

An Algorithm for Computing Primitive Relations

ANNA SCARAMUZZA (*)

Dedicated to the memory of Fabio Rossi

SUMMARY. - *Let X be a smooth complete toric variety of dimension n . It is described by its fan Σ_X . To study X from the point of view of Mori theory it is very convenient to present Σ_X enumerating its primitive relations and collections (see [2], [3]). In this paper we describe the algorithm that computes the primitive relation associated to a primitive collection.*

Introduction. A toric variety X of dimension n is a normal complex algebraic variety that contains an algebraic group T isomorphic to $(\mathbb{C}^*)^n$ (a torus), as a dense open subset, with an algebraic action $T \times X \rightarrow X$ of T on X which extends the natural action of T on itself (multiplication in T).

Throughout the paper we will assume that X is smooth and complete.

Let $N = \text{Hom}_{\mathbb{Z}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$ and $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Then X is determined by its fan Σ_X in $N_{\mathbb{Q}}$, which is a finite collection of convex rational polyhedral cones in the vector space $N_{\mathbb{Q}}$. The fan Σ_X is completely described by its maximal cones and the set $G(\Sigma_X)$ of its generators. Under the identification $N \cong \mathbb{Z}^n$, the generators can be interpreted as vectors in \mathbb{Z}^n . The fan gives a combinatorial

(*) Author's address: Anna Scaramuzza, Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy; E-mail: scaramuz@mat.uniroma3.it

description of X and so the geometrical properties of X correspond to combinatorial properties of the fan.

The fan Σ_X can also be described with the language of primitive relations and collections. It is introduced by Batyrev in [2] and [3] to classify the smooth toric Fano n -folds with $n \leq 4$. Moreover it is a very useful tool to study X by the point of view of toric Mori theory (see [5], [6], [18], [19] and [20]). We will introduce some results in toric Mori theory to explain the important role played by primitive relations and the reasons why we want to compute them.

A primitive collection P is a subset of $G(\Sigma_X)$ which does not generate a cone in Σ_X . To a given $P = \{x_1, \dots, x_k\}$, we can associate a unique relation:

$$r(P) : \quad x_1 + \dots + x_k - (b_1 y_1 + \dots + b_h y_h) = 0, \quad (1)$$

where $y_1, \dots, y_h \in (G(\Sigma_X) \setminus P)$ generate a cone in Σ_X and b_1, \dots, b_h are positive integers (see Definition 1.2).

One can compute the primitive collections easily: it is enough to apply the definition. The problem is to determine the associated primitive relations.

In this paper we will show how one can determine the coefficients b_1, \dots, b_h that appear in (1.2) considering a system of linear Diophantine equations. Then we will describe the algorithm that computes the coefficients b_1, \dots, b_h and the primitive relation $r(P)$ associated to the primitive collection P .

Let $P = \{x_1, \dots, x_k\}$ be a primitive collection. We define the following system of linear Diophantine equations:

$$\sum_{y \in G(\Sigma_X) \setminus P} v_y y = \underline{u}, \quad (2)$$

where $\underline{u} = x_1 + \dots + x_k$, y is a generator in $(G(\Sigma_X) \setminus P)$ and v_y are the unknowns of the system.

We want to determine a solution (one corresponding to each primitive relation) in the set of non negative integer numbers $\mathbb{Z}_{\geq 0}^p$, where $p = \text{card}(G(\Sigma_X) \setminus P)$.

If C is a $p \times n$ matrix with entries in $\mathbb{Z}_{\geq 0}$ and \underline{u} is a vector in \mathbb{Z}^n , then $\underline{v}C = \underline{u}$ is a non-homogeneous system of linear Diophantine equations with p unknowns represented by the vector

$\underline{v} = (v_1, \dots, v_p)$. We have to characterize the set of solutions of $\underline{v}C = \underline{u}$, denoted by $\text{Sol}(\underline{v}C = \underline{u})$. We proceed by fixing the following partial order in $\mathbb{Z}_{\geq 0}^p$:

$$(a_1, \dots, a_p) \leq (b_1, \dots, b_p) \quad \text{if and only if } a_i \leq b_i \quad \forall i \in \{1, \dots, p\}. \quad (3)$$

By the semigroups theory (see [7]), the set $\text{Sol}(\underline{v}C = \underline{u})$ can be described using two finite sets of solutions: the set of minimal solutions $\text{Min}(\underline{v}C = \underline{u})$ of $\underline{v}C = \underline{u}$ and the set of minimal non-zero solutions $\text{Min}(\underline{v}C = \underline{0})$ of the associated homogeneous system, with respect to the fixed order. In fact every solution $\underline{a} = (a_1, \dots, a_p)$ can be written as:

$$\underline{a} = \underline{\mu} + \sum_{i=1}^t \lambda_i \underline{\nu}_i, \quad (4)$$

where $\underline{\mu}$ is a particular minimal solution in $\text{Min}(\underline{v}C = \underline{u})$, $\{\underline{\nu}_1, \dots, \underline{\nu}_t\} = \text{Min}(\underline{v}C = \underline{0})$ and $\lambda_i \in \mathbb{Z}_{\geq 0}$.

We will show that the positive integers b_1, \dots, b_h which appear in (1), determine a solution of $\underline{v}C = \underline{u}$ belonging to $\text{Min}(\underline{v}C = \underline{u})$ (see Theorem 3.1). This is crucial because it allows to restrict the attention to the finite set of minimal solutions of the system, associated to the primitive collection P , to find its primitive relation.

Finally in Proposition 3.3 we will prove that there is a unique minimal solution \underline{b} such that the set $G(\underline{b}) = \{y \in G(\Sigma_X) \setminus P \mid b_y \neq 0\}$ generates a cone in Σ_X . It is called optimal solution and determines the primitive relation associated to P (see Definition 3.2).

Using Theorem 3.3 and Definition 3.2 we are able to build the algorithm `PrimRel` to find the associated primitive relation. It is built with the programming language *Mathematica*.

At the end of the paper we give two examples. In the former we consider a smooth complete non-projective toric threefold with Picard number 5 and we describe its Mori cone. By computations we prove that the Mori cone contains a linear subspace of dimension 4. In the latter we describe a smooth projective toric variety of dimension 6 and Picard number 4. It is obtained from \mathbb{P}^6 by three blow-ups in three curves. In this case the Mori cone is generated by the primitive relations (see Theorem 1.10).

The article is divided in six sections. In the first one we recall some definitions about toric varieties (see [9], [11] and [15]) and the definitions of primitive collections and relations (see [2], [3]). Moreover we introduce some notions of toric Mori theory to explain the reasons of importance of computing the primitive relation (see [15], [17], [22]). We show how one can describe the Mori cone of a toric variety introducing the sets of contractible classes and of extremal classes, special subsets of the set of numerical classes of invariant curves. In Section 2 we introduce the systems of linear Diophantine equations: $\underline{v}C = \underline{u}$ and explain how the set of solutions $\text{Sol}(\underline{v}C = \underline{u})$ can be characterized by the finite sets of minimal solutions $\text{Min}(\underline{v}C = \underline{u})$ and $\text{Min}(\underline{v}C = \underline{0})$ (see [1], [7]). In Section 3 we show that the coefficients in the negative part of the primitive relation determine an element in $\text{Min}(\underline{v}C = \underline{u})$ and we give the definition of optimal solution. Then we explain how these ones allow to compute the primitive relation associated to a primitive collection (see Theorem 2.6 and Definition 3.2). In Section 4 we describe the algorithm `PrimRel` and the commands of *Mathematica* used to compute the primitive relations. In the last two sections we present two examples. In Section 5 we describe a variety belonging to the simplest example of families of smooth, complete, toric threefolds with $\rho = 5$ which become projective after a blow-up. This family has been studied by Bonavero (see [4]) and by Fujino and Payne (see [10]). This variety is not projective and it is excluded by Fujino and Payne because the Mori cone contains a vector space of dimension 4 only. In the other example we describe a variety of dimension 6 and Picard number 4. It is obtained using the algorithms of the package *Toric Varieties*, it is projective and it has 5 contractible classes of which only 4 are rays of the Mori cone.

REMARK: The algorithm has been implemented using the programming language *Mathematica* 5.0 (see [24]). It is collected in the package *Toric Varieties* which is available together with a file of instructions of all its programs at the personal author's web page:

<http://annascaramuzza.googlepages.com/studies>.

All computations presented in this paper have been carried out on a

Pentium(R) 4 HP Pavilion ze5400 with 2.66 GHz and 256 MB RAM under Microsoft Windows XP.

1. Basic definitions on toric geometry

In this section we recall some definitions on toric geometry. More detailed references can be found in [9], [11] and [15]. For the definition of primitive collections and relations we refer to [2] and [3].

Let $N = \text{Hom}_{\mathbb{Z}}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$ be a lattice of rank n and let $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ be its rational extension. Let $\{x_1, \dots, x_k\}$ be a subset of elements of N . The **rational convex polyhedral cone (or a cone)** generated by $\{x_1, \dots, x_k\}$ is the set:

$$\langle x_1, \dots, x_k \rangle = \left\{ \sum_{i=1}^k \lambda_i x_i \mid \lambda_i \geq 0, \lambda_i \in \mathbb{Q} \right\}.$$

To every cone σ we can associate an affine toric variety denoted by U_{σ} (for the construction see [9], [11] and [15])

A **fan** Σ is a finite collection of rational convex polyhedral cones in the vector space $N_{\mathbb{Q}}$. For every cone σ in the fan Σ we define the associated affine toric variety U_{σ} . Then we can naturally glue $\{U_{\sigma} \mid \sigma \in \Sigma\}$ together to obtain an irreducible and normal variety $X = X(\Sigma)$ of dimension n . X is the variety associated to the fan Σ (see Theorem 1.4 in [15])

Throughout the paper we will use the notation Σ_X for the fan and X for the associated variety.

We deal with smooth and complete toric varieties. X is smooth and complete if and only if every cone in the fan is generated by a part of a basis and the support of the fan Σ_X is the whole vector space $N_{\mathbb{Q}}$ respectively. We define the **dimension of the cone** σ as the dimension of the smallest linear subspace $\text{Span}(\sigma)$ containing σ . Moreover we call $\text{RelInt}(\sigma)$ the interior of σ in $\text{Span}(\sigma)$. For every 1-dimensional cone σ in Σ_X we consider its primitive generator x_{σ} (for definition see [9]). Then the set:

$$\{x_{\sigma} \mid \sigma \text{ is a 1 - dimensional cone } \}$$

is the set of all generators of Σ_X . It is denoted by $G(\Sigma_X)$.

The list of maximal cones of Σ_X and the set $G(\Sigma_X)$ describe completely the fan, hence the variety X .

We observe that for every toric variety X there is a bijection between the cones of dimension k in the fan and the set of orbits of the torus T in X of dimension $n - k$. We will denote by $V(\sigma)$ the Zariski closure in X of the orbit corresponding to the cone σ . In the case of a 1-dimensional cone $\langle x \rangle$, we use the notation $V(x)$ for the divisor. We will refer to $V(\sigma)$ as an invariant subvariety.

Another combinatorial way to describe Σ_X is given by the language of primitive collections and relations introduced by Batyrev (see [2] and [3]). Next we recall their definitions.

DEFINITION 1.1. *A subset $P \subseteq G(\Sigma_X)$ is a **primitive collection** if it does not generate a cone in the fan Σ_X while every proper subset of P generates a cone in the fan. The symbol $PC(X)$ will denote the set of all primitive collections of Σ_X .*

By definition, it follows that for any subset S of $G(\Sigma_X)$, either S generates a cone in Σ_X , or S contains a primitive collection.

DEFINITION 1.2. *Let $P = \{x_1, \dots, x_k\} \subseteq G(\Sigma_X)$ be a primitive collection. Since X is complete, there exists a unique cone $\sigma_P = \langle y_1, \dots, y_r \rangle$ in Σ_X such that $x_1 + \dots + x_k \in \text{RelInt}(\sigma_P)$. Hence there exists a unique sequence of numbers $a_1, \dots, a_r \in \mathbb{Z}_{>0}$ such that*

$$r(P) : \quad x_1 + \dots + x_k - (b_1 y_1 + \dots + b_n y_n) = 0.$$

*This is called **primitive relation** associated to the primitive collection P .*

*The cone σ_P is the **cone associated** to the primitive collection P .*

*We will call $x_1 + \dots + x_k$ and $b_1 y_1 + \dots + b_n y_n$ respectively **positive part** and **negative part** of the primitive relation.*

PROPOSITION 1.3 ([3], Proposition 3.1). *Let $P \in PC(X)$ be a primitive collection in $G(\Sigma_X)$ and let $G(\sigma_P)$ be the set of generators of the cone associated to P . Then $P \cap G(\sigma_P) = \emptyset$.*

Next we introduce some notions of toric Mori theory to explain the important role of primitive relations. We refer to [15], [17] and [22] for the toric Mori theory.

Let us consider the group $\mathcal{N}_1(X)$ of algebraic 1-cycles on X modulo numerical equivalence and define the vector space $\mathcal{N}_1(X)_{\mathbb{Q}} = \mathcal{N}_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ whose dimension is equal to the Picard number ρ of the variety. In $\mathcal{N}_1(X)_{\mathbb{Q}}$ we define the Mori cone as the convex cone generated by the classes of effective curves modulo numerical equivalence, i.e.,

$$\text{NE}(X) = \left\{ \gamma \in \mathcal{N}_1(X)_{\mathbb{Q}} \mid \gamma = \left[\sum a_i C_i \right], \text{ with } a_i \in \mathbb{Q}_{\geq 0} \right\}.$$

We denote by $\text{NE}(X)_{\mathbb{Z}}$ the intersection of $\text{NE}(X)$ with $\mathcal{N}_1(X)$.

There is the following exact sequence:

$$0 \longrightarrow \mathcal{N}_1(X) \xrightarrow{\phi} \mathbb{Z}^t \xrightarrow{\psi} N \longrightarrow 0$$

where $t = \text{card}(G(\Sigma_X))$ and the maps ϕ, ψ are respectively defined by $\gamma \mapsto (\gamma \cdot V(x))_{x \in G(\Sigma_X)}$ and $(a_x)_{x \in G(\Sigma_X)} \mapsto \sum_{x \in G(\Sigma_X)} a_x x$. Then ϕ allows to identify the group $\mathcal{N}_1(X)$ with the group of integral relations among the generators of the fan Σ_X . Hence every class $\gamma \in \mathcal{N}_1(X)$ can be identified with the relation:

$$\sum_{x \in G(\Sigma_X)} (\gamma \cdot V(x))x = 0.$$

Or, equivalently, with a linear polynomial in the generators of the fan and with integer coefficients.

Reid proves that $\text{NE}(X)_{\mathbb{Z}}$ is generated as a semigroup by the finite set \mathcal{I} of all numerical classes of invariant curves. Since $\dim \text{NE}(X) = \rho$, then \mathcal{I} generates $\mathcal{N}_1(X)$ as a group.

THEOREM 1.4 ([17], Corollary 1.7). *Let X be a smooth complete toric variety of dimension n with fan Σ_X . Let \mathcal{I} be the set of all numerical classes of invariant curves of X . Then*

$$\text{NE}(X)_{\mathbb{Z}} = \sum_{\gamma \in \mathcal{I}} \mathbb{Z}_{\geq 0} \gamma.$$

When X is projective we can characterize the Mori cone using a special subset of numerical classes of curves contained in the set of primitive relations: contractible classes.

DEFINITION 1.5. Let $\gamma \in \text{NE}(X)_{\mathbb{Z}}$ be primitive in $\mathbb{Z}_{\geq 0}\gamma$ and such that there exists some irreducible curve in X having numerical class in $\mathbb{Z}_{\geq 0}\gamma$. We say that γ is **contractible** if there exist a toric variety X_{γ} and an equivariant morphism $\varphi_{\gamma} : X \rightarrow X_{\gamma}$, surjective and with connected fibers, such that for every irreducible curve $C \subset X$,

$$\varphi_{\gamma}(C) = \{pt\} \iff [C] \in \mathbb{Q}_{\geq 0}\gamma.$$

Hence contractible classes correspond to “elementary” toric morphisms with connected fibers and target X .

Observe that every primitive relation is a relation among the generators of Σ_X , hence it can be interpreted as an element of $\mathcal{N}_1(X)$. Moreover we have the following result:

PROPOSITION 1.6 ([12], Proposition 2.1). Let $\gamma \in \mathcal{N}_1(X)$ be given by the relation

$$a_1x_1 + \dots + a_kx_k - (b_1y_1 + \dots + b_hy_h) = 0,$$

with $a_i, b_j \in \mathbb{Z}_{>0}$ for each i, j . If $\langle y_1, \dots, y_h \rangle \in \Sigma_X$, then $\gamma \in \text{NE}(X)$.

Proposition 1.6 says that for every primitive collection $P \in PC(X)$ the primitive relation $r(P)$ belongs to $\text{NE}(X)_{\mathbb{Z}}$. Moreover every contractible class is also a primitive relation (see Theorem 2.2 in [5]) and it is always the class of some invariant curve. Thus we have the following important subsets of $\text{NE}(X)_{\mathbb{Z}}$, which are all finite:

$$\begin{aligned} \mathcal{I} &= \{\text{classes of invariant curves}\}, \\ &\quad \cup \\ \mathcal{C} &= \{\text{contractible classes}\}, \\ &\quad \cap \\ \text{PR} &= \{\text{primitive relations}\}. \end{aligned}$$

It is easy to determine the set of all contractible classes of X as a subset of the set of primitive relations. In fact there is the following combinatorial criterion.

PROPOSITION 1.7 ([18], Theorem 4.10 - [5], Proposition 3.4). Let $P = \{x_1, \dots, x_k\}$ be a primitive collection in Σ_X , with primitive relation:

$$r(P) : \quad x_1 + \dots + x_k - b_1y_1 - \dots - b_hy_h = 0.$$

Then $r(P)$ is contractible if and only if for every primitive collection Q of Σ_X such that $Q \cap P \neq \emptyset$ and $Q \neq P$, the set $(Q \setminus P) \cup \{y_1, \dots, y_h\}$ contains a primitive collection.

Moreover given a contractible class γ , the corresponding primitive relation describes the variety X_γ and the associated morphism φ_γ . Indeed suppose that

$$x_1 + \dots + x_k - (b_1 y_1 + \dots + b_h y_h) = 0,$$

is the primitive relation associated to γ .

If $h = 0$, that is the relation has the form:

$$x_1 + \dots + x_k = 0,$$

then X_γ is smooth and $\varphi_\gamma : X \rightarrow X_\gamma$ is a \mathbb{P}^{k-1} -bundle.

Therefore if $h > 0$, then φ_γ is birational and its exceptional locus is $V(\langle y_1, \dots, y_h \rangle)$. Hence φ_γ is divisorial if and only if $h = 1$. Moreover if $h = 1$ and $b_1 = 1$, i. e. if the relation has the form

$$x_1 + \dots + x_k - y_1 = 0,$$

then X_γ is smooth and φ_γ is the blow-up of an invariant subvariety of codimension k in X_γ .

Finally we define an **extremal ray** as a 1-dimensional face R of $\text{NE}(X)$ and the **extremal class** as the primitive element of $R \cap \text{NE}(X)_\mathbb{Z}$. Let \mathcal{E} be the set of all extremal classes of X .

Then we can reformulate Reid's results in toric Mori theory as follows.

THEOREM 1.8 ([17], Theorem 1.5). *Let X be a projective, smooth, toric variety. Any extremal class is contractible.*

The difference among contractible and extremal classes is given by the following result.

PROPOSITION 1.9 ([4], Lemma 1 - [5], Corollary 3.3). *Let X be a projective, smooth, toric variety. Let $\gamma \in \text{NE}(X)$ be a contractible class and let $\varphi_\gamma : X \rightarrow X_\gamma$ be the associated morphism. Then γ is not extremal if and only if φ_γ is birational and the variety X_γ is not projective.*

Finally we notice that by definition of extremal classes, we have:

$$\mathrm{NE}(X) = \sum_{\gamma \in \mathcal{E}} \mathbb{Q}_{\geq 0} \gamma,$$

but it is not known whether the same holds over \mathbb{Z} , i. e. whether the set \mathcal{E} generates $\mathrm{NE}(X)_{\mathbb{Z}}$ as a semigroup. If we consider all contractible classes we have:

THEOREM 1.10 ([5]). *Let X be a projective, smooth, toric variety. Let \mathcal{C} be the set of all contractible classes. Then*

$$\mathrm{NE}(X)_{\mathbb{Z}} = \sum_{\gamma \in \mathcal{C}} \mathbb{Z}_{\geq 0} \gamma,$$

that is \mathcal{C} generates $\mathrm{NE}(X)_{\mathbb{Z}}$ as a semigroup.

This means that the set \mathcal{C} of contractible classes, so the set of all primitive relations, generates $\mathrm{NE}(X)_{\mathbb{Z}}$ as a semigroup and $\mathcal{N}_1(X)$ as a group.

2. Systems of linear Diophantine equations

In this section we recall some known results about systems of linear Diophantine equations. The problem to describe the set of solutions of a system of linear Diophantine equations and write an efficient algorithm for solving it is analyzed by Ajili and Contejean in [1]. There the authors remark that their algorithm solves a linear Diophantine system of both equations and inequations directly, that is, without adding slack variables for encoding inequalities as equations. Then it is a generalization of the algorithm due to Contejean and Devie for solving systems of linear Diophantine equations (see [8]). Recently the same problem has been studied introducing the theory of Gröbner bases (see [16]).

For more detailed references about the study of systems of linear Diophantine equations we refer to [1], [8] and [23]. In [7] there are some references on the theory of semigroups.

Let us consider a system of linear Diophantine equations:

$$\underline{v}C = \underline{u}, \quad (5)$$

where $\underline{v} = (v_1, \dots, v_p)$ is the vector of unknowns, $C = (c_{ij})$ is a $p \times n$ matrix with $c_{ij} \in \mathbb{Z}$, and \underline{u} is a vector in \mathbb{Z}^n .

Next we look for all solutions whose coordinates are non-negative integers, i.e. the solutions of the system that are in $\mathbb{Z}_{\geq 0}^p$.

We will distinguish whether the linear system is homogeneous or not.

Consider the semigroup of natural numbers $(\mathbb{Z}_{\geq 0}^p, +)$ and fix in $\mathbb{Z}_{\geq 0}^p$ the following order:

$$(a_1, \dots, a_p) \leq (b_1, \dots, b_p) \quad \text{if and only if } a_i \leq b_i \quad \forall i \in \{1, \dots, p\}.$$

REMARK 2.1. *This is a partial order on $(\mathbb{Z}_{\geq 0}^p, +)$ and the element $\underline{0} \in \mathbb{Z}_{\geq 0}^p$ is the minimum with respect to this order. So given a decreasing chain of elements in $\mathbb{Z}_{\geq 0}^p$, it has a minimal element.*

Moreover the order is clearly compatible with the sum, this means that, given $\alpha = (a_1, \dots, a_p)$ and $\beta = (b_1, \dots, b_p)$ such that $\alpha \geq \beta$ then for every $\gamma = (c_1, \dots, c_p) \in \mathbb{Z}_{\geq 0}^p$, we have $\alpha + \gamma \geq \beta + \gamma$.

2.1. Homogeneous systems

Let's consider the case of a homogeneous system:

$$\underline{v}C = \underline{0}. \quad (6)$$

The set of solutions of the homogeneous system (6) is a sub-semigroup of $\mathbb{Z}_{\geq 0}^p$ and we denote it by $\text{Sol}(\underline{v}C = \underline{0})$.

We want to show that $\text{Sol}(\underline{v}C = \underline{0})$ is a finitely generated semi-group.

THEOREM 2.2 ([7], Theorem 9.18). *Let $A \subset \mathbb{Z}_{\geq 0}^p$ be a subset. Then the set M of all minimal elements of A is finite. Moreover, if $\alpha \in A$, then there exists $\nu \in M$ such that $\nu \leq \alpha$.*

The semigroup $\mathbb{Z}_{\geq 0}^p$ is contained in the free additive abelian group \mathbb{Z}^p and the partial order on $\mathbb{Z}_{\geq 0}^p$ extends naturally on \mathbb{Z}^p setting:

$$\alpha \geq \beta \iff \alpha - \beta \in \mathbb{Z}_{\geq 0}^p$$

PROPOSITION 2.3 ([7], Corollary 9.19). *Let G be a subgroup of \mathbb{Z}^p and suppose that the set $S = G \cap \mathbb{Z}_{\geq 0}^p$ contains at least one non-zero element. Then S is a finitely generated sub-semigroup of $\mathbb{Z}_{\geq 0}^p$; in fact the set of minimal elements of S finitely generates S .*

Applying these results, we can prove that $\text{Sol}(\underline{v}C = \underline{0})$ is finitely generated as a semigroup.

First of all we observe that the zero-element $\underline{0}$ is always a solution of the system $\underline{v}C = \underline{0}$ and it is the minimum among the solutions.

PROPOSITION 2.4. *$\text{Sol}(\underline{v}C = \underline{0})$ is finitely generated as sub-semigroup in $\mathbb{Z}_{\geq 0}^p$.*

Proof. If $\underline{0}$ is the unique solution of $\underline{v}C = \underline{0}$, then $\text{Sol}(\underline{v}C = \underline{0}) = \{\underline{0}\}$ and it is finitely generated.

Suppose that $\text{Sol}(\underline{v}C = \underline{0}) \neq \{\underline{0}\}$, namely the homogeneous system has a non-zero solution.

Let $\text{Min}(\underline{v}C = \underline{0})$ be the set of minimal elements of $\text{Sol}(\underline{v}C = \underline{0}) \setminus \{\underline{0}\}$. By Theorem 2.2 and Proposition 2.3, $\text{Min}(\underline{v}C = \underline{0})$ is a finite set and generates $\text{Sol}(\underline{v}C = \underline{0})$. \square

We will always denote by $\text{Min}(\underline{v}C = \underline{0})$ the set of minimal non-zero solutions of $\text{Sol}(\underline{v}C = \underline{0})$. It is empty when $\text{Sol}(\underline{v}C = \underline{0}) = \{\underline{0}\}$.

Then every non-zero solution \underline{a} of the system $\underline{v}C = \underline{0}$ is a $\mathbb{Z}_{\geq 0}$ -linear combination of elements of $\text{Min}(\underline{v}C = \underline{0})$:

$$\underline{a} = \sum_{i=1}^t \lambda_i \underline{\nu}_i,$$

where $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\text{Min}(\underline{v}C = \underline{0}) = \{\underline{\nu}_1, \dots, \underline{\nu}_t\}$.

2.2. Non-homogeneous systems

Let's consider the case of non-homogeneous systems:

$$\underline{v}C = \underline{u}. \tag{7}$$

Computing the set of solutions and their representation is more complicated because the set of solutions is not a semigroup.

However, we can characterize the solutions using two sets: $\text{Min}(\underline{v}C = \underline{0})$ and $\text{Min}(\underline{v}C = \underline{u})$. This last is the set of minimal solutions of the system $\underline{v}C = \underline{u}$.

Moreover in this subsection we will suppose that $\text{Min}(\underline{v}C = \underline{0}) \neq \emptyset$, that is $\text{Sol}(\underline{v}C = \underline{0}) \neq \{\underline{0}\}$.

By Theorem 2.2 we have that $\text{Min}(\underline{v}C = \underline{0})$ and $\text{Min}(\underline{v}C = \underline{u})$ are finite subsets of $\mathbb{Z}_{\geq 0}^p$.

REMARK 2.5. *We observe that the element*

$$\underline{\mu} + \sum_{i=1}^t \lambda_i \underline{\nu}_i$$

where $\underline{\mu} \in \text{Min}(\underline{v}C = \underline{u})$, $\text{Min}(\underline{v}C = \underline{0}) = \{\underline{\nu}_1, \dots, \underline{\nu}_t\}$ and $\lambda_i \in \mathbb{Z}_{\geq 0}$, is always a solution of the system $\underline{v}C = \underline{u}$.

The converse is also true.

PROPOSITION 2.6. *Any solution $\underline{a} \in \mathbb{Z}_{\geq 0}^p$ of the system $\underline{v}C = \underline{u}$ is the sum of an element of $\text{Min}(\underline{v}C = \underline{u})$ and a $\mathbb{Z}_{\geq 0}$ -linear combination of elements of $\text{Min}(\underline{v}C = \underline{0})$.*

The proof of this proposition reminds the process used to find the solutions of a linear non-homogeneous system over a field (see [13], [21]). The difference is that here we are working over the semigroup $\mathbb{Z}_{\geq 0}^p$ and not over a group.

Proof. Let \underline{a} be a solution of the system (7). By definition of $\text{Min}(\underline{v}C = \underline{u})$, we have

$$\exists \underline{\mu} \in \text{Min}(\underline{v}C = \underline{u}) \quad \text{such that} \quad \underline{\mu} \leq \underline{a} \quad \text{and} \quad \underline{\mu}C = \underline{u}. \quad (8)$$

Consider the following element: $\underline{a} - \underline{\mu}$. Since $\underline{\mu} \leq \underline{a}$, then $\underline{a} - \underline{\mu} \in \mathbb{Z}_{\geq 0}^p$. Moreover

$$(\underline{a} - \underline{\mu})C = \underline{a}C - \underline{\mu}C = \underline{u} - \underline{u} = 0$$

that is, the element $\underline{a} - \underline{\mu}$ is a solution of the linear homogeneous system associated to the system (7). So $\underline{a} - \underline{\mu} \in \text{Sol}(\underline{v}C = \underline{0})$. The set $\text{Sol}(\underline{v}C = \underline{0})$ is a semigroup finitely generated by $\text{Min}(\underline{v}C = \underline{0})$ and if we suppose that $\text{Min}(\underline{v}C = \underline{0}) = \{\underline{\nu}_1, \dots, \underline{\nu}_k\}$, then we have

$$\underline{a} - \underline{\mu} = \sum_{i=1}^k \lambda_i \underline{\nu}_i \quad \text{with} \quad \lambda_i \in \mathbb{Z}_{\geq 0}.$$

Therefore

$$\underline{a} = \underline{\mu} + \sum_{i=1}^k \lambda_i \underline{\nu}_i \quad \text{with } \lambda_i \in \mathbb{Z}_{\geq 0},$$

and this proves the statement. \square

3. Primitive relations and linear Diophantine equations

In this section we will show how systems of linear Diophantine equations are related with primitive relations.

Let Σ_X be the fan of X and suppose that it is described by the set of generators $G(\Sigma_X)$ and the list of its maximal cones. Fix an isomorphism $N \cong \mathbb{Z}^n$, so that we can think of elements in N as vectors in \mathbb{Z}^n . The problem is to determine the description of Σ_X from the point of view of primitive relations and collections.

Given a primitive collection $P = \{x_1, \dots, x_k\}$ of Σ_X , we want to compute its primitive relation $r(P)$.

By Proposition 1.3, we know that $G(\sigma_P) \subseteq G(\Sigma_X) \setminus P$. In order to find the set $G(\sigma_P)$ and the negative part of $r(P)$, we consider the linear system:

$$\sum_{y_i \in (G(\Sigma_X) \setminus P)} v_i y_i = x_1 + \dots + x_k, \quad (9)$$

where the elements x_i and y_i are known, and the elements v_i are the unknowns of the system. Rewrite (9) as $\underline{v}C = \underline{a}$, where $\underline{a} = x_1 + \dots + x_k$ and C is the matrix of coordinates of y_i , for $i = 1, \dots, \text{card}(G(\Sigma_X) \setminus P)$. We refer to it as *the linear system associated to the primitive collection P* .

For every solution \underline{a} of this linear system, we define

$$G(\underline{a}) = \{y_i \in (G(\Sigma_X) \setminus P) \mid a_i \neq 0\}.$$

This is the set of generators y_i whose coefficients in (9) are non-zero.

The following result characterizes a primitive relation using the linear system associated to P :

THEOREM 3.1. *Let $P = \{x_1, \dots, x_k\}$ be a primitive collection in $PC(X)$. Then the set of the coefficients of the negative part of $r(P)$ is a minimal solution of the system associated to P .*

Proof. Write:

$$r(P) : \quad x_1 + \cdots + x_k = a_1 y_1 + \cdots + a_s y_s.$$

Moreover suppose that the system associated to P is:

$$\underline{v}C = \underline{u}.$$

Write $G(\Sigma_X) \setminus P = \{y_1, \dots, y_s, y_{s+1}, \dots, y_p\}$. The vector:

$$\underline{a} = (a_1, \dots, a_p) \quad \text{with } a_i = 0 \text{ for } i = s+1, \dots, p$$

is a solution of the system associated to P .

Suppose that $\text{Min}(\underline{v}C = \underline{u})$ is the set of minimal solutions of $\underline{v}C = \underline{u}$ and that $\text{Min}(\underline{v}C = \underline{0}) = \{\underline{\nu}_1, \dots, \underline{\nu}_t\}$ is the set of minimal non-zero solutions of the homogeneous system associated to $\underline{v}C = \underline{u}$.

By Proposition 2.6, there exist $\lambda_1, \dots, \lambda_t \in \mathbb{Z}_{\geq 0}$ and $\underline{\mu} \in \text{Min}(\underline{v}C = \underline{u})$ such that:

$$\underline{a} = \underline{\mu} + \sum_{i=1}^t \lambda_i \underline{\nu}_i. \quad (10)$$

Let us show that $\lambda_i = 0$ for every $i = 1, \dots, t$.

Since $\underline{\mu}$ is minimal in $\text{Sol}(\underline{v}C = \underline{u})$, then $\underline{\mu} \leq \underline{a}$ and $\underline{\mu}C = \underline{u}$. Consider the set $G(\underline{\mu})$ and recall that $G(\sigma_P) = \{y_1, \dots, y_s\}$. Thus $G(\underline{\mu}) \subseteq G(\sigma_P)$ and

$$\mu_1 y_1 + \cdots + \mu_s y_s = x_1 + \cdots + x_k.$$

Let H be the subgroup of N generated by $G(\sigma_P)$. Since X is smooth, $G(\sigma_P)$ is a basis for H . By definition $\underline{u} = x_1 + \cdots + x_k \in H$, and its coordinates with respect to this basis are uniquely determined. Hence $\underline{a} = \underline{\mu}$ and $\lambda_i = 0$ for every $i = 1, \dots, t$. \square

Theorem 3.1 explains the importance of the set $\text{Min}(\underline{v}C = \underline{u})$ associated to the system of linear Diophantine equations: it allows to find the primitive relation reducing the search to a finite set of $\mathbb{Z}_{\geq 0}^p$.

Once constructed $\underline{v}C = \underline{u}$, we want to compute the set $\text{Min}(\underline{v}C = \underline{u})$ of its minimal solutions. Since the cardinality of $\text{Min}(\underline{v}C = \underline{u})$ can be greater than 1, we introduce the following definition.

DEFINITION 3.2. A minimal solution (a_1, \dots, a_p) is called an **optimal solution** if $G(\underline{a}) = \{y_i \mid a_i \neq 0\}$ is a set of generators of a cone of Σ_X .

Then we can prove:

PROPOSITION 3.3. There exists a unique optimal solution and it gives the coefficients of the negative part of the primitive relation $r(P)$.

Proof. By Theorem 3.1, the set of coefficients of the negative part of $r(P)$ is a minimal solution \underline{a} of the system associated to P ; clearly it is also optimal. Suppose now that \underline{b} is another optimal solution and define $\sigma_{\underline{b}} = \langle G(\underline{b}) \rangle \in \Sigma_X$. Then

$$x_1 + \dots + x_k = \sum_{y_i \in G(\underline{b})} b_i y_i,$$

with all b_i 's strictly positive, that is $x_1 + \dots + x_k \in \text{RelInt}(\sigma_{\underline{b}})$. The cone of Σ_X containing $x_1 + \dots + x_k$ in its relative interior is unique, hence $\sigma_{\underline{b}} = \sigma_P$ and $G(\underline{b}) = G(\sigma_P) = G(\underline{a})$. Since the elements of $G(\sigma_P)$ are linearly independent, we get $\underline{a} = \underline{b}$. \square

4. Algorithm computing a primitive relation

Let X be a smooth complete toric variety and suppose that its fan Σ_X is described by its maximal cones and its generators. Moreover suppose that the set of all primitive collections $PC(X)$ is known, then we want to compute the primitive relation $r(P)$ associated to a primitive collection $P \in PC(X)$.

We compute the primitive relation $r(P)$ with the programming language *Mathematica*. The built-in function `Reduce` of *Mathematica* uses the algorithm implemented by Ajili and Contejean (see [1]) to solve the linear system associated to P . The command gives a complete description of the set of solutions and answers in the following way:

$$\{ \mathbf{C}[1], \dots, \mathbf{C}[\mathbf{h}] \in \text{Integers}, \mathbf{C}[1] \geq 0, \dots, \mathbf{C}[\mathbf{h}] \geq 0, \\ \underline{a}_1(\mathbf{C}[1], \dots, \mathbf{C}[\mathbf{h}]), \dots, \underline{a}_k(\mathbf{C}[1], \dots, \mathbf{C}[\mathbf{h}]) \}$$

Here $\mathbf{C}[j]$ are parameters in $\mathbb{Z}_{\geq 0}$, and $a_i(\mathbf{C}[1], \dots, \mathbf{C}[h])$ for $i = 1, \dots, k$ are all solutions of the linear system (see [25]).

By replacing every parameter $\mathbf{C}[j]$ with zero, we find the k distinct minimal solutions of $\underline{v}C = \underline{u}$. If $\underline{v}C = \underline{u}$ is the linear system associated to a primitive collection P of a variety X , then the command **Reduce** can be used to compute the set of all minimal solutions $\text{Min}(\underline{v}C = \underline{u})$.

Then for every minimal element $\underline{\mu} = (\mu_1, \dots, \mu_p) \in \text{Min}(\underline{v}C = \underline{u})$ we consider the set:

$$G(\underline{\mu}) = \{y_i \in (G(\Sigma_X) \setminus P) \mid \mu_i \neq 0\}.$$

The optimal solution is the solution corresponding to the unique set $G(\underline{\mu})$ which verifies the following condition:

$$\text{no primitive collection in } PC(X) \setminus \{P\} \text{ is contained in } G(\underline{\mu}). \quad (11)$$

Notice that the condition (11) is equivalent to say that $G(\underline{\mu})$ generates a cone in the fan of X .

In this way we can determine the primitive relation associated to the primitive collection P .

REMARK 4.1. *We observe that the command **Reduce** can have the following answers:*

- 1: a unique solution: in this case $\text{Min}(\underline{v}C = \underline{u})$ has cardinality 1 and $\text{Min}(\underline{v}C = \underline{0})$ is empty;
- 2: a unique solution depending on h positive integer parameters: in this case $\text{Min}(\underline{v}C = \underline{u})$ has cardinality 1 and $\text{Min}(\underline{v}C = \underline{0})$ has cardinality h ;
- 3: a set of k different solutions and no parameters, this means that $\text{Min}(\underline{v}C = \underline{u})$ has cardinality k while $\text{Min}(\underline{v}C = \underline{0})$ is empty;
- 4: the more general case when there are k solutions written in terms of h parameters: in this case $\text{Min}(\underline{v}C = \underline{u})$ has cardinality k and $\text{Min}(\underline{v}C = \underline{0})$ has cardinality h .

Hence in case 1 the minimal solution gives the optimal solution directly. In case 2 it is sufficient to replace the parameters $\mathbf{C}[j]$ with zero to obtain the optimal solution.

Using the command `Reduce` and the condition (11) we implement the algorithm `PrimRel` which computes the primitive relation associated to every primitive collection:

ALGORITHM 4.2: **PrimRel**

INPUT: the variety X and the set $PC(X)$.

OUTPUT: the list of all primitive relations.

1. Let $P \in PC(X)$ be a primitive collection. *Mathematica* builds the linear system:

$$\sum_{y_i \in (G(\Sigma_X) \setminus P)} v_i y_i = \sum_{x \in P} x \quad (12)$$

where x and y_i are known because they are defined by their coordinates, v_i are the unknowns of the system.

2. *Mathematica* solves the system using the command `Reduce`. Its answer is:

$$\{ \mathbf{C}[1], \dots, \mathbf{C}[h] \in \text{Integers}, \mathbf{C}[1] \geq 0, \dots, \mathbf{C}[h] \geq 0, \\ \underline{a}_1(\mathbf{C}[1], \dots, \mathbf{C}[h]), \dots, \underline{a}_k(\mathbf{C}[1], \dots, \mathbf{C}[h]) \}$$

where $\mathbf{C}[j]$ are parameters in $\mathbb{Z}_{\geq 0}$, and $\underline{a}_i(\mathbf{C}[1], \dots, \mathbf{C}[h])$ are all solutions of the linear system.

3. *Mathematica* verifies if there are parameters which describe the solutions:

- (a) If there is no parameter, then it computes the cardinality of the set of solutions:
 - i. if the cardinality equals one then the algorithm gives the primitive relation;
 - ii. if the cardinality equals $k \geq 2$ then the algorithm has to compute the optimal solution:

- A. for every solution $\underline{a}_i(\mathbf{C}[1], \dots, \mathbf{C}[\mathbf{h}])$ defines the set $G(\underline{a}_i)$ as in Definition 3.2;
 - B. it tests if $G(\underline{a}_i)$ generates a cone in Σ_X using the condition (11);
 - C. once found the set $G(\underline{a}_i)$ which generates a cone in Σ_X , the algorithm gives the primitive relation.
- (b) If there are parameters, then *Mathematica* computes the set of minimal solutions replacing the parameters with 0.
- (c) It computes the cardinality of the set of minimal solutions:
- i. if the cardinality equals one then the algorithm gives the primitive relation;
 - ii. if the cardinality equals $k \geq 2$ then the algorithm has to compute the optimal solution:
 - A. for every solution $\underline{a}_i(\mathbf{C}[1], \dots, \mathbf{C}[\mathbf{h}])$ defines the set $G(\underline{a}_i)$ as in Definition 3.2;
 - B. it tests if $G(\underline{a}_i)$ generates a cone in Σ_X using the condition (11);
 - C. once found the set $G(\underline{a}_i)$ which generates a cone in Σ_X , the algorithm gives the primitive relation.

5. Non projective threefolds that become projective after a blow-up

In this section we consider the simplest example of families of smooth, complete, toric threefolds with $\rho = 5$ which becomes projective after a blow-up. There is a detailed description of X in [4], [10] and [14]. In [4] Bonavero studied the projectivity of this family. In [10] Fujino and Payne observe that for some special values of the parameters the variety X of the family is a threefold with no non-trivial nef line bundles, that is their Mori cone coincides with the whole vector space $\mathcal{N}_1(X)_{\mathbb{Q}}$.

Let Σ be the fan in \mathbb{Q}^3 generated by:

$$\begin{aligned} \mathbf{x}[1] &= (-1, d, 0), & \mathbf{x}[2] &= (0, -1, 0), & \mathbf{x}[3] &= (1, 0, 0), \\ \mathbf{x}[4] &= (-1, -1, -1), & \mathbf{x}[5] &= (0, 0, -1), & \mathbf{x}[6] &= (0, 1, 0), \\ \mathbf{x}[7] &= (0, 0, 1), & \mathbf{x}[8] &= (1, 1, c), \end{aligned}$$

where c and d are two integer parameters. Its maximal cones are:

$$\begin{aligned} &\langle x[1], x[2], x[3] \rangle, \quad \langle x[1], x[3], x[4] \rangle, \\ &\langle x[1], x[4], x[5] \rangle, \quad \langle x[1], x[5], x[6] \rangle, \\ &\langle x[1], x[2], x[6] \rangle, \quad \langle x[2], x[3], x[8] \rangle, \\ &\langle x[3], x[4], x[8] \rangle, \quad \langle x[4], x[5], x[8] \rangle, \\ &\langle x[5], x[6], x[7] \rangle, \quad \langle x[5], x[7], x[8] \rangle, \\ &\langle x[6], x[7], x[8] \rangle, \quad \langle x[2], x[6], x[8] \rangle. \end{aligned}$$

Let $X_{c,d}$ be the variety described by Σ .

Observe that we have changed the names of the two parameters because in this case we use Bonavero's notation: he sets $n = (1, 0, 0)$, instead of $n = (1, -1, 0)$ when he fixes the basis of N . This choice implies that the parameter b considered by Fujino and Payne in [10] is equal to $d + 1$, while the other parameter, c , coincides with a .

Bonavero proves that X is non-projective if and only if $c \neq 0$ and $d \neq -1$ (see [4], Proposition 3).

Let us consider the variety X obtained for $c = 1$ and $d = 0$ (respectively $a = 1$ and $b = 1$). The fan of X is generated by:

$$\begin{aligned} x[1] &= (-1, 0, 0), & x[2] &= (0, -1, 0), & x[3] &= (1, 0, 0), \\ x[4] &= (-1, -1, -1), & x[5] &= (0, 0, -1), & x[6] &= (0, 1, 0), \\ x[7] &= (0, 0, 1), & x[8] &= (1, 1, 1). \end{aligned}$$

By computing its primitive relations with the algorithm X is described by:

$$\begin{array}{rcll} x[1] + x[3] & & = & 0 \\ x[1] + x[5] - x[4] - x[6] & = & 0 & \text{C } \gamma_1 \\ x[1] + x[8] - x[6] - x[7] & = & 0 & \text{C } \gamma_2 \\ x[2] + x[5] - x[3] - x[4] & = & 0 & \text{C } \gamma_3 \\ x[2] + x[6] & & = & 0 \\ x[2] + x[8] - x[3] - x[7] & = & 0 & \text{C } \gamma_4 \\ x[3] + x[6] - x[5] - x[8] & = & 0 & \text{C } \gamma_5 \\ x[4] + x[7] - x[1] - x[2] & = & 0 & \text{C } \gamma_6 \\ x[4] + x[8] & & = & 0 \\ x[5] + x[7] & & = & 0 \end{array}$$

In the table the letter C denotes the contractible classes of X .

In this case there are six contractible classes: $\gamma_1, \dots, \gamma_6$ and their associated morphisms φ_{γ_i} are all small contractions.

Next we consider the set \mathcal{I} of all numerical classes of invariant curves in X , and compute their coordinates with respect to a basis B contained in \mathcal{I} (for these computations see [19] and [20]):

$$\begin{array}{ll}
 x[1] + x[3] & \rightarrow -1 \ 1 \ 0 \ 0 \ 0 \\
 x[2] - x[3] - x[4] + x[5] & \rightarrow -1 \ 0 \ -1 \ 0 \ -1 \\
 x[1] - x[4] + x[5] - x[6] & \rightarrow -1 \ 0 \ -1 \ -1 \ 0 \\
 x[2] + x[6] & \rightarrow -1 \ 1 \ 0 \ 1 \ -1 \\
 x[3] + x[4] - x[5] + x[6] & \rightarrow 0 \ 1 \ 1 \ 1 \ 0 \\
 x[1] + x[2] + x[3] - x[7] & \rightarrow 1 \ 0 \ 0 \ 0 \ 0 \\
 -x[2] + x[3] + x[4] - x[7] & \rightarrow 0 \ 1 \ 0 \ 0 \ 0 \\
 x[5] + x[7] & \rightarrow -1 \ 1 \ -1 \ 0 \ -1 \\
 -x[1] + x[4] + x[6] + x[7] & \rightarrow 0 \ 1 \ 0 \ 1 \ -1 \\
 x[3] - x[5] + x[6] - x[8] & \rightarrow 0 \ 0 \ 1 \ 0 \ 0 \\
 x[3] + x[6] + x[7] - x[8] & \rightarrow -1 \ 1 \ 0 \ 0 \ -1 \\
 x[4] + x[8] & \rightarrow 0 \ 1 \ 0 \ 1 \ 0 \\
 x[2] - x[3] - x[7] + x[8] & \rightarrow 0 \ 0 \ 0 \ 1 \ 0 \\
 x[1] - x[6] - x[7] + x[8] & \rightarrow 0 \ 0 \ 0 \ 0 \ 1
 \end{array}$$

In the table we have collected all relations associated to all numerical classes (on the left side) and the the corresponding coordinates with respect to the fixed basis B (on the right side).

Observe that there are two classes of curves such that their coordinates are all non positive with respect to the fixed basis in $\mathcal{N}_1(X)_{\mathbb{Q}} \cong \mathbb{Q}^5$. This allows us to conclude that the variety is not projective.

In fact in this case, the Mori cone of X does not coincide with the whole space $\mathcal{N}_1(X)_{\mathbb{Q}}$, but it contains a linear subspace of $\mathcal{N}_1(X)_{\mathbb{Q}}$ of dimension 4.

If the parameters (c, d) are equal to $(1, -2)$, $(-1, 0)$, $(-1, -2)$ then the corresponding varieties $X_{c,d}$ have the same property of the variety X , that is they are non projective and their Mori cones contain a linear subspace of $\mathcal{N}_1(X)_{\mathbb{Q}}$ of dimension 4.

Notice that Fujino and Payne exclude these four cases choosing $(a, b) \neq (\pm 1, \pm 1)$ because in these cases the Mori cone does not coincide with the whole vector space $\mathcal{N}_1(X)_{\mathbb{Q}}$.

6. A 6-dimensional example

In this example, we present a smooth, complete, toric variety obtained considering some blow-ups of the projective space of dimension 6 along three invariant subvarieties.

Let S be the projective space of dimension 6. We consider the following blow-ups:

$$\begin{aligned} S_1 &= \text{Bl}_{C_1} S \\ S_2 &= \text{Bl}_{C_2} S_1 \\ S_3 &= \text{Bl}_{C_3} S_2 \end{aligned}$$

where C_1, C_2, C_3 are three curves in the varieties S, S_1, S_2 respectively. C_1, C_2, C_3 respectively correspond to the cones:

$$\begin{aligned} \langle x[1], x[2], x[3], x[4], x[5] \rangle, \\ \langle x[1], x[2], x[3], x[4], x[6] \rangle, \\ \langle x[1], x[2], x[3], x[4], x[7] \rangle. \end{aligned}$$

S_3 is a variety of dimension 6 and $\rho = 4$. Its fan is generated by:

$$\begin{aligned} x[1] &= (1, 0, 0, 0, 0, 0), & x[2] &= (0, 1, 0, 0, 0, 0), \\ x[3] &= (0, 0, 1, 0, 0, 0), & x[4] &= (0, 0, 0, 1, 0, 0), \\ x[5] &= (0, 0, 0, 0, 1, 0), & x[6] &= (0, 0, 0, 0, 0, 1), \\ x[7] &= (-1, -1, -1, -1, -1, -1), & x[8] &= (1, 1, 1, 1, 1, 0), \\ x[9] &= (1, 1, 1, 1, 0, 1), & x[10] &= (0, 0, 0, 0, -1, -1). \end{aligned}$$

and has 31 maximal cones. Then we compute the primitive collections. They are:

$$\begin{aligned} \{x[1], x[2], x[3], x[4], x[7]\}, & \{x[5], x[9]\}, \\ \{x[1], x[2], x[3], x[4], x[6]\}, & \{x[5], x[10]\}, \\ \{x[1], x[2], x[3], x[4], x[5]\}, & \{x[6], x[10]\}, \\ \{x[7], x[8], x[9]\}, & \{x[6], x[7], x[8]\}. \end{aligned}$$

Using the algorithm `PrimRel` we see that the fan is described by the

following primitive relations:

$x[5]+x[9]-x[6]-x[8]$	$=0$	C
$x[5]+x[10]-x[7]-x[8]$	$=0$	C
$x[6]+x[10]-x[7]-x[9]$	$=0$	C
$x[6]+x[7]+x[8]$	$=0$	
$x[7]+x[8]+x[9]-x[1]-x[2]-x[3]-x[4]$	$=0$	
$x[8]+x[9]+x[10]-2x[1]-2x[2]-2x[3]-2x[4]$	$=0$	C
$x[1]+x[2]+x[3]+x[4]+x[5]-x[8]$	$=0$	
$x[1]+x[2]+x[3]+x[4]+x[6]-x[9]$	$=0$	
$x[1]+x[2]+x[3]+x[4]+x[7]-x[10]$	$=0$	C

There are five contractible primitive relations, they are denoted by the letter C in the previous table. The associated morphisms are all birational: the morphisms associated to the first three classes have an exceptional locus of dimension 4 and when they are restricted to the exceptional locus, they are \mathbb{P}^1 -bundles. The morphism associated to the fourth class contracts the surface $V(\langle x[1], x[2], x[3], x[4] \rangle)$ to a singular point. The last one is a smooth blow-up with exceptional divisor $V(x[10])$.

Since S_3 is obtained by a blowup of \mathbb{P}^6 then it is projective and we can compute which contractible classes are extremals:

$x[5]-x[6]-x[8]+x[9]$	E
$x[6]-x[7]-x[9]+x[10]$	E
$-2x[1]-2x[2]-2x[3]-2x[4]+x[8]+x[9]+x[10]$	E
$x[1]+x[2]+x[3]+x[4]+x[7]-x[10]$	E
$x[5]-x[7]-x[8]+x[10]$	

There are four extremal classes (denoted by the letter E) and only one non-extremal.

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