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# Note on Elongations of Summable $p$-Groups by $p^{\omega+n}$-Projective $p$-Groups II

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Summary. - *We find a suitable condition under which a special $\omega$-elongation of a summable $p$-group by a $p^{\omega+n}$-projective $p$-group is itself a summable $p$-group. This supplies our recent result on this theme in (Rend. Istit. Mat. Univ. Trieste, 2006).*

Throughout the rest of this brief article, suppose all groups into consideration are abelian, $p$-primary for some prime $p$, written additively. Thus $A$ is an abelian $p$-group with first Ulm subgroup $A^1 = \cap_{i<\omega} p^i A$, where $p^i A = \{p^i a \mid a \in A\}$ is the $p^i$-th power of $A$, and with $p^n$-socle $A[p^n] = \{a \in A \mid p^n a = 0\}$, where $n \in \mathbb{N}$. All other unstated explicitly notions and nomenclatures are classical and agree with [11].

In [14] (see [11] too) was defined the concept of *a summable group* that is a group $A$ so that $A[p] = \oplus_{\alpha<\lambda} A_\alpha$ with $A_\alpha \setminus \{0\} \subseteq p^\alpha A \setminus p^{\alpha+1} A$ for each $\alpha < \lambda = length(A)$. It is well-known that $\lambda \leq \Omega$, the first uncountable limit ordinal not cofinal with $\omega$. Moreover, following [16], a group $A$ is said to be *$p^{\omega+n}$-projective* if there is $P \leq A[p^n]$ with $A/P$ a direct sum of cyclics.

Besides, in [1] we treat a more general situation by studying the so-called by us *strong $\omega$-elongations of summable groups by $p^{\omega+n}$-projective groups*. Specifically, the group $A$ is such a special $\omega$-elongation if $A^1$ is summable and there exists $P \leq A[p^n]$ such that

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$A/(P + A^1)$ is a direct sum of cyclics (for the corresponding variant of totally projective groups see [3] and [4]). We showed there that under certain additional circumstances on $P$ these elongations are of necessity summable groups; in fact $P \cap p^n A \subseteq A^1$ was taken, that is, $P$ has a finite number of finite heights as calculated in $A$. In this way, the following simple technicality is a direct consequence of Dieudonné criterion from [10], but it also possesses an easy proof like this: Let $C \leq A$ such that $A/C$ is a direct sum of cyclics. If there exists a positive integer $n$ with $C \cap p^n A = 0$, then $A$ is a direct sum of cyclics. Indeed write $A = \cup_{i<\omega} A_i, C \subseteq A_i \subseteq A_{i+1} \leq A$ and $A_i \cap p^i A \subseteq C$ for all $i$: $n \leq i < \omega$. Therefore, $A_i \cap p^i A \subseteq C \cap p^i A = 0$. Thus Kulikov's criterion from [15] works to conclude the wanted property for $A$. Note that according to this claim, we may directly argue the Main Theorem in [1].

On the other hand, in [4] was introduced the class of *$n$-$\Sigma$-groups*, which is a proper subclass of the class of $\Sigma$-groups, as follows: $A$ is an $n$-$\Sigma$-group if $A[p^n] = \cup_{i<\omega} A_i, A_i \subseteq A_{i+1} \leq A[p^n], \forall\ i \geq 1 : A_i \cap p^i A \subseteq A^1$. We also proved there that every $n$-$\Sigma$-group which is a strong $\omega$-elongation of a totally projective group by a $p^{\omega+n}$-projective group is totally projective and vice versa; in particular each $n$-$\Sigma$-group is $p^{\omega+n}$-projective uniquely when it is a direct sum of countable groups of length at most $\omega + n$.

The aim of the present paper is to examine what is the relationship between the classes of $n$-$\Sigma$-groups and strong $\omega$-elongations of summable groups by $p^{\omega+n}$-projective groups, i.e. how $n$-$\Sigma$-groups are situated inside these special $\omega$-elongations of summable groups by $p^{\omega+n}$-projective groups, and whether there is an analogue with the strong $\omega$-elongations of a totally projective groups by $p^{\omega+n}$-projective groups.

Before doing that, we need some crucial preliminaries.

Following Hill, a group $A$ is known to be *pillared* provided that $A/A^1$ is a direct sum of cyclics. Clearly such a group is necessarily an $n$-$\Sigma$-group, and hence a $\Sigma$-group (see [4] too), whereas the converse implication fails. The next affirmation answers under which extra limitations this holds true. Besides, a group $A$ is said to be *a strong $\omega$-elongation (of a summable group) by a $p^{\omega+n}$-projective group* if there exists $P \leq A[p^n]$ with $A/(P + A^1)$ a direct sum of cyclics (and $A^1$ is summable). Such a group has first Ulm factor which is of necessity $p^{\omega+n}$-projective, while this property is not retained in a converse way that is there is a group with $p^{\omega+n}$-projective first Ulm factor which is not a strong $\omega$-elongation by a $p^{\omega+n}$-projective group. That is why we have also named these groups as groups with *strongly $p^{\omega+n}$-projective* first Ulm factor.

We are now endowed with enough information to proceed by proving the following main statement.

**Theorem 1.** *An $n$-$\Sigma$-group is a strong $\omega$-elongation by a $p^{\omega+n}$-projective group if and only if it is a pillared group.*

*Proof.* Write down $A[p^n] = \cup_{i<\omega} A_i, A_i \subseteq A_{i+1} \leq A[p^n]$ and $A_i \cap p^i A \subseteq A^1$ for all $i \geq 1$ along with $A/(P + A^1)$ a direct sum of cyclics for some existing $P \leq A[p^n]$. Furthermore, we observe that $A/A^1/(P+A^1)/A^1 \cong A/(P+A^1)$. Because $P \subseteq \cup_{i<\omega} A_i$, we deduce that $P = \cup_{i<\omega}(A_i \cap P)$ and thus $(P + A^1)/A^1 = \cup_{i<\omega}[(P_i + A^1)/A^1]$ by setting $P_i = A_i \cap P$. With the modular law in hand we compute that $[(P_i + A^1)/A^1] \cap p^i(A/A^1) = [(P_i \cap p^i A) + A^1]/A^1 = \{0\}$. That is why, appealing to [10], $A/A^1$ is a direct sum of cyclics. Finally, we conclude that $A$ is pillared, in fact. The opposite implication is straightforward. □

As a non-trivial consequence, we obtain the following.

**Proposition 2.** *An $n$-$\Sigma$-group is a strong $\omega$-elongation of a summable group by a $p^{\omega+n}$-projective group if and only if it is a summable pillared group.*

*Proof.* Assume that $A$ is the group in question. Since $A$ is a $\Sigma$-group and $A^1$ is summable, it follows from our criterion for summability in [5] that $A$ has to be summable as well. Moreover, we can also precise this statement by using Theorem 1 which ensures that $A$ must be even pillared.

The converse implication is self-evident since $A$ as summable assures that $A^1$ is so, and pillared groups are both $n$-$\Sigma$-groups and strong $\omega$-elongations by $p^{\omega+n}$-projective groups by taking $P = 0$. □

**Remark 3.** *As the referee indicated "summable" could be replaced by any property of groups, $\mathcal{P}(G)$, such that $\mathcal{P}(G)$ holds whenever*

*$\mathcal{P}(G^1)$ holds and $G/G^1$ is a direct sum of cyclics. For example, $\mathcal{P}(G)$ might be "G is totally projective" (see for instance [3]) or "G is fully starred".*

As an immediate consequence, we derive the following assertion.

**Corollary 4.** *Suppose $A$ is a $\Sigma$-group which is a strong $\omega$-elongation of a summable group by a $p^{\omega+n}$-projective group and the $(\omega+m)$-th Ulm-Kaplansky invariants of $A$ are zero for each $m$ so that $0 \leq m < n-1$ if $n > 1$. Then $A$ is a summable pillared group.*

*Proof.* The vanishing of the Ulm-Kaplansky invariants gives that $A[p^n] = H[p^n] \oplus A^1[p^n]$ where $H$ is a high subgroup of $A$. Since it is a direct sum of cyclics, one may write $H[p^n] = \cup_{i<\omega} H_i, H_i \subseteq H_{i+1} \leq H[p^n]$ where $H_i \cap p^i H = 0$. Furthermore, we obtain that $A[p^n] = \cup_{i<\omega} A_i$ by putting $A_i = H_i \oplus A^1[p^n]$. Knowing this, we compute with the help of modular law that $A_i \cap p^i A \subseteq A^1 + H_i \cap p^i A = A^1 + H_i \cap p^i H = A^1$ since $H$ is pure in $A$. Consequently, $A$ is an $n$-$\Sigma$-group and thus Proposition 2 works to infer the claim. $\square$

Before stating and proving our next result as well as a new proof of the previous corollary, we proceed with an assertion of independent interest (see [9] for more details).

**Proposition 5.** *A group of length not exceeding $\omega + n - 1$ is an $n$-$\Sigma$-group if and only if it is a direct sum of countable groups.*

*Proof.* The sufficiency is obvious (see [4]). As for the necessity, we observe that, for such a group $A$, $A^1 \subseteq A[p^{n-1}]$ and hence $(A/A^1)[p] = \cap_{i<\omega}(p^i A + A[p])/A^1 \subseteq A[p^n]/A^1$ since $p(\cap_{i<\omega}(p^i A + A[p])) \subseteq A^1$. Moreover, we write $A[p^n] = \cup_{i<\omega} A_i, A_i \subseteq A_{i+1} \leq A[p^n]$ and $A_i \cap p^i A \subseteq A^1$. Consequently, $(A/A^1)[p] = \cup_{i<\omega} S_i$, where $S_i = ((A_i + A^1)/A^1) \cap (A/A^1)[p]$. But with the modular law at hand we have $S_i \cap p^i(A/A^1) = S_i \cap (p^i A/A^1) = [(A_i + A^1) \cap p^i A]/A^1 = (A_i \cap p^i A + A^1)/A^1 = \{0\}$, whence $A$ is pillared. Referring to [11], because $A^1$ is bounded, we derive the desired claim. $\square$

We now intend to prove the following

**Corollary 6.** *A group is an $n$-$\Sigma$-group if and only if one (and hence each) of its $p^{\omega+n-1}$-high subgroups is a direct sum of countable groups.*

*Proof.* Let $A$ be such a group and $H$ its $p^{\omega+n-1}$-high subgroup. In [4] we showed that $A$ is an $n$-$\Sigma$-group precisely when $H$ is an $n$-$\Sigma$-group. Henceforth, we wish apply the preceding Proposition to infer the claim. □

Employing the last statement we can verify once again the validity of Corollary 4 because it is readily checked that a subgroup $H$ of $A$ is $p^{\omega}$-high (i.e. high) in $A$ if and only if $H$ is $p^{\omega+n-1}$-high in $A$ whenever the $(\omega+m)$-th Ulm-Kaplansky invariants of $A$ are zero for $0 \leq m < n-1$, that is $(p^{\omega}A)[p] = \cdots = (p^{\omega+n-1}A)[p]$.

Imitating [12], a group $A$ is said to be *a strong* $(\omega+n-1)$-*elongation of a summable group by a totally projective group* if $p^{\omega+n-1}A$ is summable and there is a nice subgroup $N \leq A$ such that $N \cap p^{\omega+n-1}A = 0$ and $A/(N \oplus p^{\omega+n-1}A)$ is totally projective.

So, we are now in a position to prove our final claim which is parallel to Proposition 2 (for the corresponding variant of totally projective groups see [8]).

**Theorem 7.** *An $n$-$\Sigma$-group is a strong $(\omega+n-1)$-elongation of a summable group by a totally projective group if and only if it is a summable pillared group.*

*Proof.* Observe that $A/(N \oplus p^{\omega+n-1}A) \cong A/p^{\omega+n-1}A/(N \oplus p^{\omega+n-1}A)/p^{\omega+n-1}A$ is totally projective. Moreover, since $N \cap p^{\omega+n-1}A = 0$, $N$ is contained in some $p^{\omega+n-1}$-high subgroup of $A$, say $H$. In accordance with Corollary 6, $H$ is totally projective of length at most $\omega+n-1$. Hence by [13] we may write that $H = \cup_{i<\omega} H_i$, where $H_i \subseteq H_{i+1} \leq H$ and all $H_i$ are height-finite in $H$, whence in $A$ because $H$ is isotype in $A$. Therefore, $(N \oplus p^{\omega+n-1}A)/p^{\omega+n-1}A = \cup_{i<\omega}[((H_i + p^{\omega+n-1}A)/p^{\omega+n-1}A) \cap ((N \oplus p^{\omega+n-1}A)/p^{\omega+n-1}A)]$. Likewise, it is not hard to verify that $(H_i+p^{\omega+n-1}A)/p^{\omega+n-1}A$ are height-finite in $A/p^{\omega+n-1}A$. Moreover, $(N \oplus p^{\omega+n-1}A)/p^{\omega+n-1}A$ is nice in $A/p^{\omega+n-1}A$ by consulting with [12] and [11]. Thus, in view of [6] or [7], we deduce that $A/p^{\omega+n-1}A$ is totally projective. But then $A/p^{\omega}A \cong A/p^{\omega+n-1}A/p^{\omega}A/p^{\omega+n-1}A = A/p^{\omega+n-1}A/p^{\omega}(A/p^{\omega+n-1}A)$ should be a direct sum of cyclics in virtue of [11]. That is why, $A$ is pillared.

On the other hand, $p^{\omega+n-1}A$ being summable implies that so is $p^{n-1}(p^{\omega}A)$ which implies by [5] that $p^{\omega}A$ is summable. Finally, by what we have just shown above, again [5] applies to conclude that $A$ has to be summable, thus it is summable pillared as asserted. $\square$

As an immediate consequence for $n = 1$ we yield the following (compare with Corollary 4).

**Corollary 8.** *A $\Sigma$-group is a strong $\omega$-elongation of a summable group by a totally projective group if and only if it is a summable pillared group.*

**Remark 9.** *It is well-known that there is a $\Sigma$-group which is not pillared; in fact it is well-known that there exists a $\Sigma$-group with unbounded torsion-complete first Ulm factor. Even more, there is a $\Sigma$-group which is not an $n$-$\Sigma$-group for any $n \geq 2$ (see [2], [3] and [4] too). The above Corollaries 4 and 8 provide us with some natural conditions under which a $\Sigma$-group is a pillared group and thereby an $n$-$\Sigma$-group. These restrictions on the Ulm-Kaplansky invariants are essential and cannot be dropped off (we note once again that in [2] and [3] it was constructed a $p^{\omega+2}$-projective $\Sigma$-group with nonzero $(\omega+1)$-th Ulm-Kaplansky invariant which is not a 2-$\Sigma$-group, whence it is not pillared).*

The expert referee suggests the author the following original approach to summarize in one single statement Theorems 1 and 7. To begin, we elementarily observe that a group A is pillared, i.e., $A/p^{\omega}A$ is a direct sum of cyclics, if and only if for some $n < \omega$ (and hence for all such $n$) $A/p^{\omega+n}A$ is a direct sum of countables. It appears that both main theorems are consequences of the following central statement, which is essentially Theorem 7 for the case of groups of length at most $\omega + n$ (for lengths less than or equal to $\omega + n - 1$ see Proposition 5).

**Theorem 10.** *Suppose $0 < n < \omega$ and $H$ is an $n$-$\Sigma$-group of length not exceeding $\omega + n$. Then $H$ is a direct sum of countables if and only if it has a nice subgroup $K$ such that $K \cap p^{\omega+n-1}H = 0$ and $H/K$ is a direct sum of countables.*

This formulation has several other advantages: First, in this form, Theorem 1 and Theorem 7 follow by considering $H = A/p^{\omega+n}A$, and either, in Theorem 1, $K = (P + p^{\omega+n}A)/p^{\omega+n}A \cong$

$P/(P \cap p^{\omega+n}A)$, or in Theorem 7, $K = (N + p^{\omega+n}A)/p^{\omega+n}A \cong N/(N \cap p^{\omega+n}A)$.

Second, it visually clarifies that what we are looking at this is a generalization from the case of groups of length $\omega$, considered by Dieudonné, to those of length $\omega + n$ considered here.

Third, this new version proposes a proof that more clearly indicates the relationship to Dieudonné's theorem from [10]. So, we come to

*Sketch of proof of Theorem 10.* Note that in virtue of [11] we have that $H$ is a direct sum of countables if and only if $H/p^\omega H$ is a direct sum of cyclics, since $p^\omega H$ is bounded by $p^n$. Given such a nice subgroup $K$, then similarly to above the hypothesis that $H$ is an $n$-$\Sigma$-group implies that Dieudonné's theorem applies to the exact sequence

$$0 \to K/(K \cap p^\omega H) \to H/p^\omega H \to H/K/p^\omega(H/K) \to 0,$$

where $K/(K \cap p^\omega H) \cong (K + p^\omega H)/p^\omega H$ and $H/K/p^\omega(H/K) = H/K/(K + p^\omega H)/K \cong H/(K + p^\omega H) \cong H/p^\omega H/(K + p^\omega H)/p^\omega H$, to show that $H/p^\omega H$ is a direct sum of cyclics, thus showing that $H$ is, indeed, a direct sum of countables, as required. □

We close with the following challenging

**Problem.** Decide whether or not a group is an $n$-$\Sigma$-group for every $1 \leq n < \omega$ if and only if it is pillared, i.e., its first Ulm factor is a direct sum of cyclics.

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