

On Hyperbolic π -Orbifolds with Arbitrary many Singular Components

ANDREI VESNIN (*)

SUMMARY. - *We construct a family of $(n + 1)$ -component links \mathcal{L}_n which are closures of rational 3-string braids $(\sigma_1^{-1/2}\sigma_2^2)^n$, and show that for $n \geq 3$ they arise as singular sets of hyperbolic π -orbifolds. Moreover, their 2-fold branched coverings are described by Dehn surgeries.*

1. Introduction

The concept of a hyperelliptic involution came originally from the theory of Riemann surfaces. Let S_g be a Riemann surface of genus g , $g > 1$. An involution $\tau \in \text{Iso}^+(S_g)$ is said to be *hyperelliptic* if the quotient space $S_g/\langle\tau\rangle$ is homeomorphic to the 2-dimensional sphere S^2 . A Riemann surface is said to be *hyperelliptic* if it admits a hyperelliptic involution, i.e. if it can be obtained as a 2-fold branched covering of S^2 . For properties of hyperelliptic Riemann surfaces see [4].

This concept can be generalized to higher dimensions in the natural way. Let M be an n -dimensional manifold. Suppose that there exists an involution $\tau : M \rightarrow M$ such that the quotient space $M/\langle\tau\rangle$ is homeomorphic to the n -dimensional sphere S^n . Then, τ is said to be a *hyperelliptic* involution and M is said to be a *hyperelliptic*

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Author's address: Andrei Vesnin, Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia; E-mail: vesnin@math.nsc.ru

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manifold. If M admits a geometric structure then we assume in the definition that τ is an isometry.

Three-dimensional hyperelliptic manifolds are objects of a special interest because of the relation with knot theory. If M is a 3-dimensional hyperelliptic manifold, with a hyperelliptic involution τ , then M is the 2-fold branched covering of S^3 branched over some link (in particular, a knot) L . The covering is given by the action of τ and each point of L has branching index 2. According to the terminology of orbifold theory (see [16, 19]), this situation means that M is the 2-fold covering of a π -orbifold $\mathcal{O} = S^3(L)$ with underling set S^3 and singular set L with singular angle π at each point of L .

It is known that in the 3-dimensional case there are eight model geometries: \mathbb{E}^3 , \mathbb{H}^3 , \mathbb{S}^3 , $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{S}^2 \times \mathbb{E}^1$, Sol , Nil , and $\widetilde{PSL}(2, \mathbb{R})$ [16, 19]. It was shown in [8] that for each of these geometries there exist hyperelliptic manifolds (with τ be an isometry).

Examples of hyperbolic 3-manifolds of small volume admitting one, two, or three hyperelliptic involutions can be found in [11]; we note that the maximal number of non-conjugate hyperelliptic involutions of a hyperbolic manifold is nine, see [12], [6].

Let M be a hyperbolic hyperelliptic 3-manifold with hyperelliptic involution τ . Then, the quotient π -orbifold $M/\langle\tau\rangle = S^3(L)$ is also hyperbolic.

A link L in S^3 is said to be *hyperbolic* if the complement $S^3 \setminus L$ is a hyperbolic manifold. We will say that L is π -*hyperbolic* if the π -orbifold $\mathcal{O} = S^3(L)$ is hyperbolic. Obviously, hyperbolicity of a link does not imply π -hyperbolicity of it (for example, hyperbolic 2-bridge links are not π -hyperbolic).

Most of known examples of π -hyperbolic links have few components. Among them are knots 8_{18} and 9_{49} , 2-component link 10_{138}^2 , knots and 3-component links arising as closed 3-string braids $(\sigma_1\sigma_2^{-1})^n$, $n \geq 4$ (here we use standard notations for knots and links according to [15] and for braids according to [1]). Discussions of the 2-fold branched coverings of these knots and links can be found in [9, 10, 11].

In the present paper, we construct explicit examples of π -hyperbolic links with an arbitrary number n of components, for any positive integer n . We will present quite simple examples of such a type.

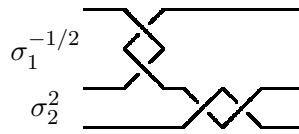


Figure 1: The rational braid $\sigma_1^{-1/2}\sigma_2^2$.

Moreover we describe the 3-manifolds that are the 2-fold branched coverings of the links under consideration.

2. π -hyperbolic links

To define a family of links we start with the notion of a rational 3-string braid.

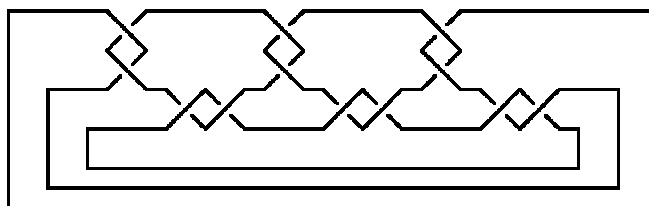
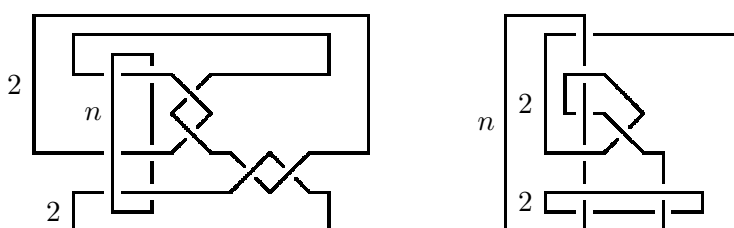
Let σ_1 and σ_2 be standard generators of the *braid group* \mathcal{B}_3 on 3 strings. Elements of \mathcal{B}_3 are of the form $\omega = \sigma_{i_1}^{p_1} \dots \sigma_{i_k}^{p_k}$, where i_1, \dots, i_k are equal to 1 or 2, and p_1, \dots, p_k are integers. To construct a geometric braid corresponding to ω , with each multiplier $\sigma_{i_j}^{p_j}$ we associate $|p_j|$ half-twists on strings i_j and $i_j + 1$ in the direction depending of sign of p_j . In other words, we are putting p_j -tangle with strings i_j and $i_j + 1$ as incoming arcs.

We generalize this construction in the following way (see also [7]). Let p_j and q_j be coprime integers. By $\sigma_{i_j}^{p_j/q_j}$ we denote the geometrical object called a *rational braid*, which is obtained by putting the rational p_j/q_j -tangle with strings i_j and $i_j + 1$ as incoming arcs. The product of two rational braids is defined similarly to the product of usual braids. Thus, an expression $\omega = \sigma_{i_1}^{p_1/q_1} \dots \sigma_{i_k}^{p_k/q_k}$, with i_1, \dots, i_k equal to 1 or 2, and p_j and q_j be coprime for each $j = 1, \dots, k$, defines a *rational braid* obtained by putting rational tangles in respect to each multiplier.

Consider a rational 3-string braid $\sigma_1^{-1/2}\sigma_2^2$ pictured in Figure 1.

Denote by \mathcal{L}_n , $n \geq 1$, the closure of the rational 3-string braid $(\sigma_1^{-1/2}\sigma_2^2)^n$ (see Figure 2, where the 4-component link \mathcal{L}_3 is pictured). Obviously, \mathcal{L}_n has $(n + 1)$ components.

THEOREM 2.1. *For any integer $n \geq 3$ the $(n + 1)$ -component link \mathcal{L}_n is π -hyperbolic.*

Figure 2: Link \mathcal{L}_3 .Figure 3: Link \mathcal{R} .

Proof. Let $\mathcal{O}_n = S^3(\mathcal{L}_n)$ be the π -orbifold with singular set \mathcal{L}_n . By the definition \mathcal{L}_n has a cyclic symmetry ρ of order n which permutes blocks $\sigma_1^{-1/2}\sigma_2^2$. The symmetry ρ induces a cyclic symmetry of order n of the orbifold \mathcal{O}_n ; we denote this symmetry also by ρ . The singular set of the quotient orbifold $\mathcal{O}'_n = \mathcal{O}_n/\langle\rho\rangle$ is the 3-component link \mathcal{R} presented in the left part of Figure 3, i.e. $\mathcal{O}'_n = S^3(\mathcal{R})$. One of its components is the image of the axis of ρ and has singularity index n . Two other components are images of \mathcal{L}_n and have singularity index 2.

Using Reidemeister moves one can redraw \mathcal{R} as in the right part of Figure 3, and then as in the left part of Figure 4.

Let \mathcal{O}''_n be the 2-fold covering of \mathcal{O}'_n , branched over one component of \mathcal{R} having singularity index 2. The singular set of \mathcal{O}''_n is the 2-component link \mathcal{Q} presented in the right part of Figure 4, i.e. $\mathcal{O}''_n = S^3(\mathcal{Q})$. One its component, say \mathcal{Q}_1 , has singularity index n , and other, say \mathcal{Q}_2 , has singularity index 2.

Now we construct a 2-fold covering of \mathcal{O}''_n branched over \mathcal{Q}_2 as

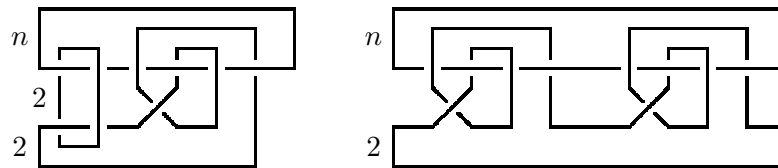


Figure 4: Links \mathcal{R} and \mathcal{Q} .

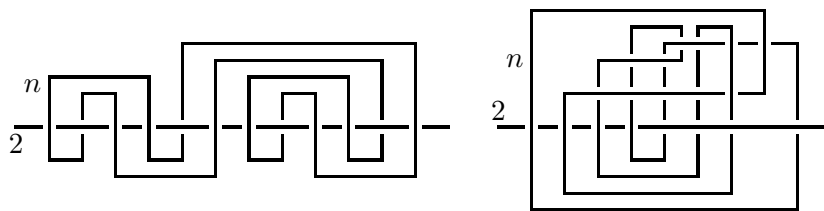


Figure 5: Link \mathcal{Q} .

follows. Using Reidemeister moves one can redraw \mathcal{Q} as in the left part of Figure 5, and then as in the right part of Figure 5.

Let us denote by \mathcal{O}_n''' the 2-fold covering of \mathcal{O}_n'' branched over \mathcal{Q}_2 . The singular set of \mathcal{O}_n''' is the 2-component link \mathcal{P} presented in Figure 6, i.e. $\mathcal{O}_n''' = S^3(\mathcal{P})$. Both its component have singularity index n .

Using Reidemeister moves \mathcal{P} can be redrawn as in the left part of Figure 7, and then as in the right part of Figure 7. Comparing Figure 7 with the standard picture for a 2-bridge link (see, for example [3, p. 195], one can conclude that \mathcal{P} is the 2-bridge link corresponding to the rational parameter $40/9 = 4 + \frac{1}{2 + \frac{1}{4}}$.

Thus \mathcal{O}_n''' is the orbifold with the singular set the 2-bridge $40/9$ -link and the singularity index n on both components. The hyperbolicity of orbifolds $\alpha/\beta(n)$ with singular set a 2-bridge knot or link α/β and singularity index n is described in [2, Example A.0.2, p. 174] and in [5]. In particular, $\alpha/\beta(n)$ is hyperbolic if $\alpha > 5$, $|\beta| > 1$, and $n \geq 3$. Therefore, the orbifold \mathcal{O}_n''' is hyperbolic if $n \geq 3$. Since by the construction \mathcal{O}_n''' is commensurable with \mathcal{O}_n , the π -orbifold \mathcal{O}_n

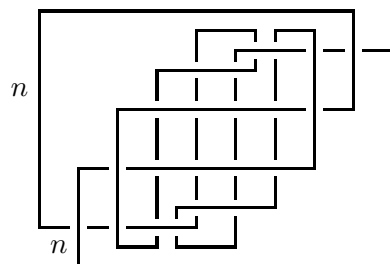


Figure 6: Link \mathcal{P} .

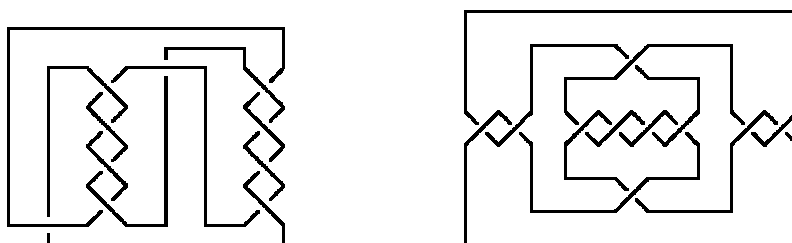


Figure 7: Link \mathcal{P} as the 2-bridge link $40/9$.

is also hyperbolic, and the link \mathcal{L}_n is π -hyperbolic for $n \geq 3$. □

Geometrical invariants of manifolds and orbifolds from the proof can be found by using a computer program *SnapPea* [17]. Thus, one can see that $vol(S^3 \setminus \mathcal{R}) = 7.70691\dots$ and $vol(S^3 \setminus \mathcal{P}) = 8.51908\dots$. Moreover, for initial values of n the following table of volumes holds:

n	$vol(S^3 \setminus \mathcal{L}_n)$	$vol \mathcal{O}_n$	$vol \mathcal{O}'_n$	$vol \mathcal{O}'''_n$
3	16.59112...	2.56897...	0.85632...	3.42529...
4	25.76187...	5.60143...	1.40036...	5.60143...
5	34.42142...	8.32706...	1.66541...	6.66165...

3. 2-fold branched coverings of links

In this section we will describe 3-manifolds M_n that are 2-fold coverings of S^3 branched over links \mathcal{L}_n .

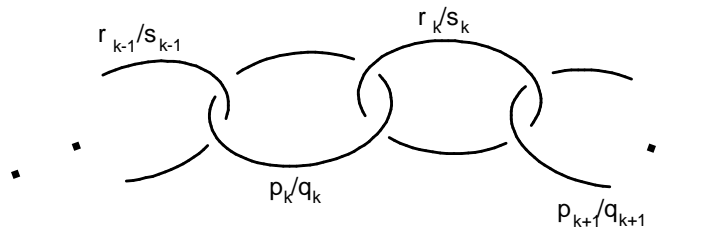


Figure 8: Surgeries along the link \mathcal{T}_n .

In [18] there was introduced a family of closed orientable 3-manifolds *Takahashi manifolds* obtained by Dehn surgery with rational coefficients p_k/q_k and r_k/s_k , $k = 1, \dots, n$, on S^3 , along the $2n$ -component link \mathcal{T}_n (see Figure 8) which is a closed chain of $2n$ unknotted components. These manifolds have been studied and generalized in [7, 13].

A Takahashi manifold is said to be *periodic* when the surgery coefficients have the same cyclic symmetry of order n as the $2n$ -component link \mathcal{T}_n , i.e. the coefficients are $p_k/q_k = p/q$ and $r_k/s_k = r/s$ alternately, for $k = 1, \dots, n$. Let us denote such Takahashi manifold by $M_n(p/q; r/s)$. By [7, 18] the manifold $M_n(p/q; r/s)$ is a 2-fold branched covering of S^3 branched over the link that is the closure of a rational 3-string braid $(\sigma_1^{p/q} \sigma_2^{r/s})^n$. By the definition, if $p/q = -1/2$ and $r/s = 2/1$ then we get the link \mathcal{L}_n from the previous section. Therefore, the following description of 2-fold branched coverings of \mathcal{L}_n holds.

PROPOSITION 3.1. *For any $n \geq 1$ the two-fold covering of S^3 branched over \mathcal{L}_n is the periodic Takahashi manifold $M_n = M_n(-1/2; 2/1)$.*

In virtue [13, 18] the fundamental group of $M_n(p/q; r/s)$ has the following presentation:

$$\langle x_1, \dots, x_n, y_1, \dots, y_n \mid y_i^{-p} = x_{i-1}^s x_i^{-s}, \\ x_i^{-r} = y_{i+1}^q y_i^{-q}, \quad i = 1, \dots, n \rangle,$$

where all indices are taken by mod n . Hence the following cyclic

presentation holds:

$$\pi_1(M_n(-1/2; 2/1)) = \langle x_1, \dots, x_n \mid w(x_i, x_{i+1}, x_{i+2}) = 1, \\ i = 1, \dots, n \rangle.$$

with the defining word $w(x_i, x_{i+1}, x_{i+2}) = x_i^2(x_i x_{i+1}^{-1})^2(x_i x_{i-1}^{-1})^2$.

4. Covering diagram

To complete the discussion of links \mathcal{L}_n and manifolds M_n let us describe a covering diagram in which they are involved.

Before formulating the main result of this section we have to talk about the types of n -fold cyclic branched coverings of links we want to consider. Obviously, a knot has an unique n -fold cyclic branched covering. Let $L = K_1 \cup K_2$ be a link in the 3-sphere with two components. Denote by $\pi_1(S^3 \setminus L)$ the fundamental group of the link complement and by m_1 and m_2 meridians of the components $K_1 \cup K_2$ of the link, oriented in an arbitrary way. The homology group $H_1(S^3 \setminus L)$ of the link complement is isomorphic to \mathbb{Z}^2 and generated by the homology classes of the meridians. Each surjection $\psi : \pi_1(S^3 \setminus L) \rightarrow H_1(S^3 \setminus L) \rightarrow \mathbb{Z}_n$ onto the cyclic groups \mathbb{Z}_n of order n defines a cyclic n -fold branched covering $M = M(\psi)$ of S^3 branched over L . According to [14] we call M a *strictly-cyclic* n -fold covering of L if the corresponding surjection ψ maps (the homotopy class of) meridians m_1 and m_2 of L to the same generator of the cyclic group \mathbb{Z}_n . Note that strictly-cyclic coverings are also called *uniform* coverings in [20].

Let us denote by M'_n the strictly-cyclic n -fold covering of S^3 branched over the 2-component 2-bridge link 40/9. Remark that M'_n is a generalized periodic Takahashi manifold in the sense of [13].

THEOREM 4.1. *For the above described manifolds and orbifolds the following diagram of coverings holds:*

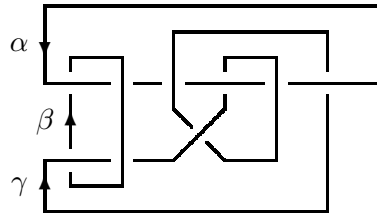
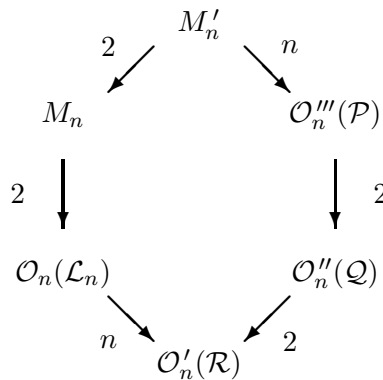


Figure 9: Generators of $\pi_1(S^3 \setminus \mathcal{R})$.



where singular sets $\mathcal{L}_n, \mathcal{R}, \mathcal{Q}$, and \mathcal{P} of orbifolds $\mathcal{O}_n, \mathcal{O}'_n, \mathcal{O}''_n$, and \mathcal{O}'''_n are presented in Figures 2, 3, 4, and 7, respectively.

Proof. By the proof of Theorem 2.1 and by Proposition 3.1 we already have the following sequences of coverings:

$$M_n \xrightarrow{2} \mathcal{O}_n \xrightarrow{n} \mathcal{O}'_n$$

and

$$M'_n \xrightarrow{n} \mathcal{O}'''_n \xrightarrow{2} \mathcal{O}''_n \xrightarrow{2} \mathcal{O}'_n.$$

Let us denote by Γ'_n the group of the orbifold \mathcal{O}'_n , i.e. $\mathcal{O}'_n = \mathbb{H}^3/\Gamma'_n$. Let α, β , and γ be generators of Γ'_n corresponding to generators of $\pi_1(S^3 \setminus \mathcal{R})$ pictured in Figure 9.

Using the Wirtinger algorithm [3] one can see that Γ'_n has the

following presentation:

$$\langle \alpha, \beta, \gamma \mid \alpha^n = 1, \beta^2 = 1, \gamma^2 = 1, \beta\alpha\gamma = \alpha\gamma\beta \\ \alpha^{-1}\gamma\beta^{-1}\alpha^{-1}\beta\gamma^{-1}\beta^{-1}\alpha\gamma^{-1}\alpha^{-1}\beta\gamma \cdot \\ \cdot \beta^{-1}\alpha\beta\gamma^{-1}\beta^{-1}\alpha\gamma\alpha^{-1}\beta\gamma\beta^{-1}\alpha\beta\gamma^{-1} = 1 \rangle.$$

Consider a group

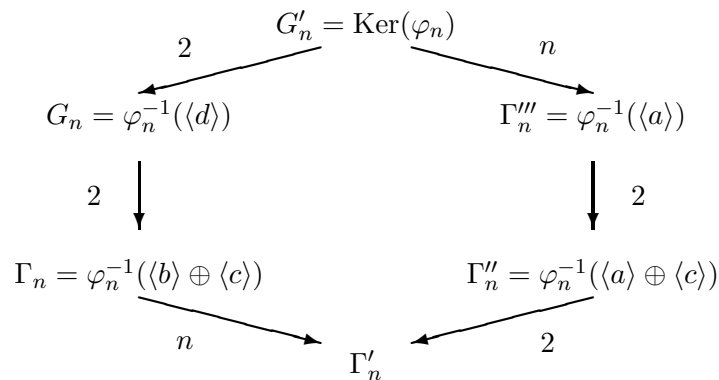
$$H_n = \mathbb{Z}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1 \rangle \oplus \langle b \mid b^2 = 1 \rangle \oplus \langle c \mid c^2 = 1 \rangle$$

and define an epimorphism $\varphi_n : \Gamma'_n \rightarrow H_n$ by setting $\varphi_n(\alpha) = a$, $\varphi_n(\beta) = b$, $\varphi_n(\gamma) = c$. Let $\Gamma_n, \Gamma''_n, \Gamma'''_n, G_n$, and G'_n be such groups that $\mathcal{O}_n = \mathbb{H}^3/\Gamma_n$, $\mathcal{O}''_n = \mathbb{H}^3/\Gamma''_n$, $\mathcal{O}'''_n = \mathbb{H}^3/\Gamma'''_n$, $M_n = \mathbb{H}^3/G_n$, and $M'_n = \mathbb{H}^3/G'_n$.

For the covering $\mathcal{O}''_n \rightarrow \mathcal{O}'_n$ a lift of the loop β is a trivial loop, lifts $\tilde{\alpha}$ and $\tilde{\gamma}$ of α and γ are loops about components of the singular set \mathcal{Q} of \mathcal{O}''_n generating subgroups \mathbb{Z}_n and \mathbb{Z}_2 , respectively. Thus, $\Gamma''_n = \varphi_n^{-1}(\langle a \mid a^n = 1 \rangle \oplus \langle c \mid c^2 = 1 \rangle)$. For the covering $\mathcal{O}'''_n \rightarrow \mathcal{O}''_n$ a lift of the loop $\tilde{\gamma}$ is a trivial loop, a lift $\tilde{\tilde{\alpha}}$ of the loop $\tilde{\alpha}$ is a loop about the singular set \mathcal{P} of \mathcal{O}'''_n generating subgroup \mathbb{Z}_n . Thus, $\Gamma'''_n = \varphi_n^{-1}(\langle a \mid a^n = 1 \rangle)$. For the covering $M'_n \rightarrow \mathcal{O}'''_n$ the preimage of the loop $\tilde{\tilde{\alpha}}$ is a trivial loop. Thus, $G'_n = \text{Ker}(\varphi_n)$.

For the covering $\mathcal{O}_n \rightarrow \mathcal{O}'_n$ a lift of the loop α is a trivial loop, lifts $\hat{\beta}$ and $\hat{\gamma}$ of loops β and γ are loops about components of the singular set \mathcal{L}_n of \mathcal{O}_n generating subgroups \mathbb{Z}_2 and \mathbb{Z}_2 . Thus, $\Gamma_n = \varphi_n^{-1}(\langle b \mid b^2 = 1 \rangle \oplus \langle c \mid c^2 = 1 \rangle)$. For the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle b \mid b^2 = 1 \rangle \oplus \langle c \mid c^2 = 1 \rangle$ we denote $d = b + c$ and consider a group $\mathbb{Z}_2 = \langle d \mid d^2 = 1 \rangle$. For the covering $M_n \rightarrow \mathcal{O}_n$ loops $\hat{\beta}$ and $\hat{\gamma}$ lift to trivial loops. Thus, $G_n = \varphi_n^{-1}(\langle d \mid d^2 = 1 \rangle)$.

Therefore we get the following diagram of subgroups (where $A \xrightarrow{m} B$ denotes that A is a subgroup of B of index m)



that implies the diagram of coverings. □

REFERENCES

- [1] J.S. BIRMAN, *Braids, links and mapping class group*, Princeton University Press, Tokyo (1974).
- [2] M. BOILEAU AND J. PORTI, *Geometrization of 3-orbifolds of cyclic type*, *Asterisque* **272** (2001).
- [3] G. BURDE AND H. ZIESCHANG, *Knots*, de Gruyter Studies in Mathematics, **5**, Berlin-New York (1985).
- [4] H. FARKAS AND I. KRA, *Riemann surfaces*, Graduate Texts in Math. **71**, Springer-Verlag (1980).
- [5] H.M. HILDEN, M.T. LOZANO AND J.M. MONTESINOS-AMILIBIA, *On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant*, *J. of Knot Theory and its Ramifications*, **4**, no. 1 (1995), 81–114.
- [6] A. KAWAUCHI, *Topological imitations and Reni-Mecchia-Zimmermann's conjecture*, *Kyungpook Math. J.* **46** (2006), 1–9.
- [7] A.C. KIM AND A. VESNIN, *The fractional Fibonacci groups and manifolds*, *Siberian Math. J.* **39**, no. 4 (1998), 655–664.
- [8] A. MEDNYKH, *Three-dimensional hyperelliptic manifolds*, *Ann. of Glob. Anal. and Geom.* **8** (1990), 13–19.
- [9] A. MEDNYKH AND A. VESNIN, *Fibonacci manifolds as 2-fold coverings over the 3-dimensional sphere and the Meyerhoff-Neumann conjecture*, *Siberian Math. J.* **37**, no. 3 (1996), 461–467.
- [10] A. MEDNYKH AND A. VESNIN, *Covering properties of small volume hyperbolic 3-manifolds*, *J. of Knot Theory and Its Ramifications* **7**, no. 3 (1998), 381–392.

- [11] A. MEDNYKH AND A. VESNIN, *Three-dimensional hyperbolic manifolds of small volume with three hyperelliptic involutions*, Siberian Math. J. **40** (1999), 873–886.
- [12] M. MECCHIA AND B. ZIMMERMANN, *The number of knots and links with the same 2-fold branched covering*, Quart. J. Math. **55** (2004), 69–76.
- [13] M. MULAZZANI AND A. VESNIN, *Generalized Takahashi manifolds*, Osaka Math. J. **39**, no. 2 (2002), 705–721.
- [14] M. MULAZZANI – A. VESNIN, *The many faces of cyclic branched coverings of 2-bridge knots and links*, Atti Sem. Mat. Fis. Univ. Modena, Supplemento al Vol. **40** (2001), 177–215.
- [15] D. ROLFSEN, *Knots and Links*, Publish or Perish Inc., Berkeley, U.S.A. (1976).
- [16] P. SCOTT, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1986), 401–487.
- [17] *SnapPea*, a computer program for creating and studying hyperbolic 3-manifolds, available from <http://geometrygames.org/SnapPea/>.
- [18] M. TAKAHASHI, *On the presentations of the fundamental groups of 3-manifolds*, Tsukuba J. Math. **13** (1989), 175–189.
- [19] W. THURSTON, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton University (1980).
- [20] B.P. ZIMMERMANN, *Genus action of finite groups on 3-manifolds*, Michigan Math. J. **43** (1996), 593–610.

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