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Some Remarks on Homogeneous Minimal Reductions

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To the memory of prof. Fabio Rossi

Summary. - *Let I be a homogeneous ideal of a graded affine k–algebra R such that there exists some homogeneous minimal reduction. We prove that the degrees (of a basis) of every homogeneous minimal reduction J of I are uniquely determined by I; moreover if the fiber cone $F(I)$ is reduced, then the last degree of J is equal to the last degree of I. Moreover, if R is Cohen–Macaulay and I is of analytic deviation one, with $0 < ht(I) := g$, it is shown that the first g degrees of J are equals to the first g degrees of I.*

These results are applied to the ideals I of $k[x_0, \ldots, x_{d-1}]$, which have scheme–th. generations of length $\leq ht(I) + 2$.

Some examples are given.

1. Introduction

In [17] the author has proved the following:

Theorem 1.1. *Let I be a homogeneous quasi–complete intersection ideal of a polynomial ring $R = k[x_0, \ldots, x_{d-1}]$ (k infinite field), with*

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$\mathrm{ht}(I) = g < d-1$. *Then the degrees (of the bases) of all the scheme–theoretic generations J of I of the minimal length (i.e. with $\mu(J) = g+1$) are uniquely determined.*

The main goal of this paper is to generalize this result, in two directions.

First of all, we observe that, in the previous theorem, the subideal $J(\subseteq I)$ is a homogeneous minimal reduction of I. Therefore we want to study the degrees (of the bases) of the homogeneous minimal reductions J (if there exist) of a homogeneous ideal I. On the other hand, we want also to work in a general positively graded affine k–algebra (not only in a polynomial ring) where k is always an infinite field.

The main result (Theorem 2.4) gives the uniqueness of the degrees of the (bases of the) homogeneous minimal reductions in a general k-algebra. Moreover, inspired by the results of Aberbach and Huneke in [1] on the special reductions and by the formula of Johnson [11, Thm.5], we can improve the cited theorem in [17], for equidimensional, generic complete intersection ideal I of analytic deviation one in a graded Cohen–Macaulay k–algebra; but, for a complete individuation of the degrees of the minimal homogeneous reductions of I we need the reducedness of the fiber cone $F(I)$.

In 3 we apply these results to the ideals I of $k[x_0, \dots, x_{d-1}]$ which are quasi–complete intersections or which have some scheme–th. generation of length $\mathrm{ht}(I) + 2$; at last, several examples and counterexamples are given.

Throughout this paper, unless stated otherwise, we denote by R a positively graded d–dimensional affine k–algebra where k is an infinite field; all ideals will be assumed to be homogeneous, and $\mathfrak{m}$ denotes the maximal homogeneous ideal of R. We define $F(I) = \bigoplus_{s=0}^{\infty} I^s/\mathfrak{m}I^s$ (with $I^0 = R$) to be the *fiber cone of* I; we denote with f^o the element f modulo $\mathfrak{m}I$ of $[F(I)]_1$, where $f \in I$. The fiber cone $F(I)$ with respect to the homogeneous ideal I has a natural bigrading on it, and the graded piece of degree (r,s) in this bigrading is $[I^r]_s/[\mathfrak{m}I^r]_s$.

We can consider the local ring $(A = R_{\mathfrak{m}}, \mathfrak{n} = \mathfrak{m}_{\mathfrak{m}})$ and the ideal $\mathfrak{a} := I_{\mathfrak{m}}$; there exist a canonical isomorphism between $I^s/\mathfrak{m}I^s$ and $\mathfrak{a}^s/\mathfrak{n}\mathfrak{a}^s$ for every s and such that to f^o correspond $(f/1)^o$ where

$f \in I$; moreover, we have also a graded isomorphism between $F(I)$ and the classical fiber cone ring $F(\mathfrak{a})$. In [15] the reader can find the definitions of reduction, minimal reduction, analytic spread and its properties; therefore, the Krull-dimension $\dim(F(I))$ is the analytic spread $l(I)$ of I (or of $\mathfrak{a}$). We denote by $\mathrm{ad}(I)$ the *analytic deviation* of I (i.e. $\mathrm{ad}(I) := l(I) - \mathrm{ht}(I)$).

We also need to observe that a homogeneous ideal I may have no homogeneous minimal reductions (i.e. its minimal reductions may be all non–homogeneous). The analytic spread is a *local* concept, but the homogeneous minimal reductions - when they exist - possess many good properties. On the other hand, a subideal $J(\subseteq I)$ with $J = (f_1, \ldots, f_s)$ is a reduction of I iff $\dim F(I)/(f_1^o, \ldots, f_s^o) = 0$.

We will write $\mu(I)$ for the minimal number of generators of the ideal I, $\sigma(I)$ for the minimal number of the scheme–theoretic generations of I, $p\sigma(I)$ for the minimal number of the punctured generations of $\mathfrak{a} := I_{\mathfrak{m}}$ (see [17]). An *unmixed* ideal is an ideal without embedded components and whose minimal primes all have the same height. We say that I has some property *generically* if it has that property locally at each $\mathfrak{p} \in \mathrm{Min}(I)$. We say that a homogeneous ideal I of R has some property *locally* if it has that property locally at each $\mathfrak{p} \in \mathrm{Proj}(R)$. For a *basis* of I we mean a minimal system of generators of I. We recall that $\mathrm{grade}(I)$ is the length of a maximal R-(regular) sequence in I.

Let $I \subset R$ be a homogeneous ideal with $g := \mathrm{ht}(I)$; we say that I is a *quasi complete intersection* (q.c.i. for short) if I is unmixed, generic complete intersection and $\sigma(I) = g + 1$.

Our general reference for the paper is [14].

Remark 1.2. *In this paper, several propositions (for ideal-reductions) can be generalized for reduction of modules, following the existing (last and not) literature.*

2. On the degrees of homogeneous minimal reductions

Throughout this section, let I be a homogeneous ideal of the d–dimensional graded affine k–algebra R (where k is an infinite field) with analytic spread $l := l(I)$, $\mu := \mu(I)$, $\sigma := \sigma(I)$ and we set $d_1 \leqslant \ldots \leqslant d_\mu$ for the sequence of the degrees (of a basis) of I.

LEMMA 2.1. *If* $J = (f_1, \ldots, f_l)$ *is a homogeneous minimal reduction of* I *where* $\deg(f_i) = \delta_i$ *with* $\delta_1 \leqslant \ldots \leqslant \delta_l$*, then:*

(i) if R *or* $F(I)$ *is reduced, then* $\delta_1 = d_1$ *;*

(ii) if $J' = (f_1{}', \ldots, f_l{}')$ *is another homogeneous minimal reduction of* I *where* $\deg(f_i{}') = \delta_i{}'$ *and* $\delta_1{}' \leqslant \ldots \leqslant \delta_l{}'$*, we have* $\delta_1 = \delta_1{}'$ *and* $\delta_l = \delta_l{}'$*. Moreover, if* $F(I)$ *is reduced, then* $\delta_l = d_\mu$*.*

Proof. (i) We recall that $F(J)$ is a subring of $F(I)$, both bigraded k–algebras, and that $F(I)$ is integral over $F(J)$. We can consider a bihomogeneous relation of integral dependence of g^o over $F(J)$ (where $g \in I \setminus \mathfrak{m}I$ with $\deg(g) = d_1$) : $(g^o)^n + b_1(g^o)^{n-1} + \ldots + b_n = 0$ where $b_i \in F(J)$ with $\deg(b_i) = (i, id_1)$. If $\delta_1 > d_1$, then we have that all $b_i = 0$ and so $(g^o)^n = 0$, i.e. $g^n \in \mathfrak{m}I^n$. But, if $F(I)$ is reduced, then $g^o = 0$ and by $g \notin \mathfrak{m}I$, this is impossible; on the other hand, if R is reduced, by the minimality of the degree of g in I, and by $g^n \in \mathfrak{m}I^n$, we have $g^n = 0$, and so $g = 0$.

(ii) We take $\delta_1 < \delta_1{}'$. Then, we consider a bihomogeneous relation of integral dependence of $f_1^o (\in F(J) \subseteq F(I))$ over $F(J')$: $(f_1^o)^n + b_1{}'(f_1^o)^{n-1} + \ldots + b_n{}' = 0$, where $b_i{}' \in F(J')$ and $\deg(b_i{}') = (i, i\delta_1)$; therefore $(f_1^o)^n = 0$, but f_1^o is transcendent over k. On the other hand, we assume that $\delta_l{}' < \delta_l$. Then, f_l^o verifies a bihomogeneous relation of integral dependence over $F(J')$: $(f_l^o)^n + b_1{}'(f_l^o)^{n-1} + \ldots + b_n{}' = 0$, where $b_i{}' \in F(J')$ with $\deg(b_i{}') = (i, i\delta_l)$; therefore $(f_l^o)^n = 0$, in contradiction with the transcendence of f_l^o over k. Finally, we can consider a bihomogeneous integral relation over $F(J)$ of f_μ^o; if $\delta_l < d_\mu$, working as above, we have that f_μ^o is nilpotent, in contrast with the reducedness of $F(I)$.

□

If I is a quasi-complete intersection of the polynomial ring $k[x_0, \ldots, x_{d-1}]$ with $\mathrm{ht}(I) \leqslant d-2$, then the author in [17] proved the uniqueness of degrees of all scheme–theoretic generations of minimal length (i.e. of length $\mathrm{ht}(I) + 1$); it is also pointed out by several examples that the condition $\mathrm{ht}(I) \leqslant d - 2$ is essential. We recall that if I is a q.c.i. of codimension two in a polynomial ring (with the usual restriction for the dimension), this uniqueness of degrees is proved, first of all, by D.Y. Kuznetsov [13, Proposition 2.10]; moreover, in codimension two also, is noteworthy the proof in [4, Theorem 1.7]. Now, from Lemma 2.1, we can easily give another proof of the uniqueness of degrees for quasi-complete intersection of codimension two. More exactly:

Corollary 2.2. *Let I be a quasi-complete intersection with $2 = ht(I) \leqslant d-2$ of the polynomial ring $k[x_0, \dots, x_{d-1}]$. If $J = (f_1, f_2, f_3)$ is a scheme-th. generation of I, then the degrees $\deg(f_i) = \delta_i$ are defined uniquely by I.*

Proof. Since J is a minimal reduction of I and if $\delta_1 \leqslant \delta_2 \leqslant \delta_3$, then Lemma 2.1 implies the uniqueness of δ_1 and δ_3. On the other hand, from the "enumerative geometry formula" (see [6], [18, Theorem 5] and [10, Theorem 4.5]) we deduce the uniqueness of δ_2. □

Following this way, also for ideals of greater codimension, we need other formulas, as, for example, the one of [10, Theorem 4.11]. Of course, there exist various formulas (all interesting); however (over the difficulty of its discovery) we have to prove that these formulas are sufficient to determine the uniqueness of the degrees for a scheme–th. generation (of minimal length) of a quasi complete intersection. This method is unhappy. On the other hand, we have just see that the momentous notion, for decision on the degrees for a scheme–th. generation of a quasi complete intersection, is the *homogeneous minimal reduction.*

The first idea, after the uniqueness of the first and the last degree of the basis of the homogeneous minimal reductions J, is to consider (for induction on the analytic spread) quotients of the minimal reductions $J/(f)$ modulo a suitable homogeneous element $f \in J \setminus \mathfrak{m}J$. Following the plan of S. Huckaba in [9], one can prove that $l(J/(f)) \geqslant l(J) - 1$; the equality $l(J/(f)) = l(J) - 1$ is verified if f is a superficial element for an ideal K with $J \subseteq K \subseteq \bar{J}$, where $\bar{J}$ denote the integral closure of J; for the existence of such a element, see [12].

In the local case, it is well–known that superficial elements exist for any (non–nilpotent) ideal I; moreover, there exists a non–empty open subset U of $I/\mathfrak{m}I$ such that whenever $x \in U$, then every preimage of x in I is a superficial element of I. (see [3, Chapitre 8, §7, n.5, Remarque 4]); moreover after [19], $l(I)$ is also the maximal length of a superficial sequence for I and every maximal superficial sequence for I generate a minimal reduction.

In the graded case there is a complication: even if I has a homogeneous minimal reduction J, it is very hard to determine a homogeneous superficial sequence for J or for some ideal K with $J \subseteq K \subseteq \bar{J}$.

We can avoid this difficulty, through the trick of the following

Lemma 2.3. *Assume that R is a graded affine k–algebra, I a homogeneous ideal of R such that there exists some homogeneous minimal reduction. If $\delta_1 \leqslant \ldots \leqslant \delta_l$ and ${\delta_1}' \leqslant \ldots \leqslant {\delta_l}'$ are the sequences of the degrees of the basis of two homogeneous minimal reductions J and J' of I, then:*

$$\{i \mid \delta_i = \delta_1\} = \{j \mid \delta'_j = \delta_1\}$$

and

$$\{i \mid \delta_i = \delta_l\} = \{j \mid \delta'_j = \delta_l\}$$

.

Proof. We set $J = (f_1, \ldots, f_l)$ and $J' = (f'_1, \ldots, f'_l)$ where $\deg(f_i) = \delta_i$ and $\deg(f'_i) = \delta'_i$. By Lemma 2.1, we have $\delta_1 = \delta'_1$ and $\delta_l = \delta'_l$. Moreover, we suppose that $\max\{i \mid \delta_i = \delta_1\} > s := \max\{j \mid \delta'_j = \delta_1\}$. We consider $J + J'$ as a reduction of I and we look for particular homogeneous minimal reduction in $J + J'$. There exists an open non empty subset U of k^{2s^2} such that the ideal $(\sum_{i=1}^s a_{1i} f_i^o + \sum_{j=1}^s b_{1j} {f'_j}^o, \ldots, \sum_{i=1}^s a_{si} f_i^o + \sum_{j=1}^s b_{sj} {f'_j}^o, f_{s+1}^o, \ldots, f_l^o)$ is irrelevant in $F(I)$ for $(a_{11}, \ldots, a_{1s}, \ldots, a_{ss}, b_{11}, \ldots, b_{1s}, \ldots, b_{ss}) \in U$. Analogously, there exists an open non empty subset U' of k^{2s^2} such that the ideal $(\sum_{i=1}^s a_{1i} f_i^o + \sum_{j=1}^s b_{1j} {f'_j}^o, \ldots, \sum_{i=1}^s a_{si} f_i^o + \sum_{j=1}^s b_{sj} {f'_j}^o, {f'_{s+1}}^o, \ldots, {f'_l}^o)$ is irrelevant in $F(I)$ for $(a_{11}, \ldots, a_{ss}, b_{11}, \ldots, b_{ss}) \in U'$. From a choice of $(a_{11}, \ldots, a_{ss}, b_{11}, \ldots, b_{ss}) \in U \cap U'$ we obtain elements $h_1, \ldots, h_s \in J + J'$ with $\deg(h_j) = \delta_1$ $(j = 1, \ldots, s)$, such that both $J_1 = (h_1, \ldots, h_s, f_{s+1}, \ldots, f_l)$ and $J'_1 = (h_1, \ldots, h_s, f'_{s+1}, \ldots, {f_l}')$ are minimal reductions of I.

We can, now, consider a bihomogeneous relation of integral dependence of f^o_{s+1} over $F(J'_1)$: $(f^o_{s+1})^n + c_1(f^o_{s+1})^{n-1} + \ldots + c_n = 0$ where $c_i \in F(J'_1)$ with $\deg(c_i) = (i, i\delta_1)$; it is necessary that c_i is a homogeneous polynomial in $h_1, \ldots, h_s$; but this is a contradiction with the k–algebraic independence of $h_1, \ldots, h_s, f^o_{s+1}$.

Proceeding in the same way gives the result for the degree δ_l. □

THEOREM 2.4. *Let I be a homogeneous ideal of a graded affine k–algebra R such that there exists some homogeneous minimal reduction. If $\delta_1 \leqslant \ldots \leqslant \delta_l$ and $\delta_1{}' \leqslant \ldots \leqslant \delta_l{}'$ are the sequences of the degrees of the basis of two homogeneous minimal reductions J and J' of I, then: $\delta_i = \delta_i{}'$ for all $i = 1, \ldots, l$.*

Proof. Working in the same way as in the trick of Lemma 2.3 we can suppose that: $J = (f_1, \ldots, f_l)$ and $J' = (f'_1, \ldots, f'_l)$ with $f_1 = f'_1, \ldots, f_{t-1} = f'_{t-1}$ and $\delta_t < \delta'_t$, where $\deg(f_i) = \delta_i, \deg(f'_i) = \delta'_i$.

We can consider a bihomogeneous relation of integral dependence of f^o_t over $F(J')$: $(f^o_t)^n + c_1(f^o_t)^{n-1} + \ldots + c_n = 0$ where $c_i \in F(J')$ with $\deg(c_i) = (i, i\delta_t)$; therefore the element c_i is a (homogeneous) polynomial in $f^o_1, \ldots, f^o_{t-1}, f'^o_t, \ldots, f'^o_l$ where some of the variables $f^o_1, \ldots, f^o_{t-1}$ is present, on account of the second degree.

Now set $\mathfrak{p} = \sum_{j=1}^{t-1} f^o_j F(J)$ prime ideal of $F(J)$, $\mathfrak{p}' = \sum_{j=1}^{t-1} f^o_j F(J')$ prime ideal of $F(J')$, and $\mathfrak{a} = \mathfrak{p}F(I) = \mathfrak{p}'F(I)$ ideal of $F(I)$.

$F(I)$ is an extension ring integral over $F(J)$ and over $F(J')$; from the lying over theorem we have that $\mathfrak{a} \cap F(J) = \mathfrak{p}$ and $\mathfrak{a} \cap F(J') = \mathfrak{p}'$.

Therefore we have $(f^o_t)^n \equiv 0 \bmod \mathfrak{a}$, and so $f^o_t \equiv 0 \bmod \mathfrak{p}$; but this is a contradiction with the k–algebraic independence of $f^o_1, \ldots, f^o_{t-1}, f^o_t$. □

Now, we apply the results [1, Lemma 6.1, Proposition 6.4] on the special (minimal) reductions, and the M.R. Johnson's formula [11, Theorem 5]; we will need additional conditions on the ring R (as Cohen–Macaulay property) and on the ideal I (as equidimensional, generic complete intersection, positive height, analytic deviation one) and so we can give a complete description of the degrees of the minimal homogeneous reductions of I.

THEOREM 2.5. *In addition to the hypothesis of the Theorem 2.4, we assume that R is Cohen–Macaulay and I is equidimensional, generic complete intersection, with* $\mathrm{ad}(I) = 1$ *and* $g = \mathrm{ht}(I) > 0$. *If* $\delta_1 \leqslant \ldots \leqslant \delta_l$ *is the sequence of the degrees of a basis of a homogeneous minimal reduction J of I, then* $\delta_1 = d_1, \ldots, \delta_{l-1} = d_{l-1}$, *(where* $l = g + 1$*), and moreover, if $F(I)$ is reduced, we have also* $\delta_l = d_\mu$.

Proof. With the usual notations, by [11, Theorem 5], we have $e(R[It]) = (1 + d_1 + \ldots + d_1 \cdots d_g)e(R) - e(R/I)$ and $e(R[Jt]) = (1 + \delta_1 + \ldots + \delta_1 \cdots \delta_g)e(R) - e(R/J)$, where e denotes the multiplicity. Moreover $e(R[It]) = e(R[Jt])$ by [11, Lemma 2]; since I is generic c.i. and equidimensional, the ideals I and J have the same primary components of height g , and so $e(R/I) = e(R/J)$. Hence, we have also: $1 + d_1 + \ldots + d_1 \cdots d_g = 1 + \delta_1 + \ldots + \delta_1 \cdots \delta_g$. The first result now follows; the second result is in 2.1. ☐

3. Results on the scheme-th. generations of small deviation

3.1. Relations between reductions and scheme-th.generations

We consider, in this section, some connexion between minimal reductions (homogeneous or not) and scheme-th. generations (of minimal length or not).

PROPOSITION 3.1. *Let I be a homogeneous ideal of* $k[x_0, \ldots, x_{d-1}]$ *with* $\sigma(I) < d$ *and J a scheme-th. generation of I with* $\mu(J) = \sigma(I)$. *Then, J is a (homogeneous) reduction of I; moreover we have:* $l(I) \leq p\sigma(I) \leq \sigma(I)$.

Proof. See [17, Theorem 3]. ☐

PROPOSITION 3.2. *Let I be a homogeneous ideal of* $R = k[x_0, \ldots, x_{d-1}]$, *with* $\mu(I_\mathfrak{p}) \leq ht(I) + 1$ *for every prime ideal* $\mathfrak{p} \in Proj(R)$, *and let K be a minimal reduction of I.*

(i) If K is homogeneous, then K is a scheme-th.generation of minimal length of I and therefore $l(I) = p\sigma(I) = \sigma(I)$*;*

(ii) if all minimal reductions of I are non-homogeneous, then K is (only) a punctured generation of minimal length of I and therefore $l(I) = p\sigma(I) \leq \sigma(I)$; in particular, if $l(I) < d$, then $l(I) < \sigma(I)$.

Proof. As $K_{\mathfrak{p}}$ is a reduction of $I_{\mathfrak{p}}$ (where $\mathfrak{p} \subsetneq \mathfrak{m}$), by [8, Theorem 3.1], it follows that $K_{\mathfrak{p}} = I_{\mathfrak{p}}$ and so K is a punctured generation of I (if K homogeneous, also a scheme-th. generation of I), and by [17, Proposition 2], we have: $l(I) = p\sigma(I) \leq \sigma(I)$. The rest of the statement is trivial. □

Several are the cases in which it can to apply the prop. 3.2; for example:

- Ideals I quasi complete intersection(q.c.i.) (i.e. where $\sigma(I) = \text{ht}(I) + 1$).
- Subcanonical ideals of codimension 2 (i.e. ideals I generically c.i., unmixed, $\text{ht}(I) = 2$ such that the canonical module $\omega_{R/I} = \text{Ext}^2_R(R/I, R)$ of R/I is scheme-th. generated by one element.)
  - By the "Syzygy problem" of Evans-Griffith and the Gorenstein-c.i. property in codimension 2 by Serre, we have that I is locally a c.i.-
- Ideals I locally non singular (i.e. such that $R_{\mathfrak{p}}/I_{\mathfrak{p}}$ are local regular rings for every $\mathfrak{p} \in \text{Proj}(R)$).
- Ideals I, saturated ideals of monomial projective curves Γ of $\mathbb{P}_3(k)$.
  - By Forster-Swan results (imiting in $\text{Proj}(R)$), we have $\sigma(I) \leq 4$; on the other hand, by [7] or [2], we have $l(I) \leq 3$; by the well-known old result of J. Herzog, we have $\mu(I_{\mathfrak{p}}) \leq 3$ for every prime ideal $\mathfrak{p} \in \text{Proj}(R)$; and so $p\sigma(I) \leq 3$ and $\sigma(I) \leq 3$ *iff* there exist a homogeneous minimal reduction of I.

    Studying the structure of the fiber cone $F(I)$ in [7] and on the existence of homogenous (or not) minimal reductions of I one can give an alternative test (see [5]) for the classification of the monomial projective curves I of $\mathbb{P}_3$, according to $\sigma(I) = 2, 3, 4$.

3.2. Some applications of uniqueness of degrees

Here, it is useful to define the *scheme-analytic deviation* to be the non negative integer sc-d$(I) := \sigma(I) - l(I)$; analogously, the *punctured-analytic deviation* is the non negative integer pu-d$(I) := p\sigma(I) - l(I)$ (these definitions are chiefly inspired by the concept of the classical second analytic deviation).

We will focus upon homogeneous ideals having scheme-analytic deviation either zero or one.

Proposition 3.3. *Assume that R is equidimensional and I such that $\sigma = \sigma(I) = l(I) < d = dim(R)$. Then, for every scheme-th. generation of I of minimal length $J = (f_1, \ldots, f_\sigma)$, its sequence of degrees is uniquely determined by I.*

Proof. Since J is a homogeneous minimal reduction of I (see [17, Theorem 3]), then we can apply the Thm. 2.4. □

Corollary 3.4. *Assume that I is a quasi complete intersection ideal with $0 < \mathrm{ht}(I) < d - 1$ where $d = \dim(R)$. Then, for every scheme-th. generation of I of minimal length $J = (f_1, \ldots, f_\sigma)$, its sequence of the degrees is uniquely determined by I, and also the first $\sigma - 1$ degrees of J are equals to the first $\sigma - 1$ degrees of I.*

Proof. The first statement follows by the previous Proposition; Theorem 2.5 implies the second assertion. □

Now, we assume that R as Cohen–Macaulay with $d = \dim(R)$ and I is unmixed, generically c.i. and such that $\sigma = \sigma(I) = \mathrm{ht}(I) + 2 < d$. We set by J a scheme–th. generation of I of length σ , and by $\eta_1 \leqslant \ldots \leqslant \eta_\sigma$ its sequence of degrees. From $l(J) \leqslant d-1$ it follows that J is a reduction of I, and so $l(I) = l(J)$. If sc-d$(I) \leqslant 1$ the following situations can happen:

1. if sc-d$(I) = 0$, the sequence of degrees of J is uniquely determined by I as proved in Proposition 3.3;

2. if sc-d$(I) = 1$ and if there exists a homogeneous minimal reduction K of J where $\delta_1 \leqslant \ldots \leqslant \delta_{\sigma-1}$ is its sequence of degrees, then $\delta_1 = \eta_1 = d_1, \ldots, \delta_{\sigma-2} = \eta_{\sigma-2} = d_{\sigma-2}$ (see Thm. 2.5), and $\delta_{\sigma-1}$ is equal to $\eta_{\sigma-1}$ or to η_σ.

3. if sc-d$(I) = 1$ and if all the minimal reductions of J are not homogeneous, then the degrees η_i are variables with the particular choice of J.

The following examples (with computation using *Macaulay*) illustrates the usefullnes of the previous propositions.

Example 3.5. *In $\mathbb{P}^4$ we consider the variety with generic point $(s^3t, st^3, t^4, tu^3, s^4)$ (where s, t, u are k–algebraic independents); its prime ideal I in $k[x, y, z, v, w]$ of $ht(I) = 2$ has $\mu(I) = \sigma(I) = 4$ and $l(I) = 3$. On the other hand, I is minimal generated by $f_1 = xy - zw, f_2 = y^3 - xz^2, f_3 = x^2z - y^2w, f_4 = x^3 - yw^2$ and the fiber cone $F(I)$ has a presentation $k[a, b, c, d]/(c^2 + bd)$ where a modulo $(c^2 + bd)$ represent f_1^o and so on; the subideal $(f_1, f_3, f_2 - f_4)$ is a homogeneous minimal reduction of I.*

Example 3.6. *As above, with the same notations, let I be the ideal associated to the generic point $(stu^2, st^3, s^2t^2, tu^3, s^4)$; we can verify that I is minimally generated by $f_1 = z^3 - y^2w, f_2 = x^3z - yv^2w, f_3 = x^3y - z^2v^2, f_4 = x^6 - zv^4w$ and that $\sigma(I) = 4$ but $l(I) = 3$. The fiber cone $F(I)$, (with the usual notations) is $k[a, b, c, d]/(ad)$ and all the minimal reductions of I are not homogeneous.*

Example 3.7. *As above, with the same notations, let I be the ideal associated to the generic point $(t^5, s^2tu^2, t^3u^2, s^2u^3, s^5)$; we can verify that I is minimally generated by $f_1 = y^2z - xv^2, f_2 = x^2y^5 - z^5w^2, f_3 = xy^7 - z^4v^2w^2, f_4 = y^9 - z^3v^4w^2$ and that $\sigma(I) = 4$ but $l(I) = 3$. The fiber cone $F(I)$ is $k[a, b, c, d]/(c^2 - bd)$ and so $K = (f_1, f_2, f_4)$ is a homogeneous minimal reduction of I.*

Example 3.8. *Counterexample: as above, with the same notations, let I be the prime ideal determined by the generic point $(t^3 - t^2u, st^2, stu, u^3, s^3)$; we can verify that I is minimally generated by 4 elements of degrees [3,4,4,5] and that $l(I) = 4$ with the fiber cone $F(I)$ isomorphic to the polynomial ring $k[a, b, c, d]$. This* *example is also a partial counterexample to the conjecture given by A.Polo and the author in* [16, 2.3 – A conjecture].

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