see [11], pag. 54 and pag. 198 and following pages)

Now we want to prove the following proposition:

Proposition 3.2. *The class F_k is closed with respect to the Hausdorff metric.*

Proof. If we suppose by contradiction that a sequence $(U_i)_i$, $U_i \in F_k$, exists such that $\lim_i U_i = U$ does not belong to F_k, we find that the thickness of U is necessarily 1.

Moreover, let our assumption be now that $\exists\, B : \sigma_U(B) + \sigma_U(-B) > k\,\lambda(B)$. The characteristic function χ_B of B, that is

$$\chi_B(z) = \begin{cases} 1 & z \in B \\ 0 & z \notin B \end{cases},$$

can be approximated by a continuous function f_ε such that $\chi_B \leq f_\varepsilon \leq \chi_V$ with $\lambda(V) \leq \lambda(B) + \varepsilon$. So we can write

$$\begin{aligned} \sigma_U(B) + \sigma_U(-B) &= \int_{S^1} (\chi_B(z) + \chi_B(-z))d\sigma_U(z) \\ &\leq \int_{S^1} (f_\varepsilon(z) + f_\varepsilon(-z))d\sigma_U(z) \\ &= \lim_{n\to\infty} \int_{S^1} (f_\varepsilon(z) + f_\varepsilon(-z))d\sigma_{U_n}(z) \\ &\leq \lim_{n\to\infty} \int_{S^1} (\chi_V(z) + \chi_V(-z))d\sigma_{U_n}(z) \\ &= \lim_{n\to\infty} (\sigma_{U_n}(V) + \sigma_{U_n}(-V)) \\ &\leq k\lambda(V) \leq k\lambda(B) + k\varepsilon. \end{aligned}$$

Because of the arbitrariness of ε we get $\sigma_U(B) + \sigma_U(-B) \leq k\lambda(B)$, which is obviuosly a contradiction. $\square$

Now, the bodies in F_k carry the following interesting geometric property:

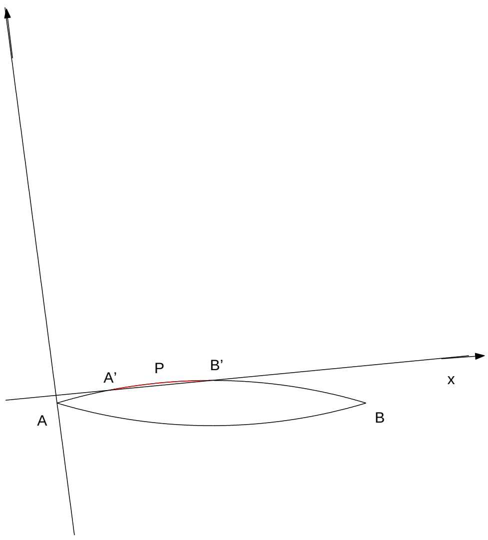

Figure 1: Diagram for proof of proposition 3.3.

PROPOSITION 3.3. *If K is a convex body such that $\sigma_K(B) \leq k\,\lambda(B)$, for all Borel sets $B \subset S^1$, then K is k- convex.*

This means that, if one takes two points A and B in K, this body contains the entire lens delimited by the two shortest arcs of radius k joining A and B.

Proof. Let A and B be two points from K.

Consider one of the 2 k-arcs (arcs of radius k) joining A and B and suppose by contradiction that a point P exists which belongs to the arc but not to K. Since K is compact, there is an open subarc $A'B'$ containing P, whose extreme points lie on ∂K and which is outside K. (see Figure 1)

Let θ_1 and θ_2 be the angles corresponding to the normals at A' and B' respectively to the k-arc $A'B'$, pointing out of K; moreover we choose a suitable coordinate system such that the x axis goes through A' and B', while the y axis is perpendicular to it (see Figure 1). Consider the integral $\int_{\theta_1}^{\theta_2} z\, d\sigma_K(z),\ z \in S^1$, whose components are $\int_{\theta_1}^{\theta_2} \cos\theta\, d\sigma_K(\theta)$ and $\int_{\theta_1}^{\theta_2} \sin\theta\, d\sigma_K(\theta)$. The arc $A'B'$ is smaller than the arc AB, so its length is surely smaller than half a circle;

moreover, the y direction stands between the directions given by θ_1 and θ_2. This means that $\cos\theta$ changes its sign from θ_1 to θ_2, while $\sin\theta$ remains positive.

Since K is convex and, from A' to B', ∂K stays below the arc $A'B'$, we have that the x-coordinate of the points C and D, from ∂K, whose outward normals correspond to θ_1 and θ_2, are not larger than those of A' and not less than those of B' respectively. So

$$\int_{\theta_1}^{\theta_2} \sin\theta\, k\, d\lambda\,(\theta) \;=\; \int_{\theta_1}^{\theta_2} \langle\theta,\, y\rangle\, k\, d\lambda\,(\theta) \;= \mid B' \,-\, A' \mid .$$

On the contrary, $\int_{\theta_1}^{\theta_2} \sin\theta\, d\sigma_K\,(\theta)$, gives the projection on the x axis of the arc CD, therefore this integral is not less than $\mid B' \,-\, A' \mid$. If we take the difference of the two integrals, we get

$$\int_{\theta_1}^{\theta_2} \sin\theta\, d\,(k\,\lambda \,-\, \sigma_K)\,(\theta) \;\le\; 0,$$

that is $\sigma_K \;=\; k\,\lambda$ in $(\theta_1,\, \theta_2)$, since $(k\,\lambda - \sigma_K)$ is positive. This means that P belongs to K, which contradicts our initial assumption. $\square$

According to condition (3), if $K \,\in\, F_k$ and the convex body L has the same width function as K, then $L \,\in\, F_k$, since $\sigma_K\,(B) \,+\, \sigma_K\,(-B) \;=\; \sigma_L\,(B) \,+\, \sigma_L\,(-B)$. So we can say that a minimizer K in F_k has also minimum area among the convex sets with the same width function. Then, the Sholander- Chakerian Theorem tells us that our extremal body is a triarc, which has the property that $\rho(z)\rho(-z) = 0$, for all $z \,\in\, S^1$. Denote with A, B and C the basic points of such a triarc which is the solution to our problem. Since the minimum width is 1, the 1-arcs γ_1, γ_2, γ_3, centered in A, B and C respectively and contained in the angles $B\hat{A}C$, $C\hat{B}A$, $B\hat{C}A$, are entirely contained in the extremal body. According to Proposition 3.3, the k-convex envelope K_A of A, B, C, γ_1, γ_2, γ_3, (that is the smallest k-convex set containing A, B, C, γ_1, γ_2, γ_3, which is also the intersection of all circles of radius k containing the 6 elements above), is contained in K. We can say even more: since $k \ge 1$, it is easy to prove that $K_A \,\in\, F_k$.

In the next step our aim is to prove that in the optimal configuration A, B and C are vertices of an equilateral triangle:

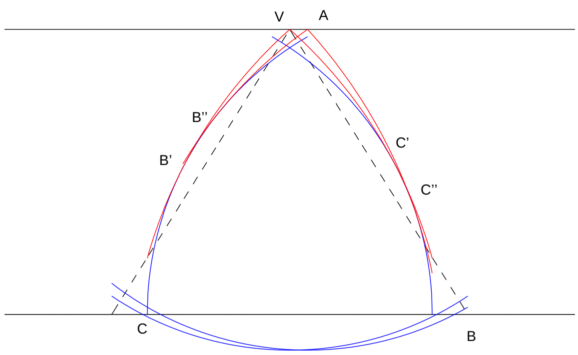

Figure 2: Diagram for proof of proposition 3.4

Proposition 3.4. *Let A,B and C be the basic points of a minimal triarc in F_k; then, A, B and C are vertices of an equilateral triangle.*

Proof. We are able to prove that the Steiner symmetral K_A^s of K_A with respect to the axis r of the segment BC still belongs to F_k. Let V be the point on the line passing through A and parallel to BC which is also on r, and let γ_1' be the 1-arc centered in V and contained in the angle $C\hat{V}B$; moreover, denote with K_V the k-convex envelope of V, B, C, γ_1', γ_2, γ_3. (see Figure 2)

A theorem by Blaschke [1, p. 124], states that the radius of curvature in a point of the Steiner symmetral of a sufficiently regular convex body is less than or equal to the maximum of the radii in the end points of the corresponding chord (which is perpendicular to the symmetrization axis) of the original body.

Since $\rho_{K_A} \leq k$ a.e., this implies that $\rho_{K_A^s} \leq k$ a.e. and , from Proposition 3.3, we can conclude that K_A^s is k-convex. The final step is to prove that $K_V \subset K_A^s$: the points B and C are obviously symmetric with respect to the axis of the segment BC, so they belong to K_A^s; $V \in K_A^s$ because it lies on the symmetrization axis. The arcs γ_2 and γ_3 are not moved by the symmetrization and , finally, $K_A \supset \gamma_1$ and $K_A^s \supset \gamma_1'$; therefore

$$K_A^s \supset \{V, B, C, \gamma_1', \gamma_2, \gamma_3\}.$$

Since $Area\,(K_A^s) = Area\,(K_A)$, the ABC triangle in the minimal figure can be supposed to be isosceles. Working in the same way on

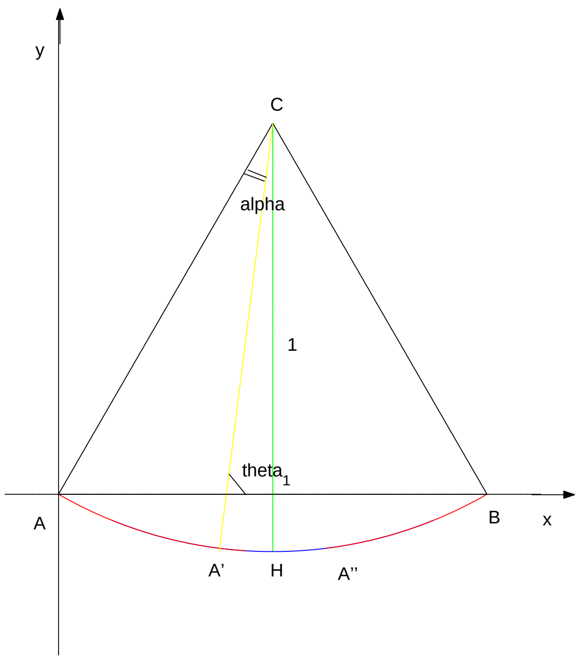

Figure 3: Diagram for proof of proposition 3.5.

B and C, the conclusion is that ABC is an equilateral triangle. □

At this point, the final statement is the following proposition. Its proof consists of showing that the 1-arcs of the k-convex envelope shrink to a single point:

PROPOSITION 3.5. *The boundary of a triarc having minimum area in F_k is formed by three circular arcs of radius k, each joining two basic points.*

Proof. Making reference to figure 3, let A, B and C be the basic points of a triarc of minimal area.

Denote with l the length of the segment AB and let $\overline{CH} = 1$; AA' and $A''B$ are k-arcs, $A'A''$ is an arc of radius 1 centered in C. We work in a coordinate system such that A is the origin, $B =$ $= (l, 0)$, $C = (\frac{l}{2}, \frac{l}{2}\sqrt{3})$. The circle $\mathcal{C}_1$ with center in C and radius

1 has the following equation:

$$x^2 + y^2 - lx - \sqrt{3}\, ly + l^2 - 1 = 0;$$

a circle $\mathcal{C}_2$ of radius k and going through A can be written as

$$x^2 + y^2 + ax - \sqrt{4k^2 - a^2}\, y = 0,$$

since its center D, as a function of the parameter a, becomes $D = (-\frac{a}{2}, \frac{\sqrt{4k^2-a^2}}{2})$.

We are looking for the one which is tangent to $\mathcal{C}_1$ in a point A' whose coordinates $(x_{A'}, y_{A'})$ satisfy $0 < x_{A'} < \frac{l}{2}$ and $y_{A'} < 0$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ have to be tangent, we have

$$\frac{2x - l}{2\sqrt{1 - (x - \frac{l}{2})^2}} = \frac{2x + a}{\sqrt{4\,k^2 - a^2 - 4\,x^2 - 4ax}}$$
$$\Leftrightarrow x = \frac{l\,k^2 + a \pm k\,(a + l)}{2\,(k^2 - 1)},$$

where $a + l \geq 0$. Our aim is to prove that $a = -l$, so that $A' \equiv H \equiv A''$. We consider just the value

$$x = \frac{l\,k^2 + a - k\,(a + l)}{2\,(k^2 - 1)} = \frac{kl - a}{2\,(k + 1)},\ a < kl,$$

since the other solution does not belong to $(0, \frac{l}{2})$; then , we are going to make A' to be a point of $\mathcal{C}_1 \cap \mathcal{C}_2$. Now, we take $\theta_1 = \arctan \frac{\sqrt{4\,(k+1)^2 - (a+l)^2}}{a+l}$ and $\alpha = \theta_1 - \frac{\pi}{3}$ (see figure 3). The area of the k-convex envelope becomes

$$\mathcal{A} = 3\left[2\,Area\,(AA'C) + \frac{\pi}{2} - \theta_1 - \frac{l^2}{4}\sqrt{3}\right] + \frac{l^2}{4}\sqrt{3},$$

where $AA'C$ is the figure delimited by the segments $A'C$ and AC and by the k-arc AA'; moreover, $\frac{\pi}{2} - \theta_1$ is the area of the sector $CA'A''$, while $\frac{l^2}{4}\sqrt{3}$ is the area of the equilateral triangle ABC. Since the arc AA' is tangent to the arc $A'A''$ in A', it has center O on $A'C$ and

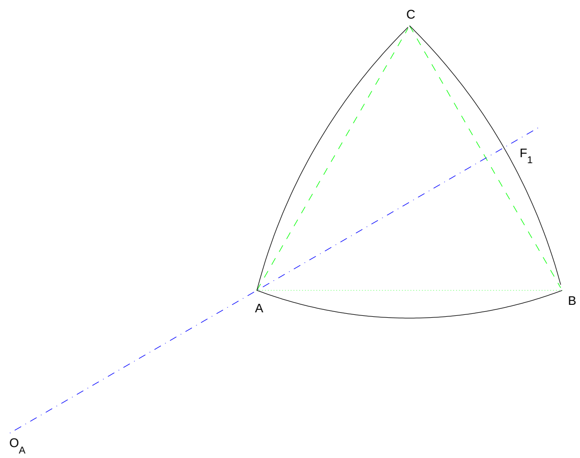

Figure 4: R_k with $\overline{O_A F_1} = k$ and $\overline{A F_1} = 1$.

we get $Area\,(AA'C) = \frac{k^2}{2}\,\beta - \frac{l}{2}\,\overline{OC}\,\sin\alpha$, where $\beta = A\hat{O}C$. As a function of k, a and l, the area $\mathcal{A}$ can be written as

$$\begin{aligned}\mathcal{A} \;=\; 6\,&\left(\frac{k^2}{2}\arccos\left(\frac{2k^2 - 2k + 1 - l^2}{2k\,(k-1)}\right)\right.\\ &\left.-\frac{l}{8}\,(k-1)\,\frac{\sqrt{4(k+1)^2-(a+l)^2}-\sqrt{3}(a+l)}{k+1}\right)\\ &+\frac{3}{2}\,\pi - 3\theta_1 - \frac{l^2}{2}\,\sqrt{3}.\end{aligned}$$

$\mathcal{A}$ attains its minimum when $a = -l$; indeed, $\arccos\beta$ does not depend on $a = a\,(l)$; $\frac{\pi}{2} - \theta_1 = \frac{\pi}{2} - \arctan\frac{\sqrt{4\,(k+1)^2-(a+l)^2}}{a+l}$ is the area of the sector $CA'A''$, which is equal to zero if $a = -l$; finally, $\frac{l}{8}\,(k-1)\,\frac{\sqrt{4\,(k+1)^2-(a+l)^2}-\sqrt{3}\,(a+l)}{k+1}$ has maximum value if $a + l = 0$. $\square$

The additional condition we get from this result is that the sides of the equilateral triangle formed by the basic points of the extremal triarc R_k measure

$$l = \frac{\sqrt{3}}{2}\,(1-k) + \frac{\sqrt{3k^2 + 2k - 1}}{2}.$$

Finally, we observe that R_k is unique up to rigid motions. Figure 4 represents R_k, with $\overline{O_A F_1} = k$ and $\overline{A F_1} = 1$.

4. A proof of Pal's Theorem

The previous result is now used to prove that the equilateral triangle T with height 1 is one of the bodies of minimum area having thickness 1. The family $(R_k)_{k>1}$ of the extremal bodies with limited radius of curvature ρ converges to T with respect to the Hausdorff metric as k tends to infinity; therefore the area A_k of R_k converges to $\frac{\sqrt{3}}{3}$, when $k \to +\infty$.[1] This follows from

$$\overline{AB} = \frac{\sqrt{3}}{2}(1 - k) + \frac{\sqrt{3k^2 + 2k - 1}}{2} \longrightarrow \frac{2\sqrt{3}}{3},$$

when $k \to +\infty$, where $\frac{2\sqrt{3}}{3}$ is the length of the side of T, and A and B are basic points in R_k. The body R_k has minimum area in the class F_k defined at the beginning of this section. Let Ω be the class of the bodies having thickness $\omega_0 = 1$. Assume that $Q \in \Omega$ is a solution to Pal's problem. Obviously we have

$$Area\,(Q) \le Area\,(T).$$

If we denote by Q_k the $\frac{k}{2}$-convex envelope of Q, it is clear that $Q_k \in F_k$ and that the family $(Q_k)_k$ converges to Q with respect to the Hausdorff metric.

Now, we have that

$$Area\,(R_k) \le Area\,(Q_k),$$

since R_k is the minimum in F_k, and

$$Area\,(T) = \lim_{k \to +\infty} Area\,(R_k) \le \lim_{k \to +\infty} Area(Q_k) = Area\,(Q).$$

Such a relation is consistent with the initial assumption if and only if $Area\,(Q) = Area\,(T)$, that is T is one of the bodies of minimum area in Ω.

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[1] $A_k = k^2 \left(\frac{\pi}{2} - 3 \arccos \left(\frac{1}{2\sqrt{3}} \frac{3k^2+2k-1-\sqrt{3}(k-1)\sqrt{3k^2+2k-1}}{k-k^2+\frac{1}{\sqrt{3}}k\sqrt{3k^2+2k-1}} \right) \right) - \sqrt{3}k + \frac{\sqrt{3}}{2},$

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Received May 28, 2007.