Rend. Istit. Mat. Univ. Trieste
Vol. XXXIX, 337–346 (2007)

On the Determination of Knots by their Cyclic Unbranched Coverings

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Dedicated to the memory of Fabio Rossi

Summary. - *We show that, for any prime p, a knot K in S^3 is determined by its p-fold cyclic unbranched covering. We also investigate when the m-fold cyclic unbranched covering of a knot in S^3 coincides with the n-fold cyclic unbranched covering of another knot, for different coprime integers m and n.*

1. Introduction

There is an extensive literature on the determination of knots in S^3 by their p-fold cyclic *branched* coverings; for example, the case of odd prime numbers p is considered in [20] for hyperbolic knots and in [2] for arbitrary prime knots, the case of 2-fold branched coverings is considered in [17], [13] and [11] (see also the survey [21]). On the other hand, less seems to be known for the case of cyclic *unbranched* coverings (that is, of the complements of the knots). For a knot K in S^3, we denote by $M_p(K)$ the p-fold cyclic unbranched covering of its complement $M_1(K) = S^3 - N(K)$ (S^3 minus the interior of a regular neighbourhood of the knot), so $M_p(K)$ is a compact orientable 3-manifold with a torus boundary. The basic case here is, of course, the case $p = 1$ or the fact that a knot K in S^3 is determined by its complement ([9]). In the present paper, we study the case of primes $p > 1$ and prove the following

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Theorem 1.1. *For any prime p, a knot K in S^3 is determined by its p-fold cyclic unbranched covering $M_p(K)$ (i.e., any other knot with the same p-fold cyclic unbranched covering is equivalent to K).*

Here two unoriented knots are equivalent if there is a diffeomorphism of S^3 which maps one to the other. Concerning different branching orders, we introduce the following:

Abelian construction. Let m and n be distinct positive integers. Let $\bar{K}$ be a knot in a lens space L which represents a generator of $\pi_1(L) \cong \mathbb{Z}_n$, and denote by K the knot which is the preimage of $\bar{K}$ in the universal covering S^3 of L. Similarly, let K' be the preimage of a knot $\bar{K}'$ in a lens space L', representing a generator of $\pi_1(L') \cong \mathbb{Z}_m$, in the universal covering S^3 of L'. Suppose that $\bar{K}$ and $\bar{K}'$ have homeomorphic complements $L - N(\bar{K}) = L' - N(\bar{K}')$; then the m-fold cyclic unbranched covering of K coincides with the n-fold cyclic unbranched covering of K', i.e. $M_m(K) = M_n(K')$ (both are equal to a regular unbranched $(\mathbb{Z}_m \times \mathbb{Z}_n)$-covering of $L - N(\bar{K}) = L' - N(\bar{K}')$).

Concentrating mainly on the basic case of hyperbolic knots, the following holds.

Theorem 1.2. *(i) Let K be a hyperbolic knot and K' be any knot in S^3 such that $M_m(K) = M_n(K')$, for coprime positive integers m and n. Then K and K' are obtained by the Abelian Construction. The same remains true for arbitray knots K and K' if m and n are different prime numbers.*

(ii) Let M be a compact orientable 3-manifold whose boundary is a torus and whose interior has a complete hyperbolic structure of finite volume. There are at most three coprime positive integers m such that M is the m-fold cyclic unbranched covering of a knot K in S^3.

We think that part (i) of Theorem 1.2 remains true for arbitrary knots K and K' and coprime integers m and n. We note that the present formulation of part (i) of Theorem 1.2 uses the recent geometrization of free cyclic group actions on S^3 after Perelman ([14], [15]); without this, one concludes that, in the Abelian Construction,

L and L' are 3-manifolds with finite cyclic fundamental groups whose universal covering is S^3.

Let K_0 be a knot in S^3 which admits two non-trivial lens space surgeries L and L', with $\pi_1(L) \cong \mathbb{Z}_n$ and $\pi_1(L') \cong \mathbb{Z}_m$. The cores of the surgered solid tori give two knots $\bar{K}$ and $\bar{K}'$ in the lens spaces L and L' whose complements coincide with the complement of K_0 in S^3. By the Abelian Construction one obtains two knots K and K' in S^3 such that $M_m(K) = M_n(K') = M_{mn}(K_0)$. Note also that $M_1(K) = M_n(K_0)$ and $M_1(K') = M_m(K_0)$. Examples of hyperbolic knots in S^3 with two non-trivial lens space surgeries can be found in [1], [6].

The case of different branching orders for cyclic *branched* coverings has been considered in [18], [21] for hyperbolic knots, see also [3]; one main difference is that the isometry group of such a cyclic branched covering (a closed hyperbolic 3-manifold) may, in principle, be much more complicated than that of a cyclic unbranched covering (a 3-manifold with torus-boundary). In particular, the situation for cyclic branched coverings is not well understood in the case of non-solvable isometry groups; for example, it is not clear which finite non-abelian simple groups may occur as groups of isometries of a cyclic branched covering.

We thank S. Friedl, L. Paoluzzi and M. Scharlemann for helpful correspondence, and A. Reid for bringing to our attention the references [7] and [16] on the related topic of commensurability classes of hyperbolic knot complements.

2. Proof of Theorem 1.1

We start with the following

LEMMA 2.1. *For a prime p, let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ be a finite group of orientation-preserving diffeomorphisms of a mod p homology 3-sphere M (i.e., for homology with coefficients in the integers mod p). Then either there are exactly two subgroups $\mathbb{Z}_p$ of G with nonempty fixed point set (two disjoint circles) or, if $p = 2$, all three involutions in G may have nonempty fixed point set (three circles intersecting in exactly two points).*

Proof. By Smith fixed point theory, G does not act freely, and each element of G has empty or connected fixed point set (see [4], Theorems 7.9 and 8.1). Let X be a nontrivial cyclic subgroup of G with nonempty fixed point set K which is a circle. Because G is abelian, K is invariant under the action of G. The projection $\bar{G}$ of G to $\bar{M} := M/X$ is a cyclic group leaving invariant the projection $\bar{K}$ of K. It is easy to see that also $\bar{M}$ is a mod p homology 3-sphere, and hence $\bar{G}$ has empty or connected fixed point set.

Suppose that there is another nontrivial cyclic subgroup $X' \neq X$ of G with nonempty fixed point set K' different from K. If K and K' intersect then they intersect in exactly two points and $p = 2$ (because K is invariant under X'), and consequently we are in the second case of the Lemma. Hence we can assume that K and K' do not intersect. Note that K' is invariant under X which acts as a group of rotations on K'. As the fixed point set of $\bar{G}$ is connected (a circle) it consists of the projection of K'. The preimage of this projection is exactly K' which implies that X and X' are the only nontrivial cyclic subgroups of G with non-empty fixed point set. Thus we are in the first case of the Lemma.

Now suppose that X is the only cyclic subgroup of G with nonempty fixed point set. Then $\bar{G}$ acts freely on the mod p homology 3-sphere $\bar{M}$. Let $N := \bar{M}/\bar{G}$ be the quotient and L the projection of $\bar{K}$ to N. Note that $M - K$ is a regular unbranched covering of $N - L$, with covering group $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By [10, page 92], $H_1(N - L; \mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p$; this implies that $N - L$ has no abelian covering with covering group $\mathbb{Z}_p \times \mathbb{Z}_p$ which is a contradiction.

This completes the proof of Lemma 2.1. □

Starting with the *Proof of Theorem 1.1* now, the p-fold cyclic unbranched covering $M = M_p(K)$ of the knot K is a compact 3-manifold whose boundary is a torus; we denote by $C \cong \mathbb{Z}_p$ the cyclic covering group acting freely on M. Suppose that M is also the p-fold cyclic unbranched covering of another knot K', i.e. $M = M_p(K')$, with covering group $C' \cong \mathbb{Z}_p$. We denote by $B_p(K)$ the p-fold cyclic *branched* covering of K, so $M = B_p(K) - N(\tilde{K})$ where $N(\tilde{K})$ denotes the interior of a regular neighbourhood of the preimage $\tilde{K}$ of K in $B_p(K)$. The action of the covering group C on M extends to an action of C on $B_p(K)$ with fixed point set $\tilde{K}$, giving the covering group of the p-fold cyclic branched covering of K which will be denoted also by C. Similarly, C' extends to the p-fold cyclic branched covering $B_p(K')$ of K' fixing the preimage $\tilde{K}'$ of K'.

We will show that, up to conjugation, the covering groups C and C' commute; Theorem 1.1 then follows from the following:

Lemma 2.2. *Suppose that the covering groups C and C' commute, generating a group $G = C \oplus C' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ of diffeomorphisms of M. Then the knots K and K' are equivalent.*

Proof. The action of C' on M extends to a free action of C' on the p-fold cyclic branched covering $B_p(K)$ of K (unless $C = C'$: in this case, K and K' have homeomorphic complements and hence are equivalent). We can assume that the actions of C and C' commute also on $B_p(K)$ and hence generate a group $G = C \oplus C' \cong \mathbb{Z}_p \times \mathbb{Z}_p$ of diffeomorphisms of $B_p(K)$.

The p-fold cyclic branched covering $B_p(K)$ of a knot K in S^3 is a mod p homology 3-sphere (see e.g. [8]). By Lemma 2.1, there are exactly two subgroups $\mathbb{Z}_p$ of G with non-empty (connected) fixed point set; one of these is C which fixes $\tilde{K}$, and we denote by $\tilde{A}$ the fixed point set of the other which is contained in M. Now G projects to a group $H \cong G/C \cong \mathbb{Z}_p$ of symmetries of $(B_p(K), \tilde{K})/C = (S^3, K)$, with non-empty fixed point set A disjoint from K (the projection of $\tilde{A}$). Hence K has cyclic period p.

By the positive solution of the Smith conjecture, H acts by standard rotations on the 3-sphere, so S^3/H is again the 3-sphere. The group H acts by rotations along K and maps a meridian of K to p disjoint meridians of K. Its fixed point set is the projection $\bar{A}$ of $\tilde{A}$ resp. A, so (S^3, K) is the cyclic branched covering of $(S^3, \bar{K})$ branched along the trivial knot $\bar{A}$ in S^3.

Now $(S^3 - N(K))/H = S^3 - N(\bar{K}) = M/G$, and by symmetry and with analogous notation, $(S^3 - N(K'))/H' = S^3 - N(\bar{K}') = M/G$, so $S^3 - N(\bar{K}) = S^3 - N(\bar{K}')$ (where $H' \cong G/C' \cong \mathbb{Z}_p$). Hence $(S^3, \bar{K}')$ is obtained from $(S^3, \bar{K})$ by $1/n$-surgery on $\bar{K}$, for some integer n, and this surgery transforms also $\bar{A}$ into $\bar{A}'$ (note that the projections $\bar{A}$ and $\bar{A}'$ of $\tilde{A} \subset M$ coincide as subsets of $M/G = S^3 - N(\bar{K}) = S^3 - N(\bar{K}'))$.

If the surgery on $\bar{K}$ is trivial (i.e., $n = 0$) then $\bar{K}' = \bar{K}$ and $\bar{A}' = \bar{A}$, so $K' = K$ and Lemma 2.2 is proved in this case.

In the case of non-trivial surgery, since the result of the surgery is again the 3-sphere, $\bar{K}$ has to be a trivial knot by [9], and hence $\bar{K} \cup \bar{A}$ is a hyperbolic link of two unknotted components. By [12, Corollary 3], a non-trivial $1/n$-surgery on one component $\bar{K}$ of a link of two unknotted components $\bar{K} \cup \bar{A}$ transforms the other component $\bar{A}$ into a non-trivial knot (obtained from $\bar{A}$ by twisting n times around a spanning disk for $\bar{K}$), and hence $\bar{A}'$ is non-trivial. This contradicts the Smith conjecture since the p-fold cyclic covering of S^3 branched along $\bar{A}'$ is the 3-sphere.

This finishes the proof of Lemma 2.2. □

The proof of Theorem 1.1 follows now from the following:

LEMMA 2.3. *Up to conjugation, the covering groups C and C' commute.*

Proof. Since the proof is much easier if K is a hyperbolic knot we will give the proof first for this case.

(i) Suppose that K is hyperbolic. Then M is a hyperbolic 3-manifold of finite volume, with one torus-boundary or cusp. It is a consequence of Mostow rigidity and Waldhausen's theorem for Haken 3-manifolds [19] that also K' is hyperbolic, and hence we can assume that both cyclic covering groups C of K and C' of K' act by isometries on M. The covering groups C and C' act freely on M, and hence by euclidean rotations on the boundary torus of the hyperbolic 3-manifold M (corresponding to a cusp of the hyperbolic 3-manifold M; the rotations lift to translations of the euclidean horospheres corresponding to the boundary torus of M in the universal covering of M). It follows that the groups C and C' of isometries of M commute elementwise (because they commute on the boundary torus of M).

This finishes the proof of Lemma 2.3 and Theorem 1.1 in the case where K is hyperbolic.

(ii) Now let K be an arbitrary knot. We will apply the methods in [2] to show that C and C' commute, up to conjugation. We consider the JSJ- or torus-decomposition of M and the graph Γ dual to this decomposition; note that Γ is a tree since $B_p(K)$ is a mod p homology sphere. By the equivariant torus-decomposition, we can assume that both C and C' respect the decomposition and are geometric on the pieces of the decomposition (i.e., isometries on the hyperbolic pieces, fiber-preserving on the Seifert fibered pieces; these pieces correspond to the vertices of the graph Γ, the decomposing tori to the edges). Then C and C' induce a finite group G of automorphisms of the tree Γ which fixes the vertex corresponding to the piece containing the boundary torus of M. Let Γ_f denote the subtree of Γ whose vertices and edges are fixed by every element of G. Now one shows as in [2] that, up to conjugation by diffeomorphisms of M, one can assume that C and C' commute elementwise on the submanifold M_f of M corresponding to the subtree Γ_f of Γ (in [2], only the case of odd primes p is considered; however, in our situation the methods work equally well for $p = 2$).

Denote by M_c the maximal connected submanifold of M corresponding to a subtree Γ_c of Γ containing Γ_f on which C and C' commute, up to conjugation by diffeomorphisms of M. If $M_c = M$ we are done, so suppose that $M_c \neq M$. Consider a boundary torus T of M_c connecting M_c with a piece U of the decomposition of $M - M_c$. By the proof of [2, Claim 9], we can assume that the orbit of T under both C and C' consists of the same p tori (in all other situations, commutativity of C and C' can be extended to the G-orbit of U, contradicting maximality of M_c). The torus T projects to a torus $\bar{T}$ of the torus-decomposition of the complement of K which seperates S^3 into a solid torus (containing K) and a knot space (the complement of a knot, containing the projection of U); in particular, there is a well-defined meridian-longitude system on $\bar{T}$, and also on each torus of the G-orbit of T which is invariant under the actions of C and C'. Now one replaces the knot complement by a solid torus such that one obtains again the 3-sphere, and similarly performs C- and C'-equivariant surgery on T and its images under C and C'. Moreover, if the piece of M_c containing T is hyperbolic, one does the surgery such that the resulting 3-manifold is still hyperbolic and the central curve of the added solid torus is a shortest geodesic; if it is Seifert fibered, one creates an exceptional fiber of high order by the surgery. Doing this for each such boundary torus T of M_c, one obtains a 3-manifold $\hat{M}_c$ with induced actions of C and C' which are still the covering groups of two p-fold cyclic unbranched coverings of two knots in S^3; moreover, by construction C and C' commute on $\hat{M}_c$. By Lemma 2.2, the two corresponding knots are equivalent, hence the actions of C and C' on $\hat{M}_c$ are conjugate; by the choice of the surgeries, the conjugating diffeomorphism restricts to M_c and then extends to M, hence we can assume that the actions of C and C' coincide on M_c. Now by [2, Lemma 10] the actions of C and C' coincide on M, up to conjugation, hence K and K' have homeomorphic complements and are equivalent.

This finishes the proof of Lemma 2.3 and of Theorem 1.1 in the general case. □

3. Proof of Theorem 1.2

The proof is along similar lines. For the proof of part (i) of Theorem 1.2, let $M = M_m(K) = M_n(K')$ and denote by $C \cong \mathbb{Z}_m$ and $C' \cong \mathbb{Z}_n$ the two covering groups. As in case (i) of the proof of Lemma 2.3, we can assume that the covering groups act by hyperbolic isometries, commute and generate a group $G = C \oplus C' \cong \mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ of isometries of M. Both groups C and C' extend to the m-fold cyclic branched covering $B_m(K)$ of K, and C fixes pointwise the preimage $\tilde{K}$ of K. The group G act freely on M, and the only non-trivial subgroup of G with non-empty fixed point set in $B_m(K)$ is C. Then G projects to a cyclic group $H \cong G/C \cong \mathbb{Z}_n$ acting freely on $(B_m(K), \tilde{K})/C = (S^3, K)$, so H is a group of free symmetries of K of order n.

Similarly, G projects to a group $H' \cong G/C' \cong \mathbb{Z}_m$ acting freely on $(B_n(K'), \tilde{K}')/C = (S^3, K')$. By the geometrization of free cyclic group actions on S^3, the quotients S^3/H and S^3/H' are lens spaces. The quotients $(S^3, K)/H = (L, \bar{K})$ and $(S^3, K')/H' = (L', \bar{K}')$ define knots $\bar{K}$ and $\bar{K}'$ in lens spaces L and L' such that $L - N(\bar{K}) = L' - N(\bar{K}') = M/G$, hence K and K' are obtained by the Abelian Construction.

This finishes the proof of Theorem 1.2(i) in the case where K is hyperbolic. Now let K be an arbitrary knot and assume that m and n are different primes. We want to show that again the covering groups of the two knots commute, up to conjugation; then the proof finishes as in the hyperbolic case. Now, in the situation of different primes m and n, the methods in [3] apply and as in the proof of case (ii) of Theorem 1.1 one obtains commutativity on the submanifold

M_f corresponding to the subtree Γ_f of Γ fixed by the two covering groups, and then one pushes out commutativity to all of M (this is, in fact, easier than in the situation of two equal primes considered in [2] (as applied in the proof of Theorem 1.1) where in principle some obstruction against commutativity may arise). We remark that these methods probably can be generalized to prove Theorem 1.2(i) for the case of arbitrary knots and arbitrary coprime integers.

For the proof of part (ii) of the theorem, suppose that M is the n_i-fold cyclic unbranched covering of knots K_i in S^3, for pairwise coprime positive integers $n_1, \ldots, n_\alpha$. Denoting by $n = n_1 \ldots n_\alpha$ their product, the manifold M has now a free action of $G \cong \mathbb{Z}_n$. This G-action on M induces a free action of $G_i \cong \mathbb{Z}_{n/n_i}$ on $S^3 - N(K_i)$ which extends to a free action on S^3. The quotient S^3/G_i is a lens space L_i, with fundamental group $\pi_1(L_i) \cong G_i$, which contains the projection $\bar{K}_i$ of K_i. Now $L_i - N(\bar{K}_i) = M/G$, so all lens spaces L_i are obtained by surgery (Dehn filling) on M/G. By [5], there are at most three surgeries on a compact hyperbolic 3-manifold of finite volume and with a single torus-boundary resulting in lens spaces, hence $\alpha \leq 3$.

This finishes the proof of Theorem 1.2.

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Received December 24, 2007.