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Some Remarks on Fixed Points for Maps which are Expansive along one Direction

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Dedicated to the memory of Fabio Rossi

Summary. - *We present some fixed point theorems for planar maps which satisfy a property of path–expansion along a certain direction. We also show some links between these fixed point theorems and other recent results about covering relations and topological horseshoes.*

1. Introduction

This paper follows a line of research initiated in [23] and motivated by the study in [22] of the Poincaré map associated to some second order nonlinear ODEs with periodic coefficients. In [22], analyzing in the phase–plane the solutions $(u(t), u'(t))$ of the nonlinear scalar Hill's type second order ODE

$$u'' + a(t)g(u) = 0, \tag{1}$$

a stretching property along the paths was detected. More precisely, two topological planar rectangles were found such that every path

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joining two opposite sides of any one of the two rectangles contains a sub–path which is expanded by the flow across the same rectangle or the other one. This was the key lemma in [22] in order to prove the presence of a complicated oscillatory behavior for the solutions of (1), in the case that $a(t)$ is a sign–changing continuous weight and $g : \mathbb{R} \to \mathbb{R}$ is a function having superlinear growth at infinity. It seems rather surprising to notice that similar geometric features were already considered by Kennedy and Yorke in [12] in the framework of the theory of fluid mixing, studying planar functions obtained as composition of a squeezing map and a stirring rotation. The above quoted arc–expansive property along the flow of

$$x' = y, \quad y' = -a(t)g(x)$$

resembles different notions of covering relations associated to Markov partitions [10, 39, 41], as well as some expansive–type conditions [11, 13, 14] which arise in the theory of topological horseshoes. Such concepts have been introduced by many authors in order to find geometrical features associated to chaotic–like dynamics in absence of the more classical hyperbolicity conditions based on the Smale horseshoe [20, Ch. 3].
In [23] and [24] some fixed point theorems for path–stretching maps and their iterates were obtained. Such results were applied to prove the existence of infinitely many periodic solutions and complex dynamics for some second order nonlinear ODEs with periodic coefficients [8, 25]. In [23, 24, 25] the term " path " was used to designate the image of a compact interval through a continuous map: however, this choice turns out to be a little awkward when dealing with the composition of path–stretching maps. It also appears somehow unnatural in the application of the abstract theorems to concrete ODE models. In fact, suppose that we denote by $\zeta(\cdot\,; t_0, z)$ the solution of the first order planar differential system [1]

$$x' = F(t, x),$$

[1] For simplicity we discuss only the two–dimensional case. The theory, however, has been extended to the higher dimension in [30].

satisfying the initial condition $x(t_0) = z$ (for some $t_0 \in \mathbb{R}$) and we look at the Poincaré map

$$\phi : z \mapsto \zeta(t_1\,; t_0, z)$$

(for a fixed $t_1 > t_0$). If we are interested in detecting how a certain curve is transformed by ϕ, a very natural approach consists in studying the composite map

$$\theta \mapsto \phi(\gamma(\theta)) = \zeta(t_1\,; t_0, \gamma(\theta)),$$

where $\gamma : [0, 1] \to \mathbb{R}^2$ is a continuous map which is convenient to call now a " path " (actually, this procedure was applied also in [8, 22, 25], even if therein the path was considered as the range of γ). A clarification of the underlying setting led us in [30] to a more precise definition of the stretching property and, consequently, to a corresponding fixed point theorem which, although equivalent to the main result of [24, 25], looks more feasible from the point of view of the applications. The advantage of such an alternative approach is evident when periodic points (obtained by iterating a given map) are involved. In this direction, new results and applications can be found in [27, 28, 30, 38].

As mentioned before, our results may be classified in the framework of the so–called *topological horseshoes*. Under this name some authors [5, 9, 14] usually mean suitable geometric or topological features for maps, which are weaker than the classical hyperbolicity conditions but are strong enough to imply some kind of chaotic dynamics (like the presence of arbitrary itineraries of coin–tossing type for the iterates of a discrete dynamical system, or the existence of a compact invariant set on which a given map is semiconjugate to the Bernoulli shift on some symbols). The theory of topological horseshoes has encountered a fast growth in the past decades. The main developments have been directed toward two directions.

On the one side the investigation has been addressed to the discovering of very general conditions (both on spaces and maps) which guarantee the existence of a semiconjugation to the Bernoulli shift. With this respect, starting with [13], Kennedy and Yorke produced a theory for homeomorphisms in metric spaces, further extended in [11, 14] to continuous maps. Their main assumptions [14, horseshoe

hypotheses Ω] concern a continuous map f defined on a compact and locally connected subset Q of a separable metric space X. Two nonempty disjoint compact sets $end_0\,, end_1 \subseteq Q$ are selected such that each component of Q intersects both of them. The map f is required to act so that each continuum $\Gamma \subseteq Q$ joining end_0 and end_1 contains at least $m \geq 2$ mutually disjoint sub–continua $P_1\,, \dots, P_m$ [2] such that $f(P_i)$ is a continuum joining end_0 and end_1 in Q, for each $i = 1, \dots, m$. Under these hypotheses the authors prove the existence of a compact invariant set $Q_I \subseteq Q$ on which f is semiconjugate to an m–shift. Sensitive dependence on initial data was shown in [11] as well. However, such general assumptions are not sufficient to guarantee the existence of fixed points (or periodic points) for the map f in the set Q_I. In [14, Example 10] a specific situation of a map defined on $\mathbb{R}^2 \times S^1$ and without any periodic point is presented.
On the other side, by imposing more restrictive conditions on the class of spaces and maps under consideration, various authors have focused their attention on the search of suitable assumptions ensuring the existence of periodic points (possibly fixed points) in the inverse image of a periodic sequence of symbols through the semiconjugation map. Typical results in this direction require some kind of splitting of the euclidean space $\mathbb{R}^N$ into an expansive u–dimensional and a contractive s–dimensional directions. The domain Q of the continuous map f is a homeomorphic image of a rectangular set $B_u \times B_s$, where $B_u \subseteq \mathbb{R}^u$ and $B_s \subseteq \mathbb{R}^s$ are the closed unit balls of the corresponding spaces. The mathematical tools employed for the proof of the existence of fixed points and periodic points come from the Conley index theory and associated homological (or cohomological) invariants [18, 21, 35, 36], from the Lefschetz theory [16, 33, 34], from the topological degree [10, 29, 32, 39, 40, 41]. Recent results based on the use of the Brouwer fixed point theorem or some of its equivalent versions (like the Poincaré–Miranda theorem) can be found in [4, 17, 30]. The general theorems developed via all these different approaches have already found useful applications in many subsequent works (see, for instance, [3, 6, 7, 19, 31, 37, 40] just to quote a few contributions).

[2] In a variant of the theory presented in [11] Γ and its subsets P_i's are required to be only compact.

Our results are intermediate with respect to the two main directions described above. In fact, we follow an approach which, although related to the one by Kennedy and Yorke, specializes their setting to domains Q which are homeomorphic to the unit square of the plane (a particular case of $B_u \times B_s$, with $u = s = 1$). Within such a configuration, we are able to obtain fixed point theorems via elementary tools. Actually, our results still hold for cylindrical domains in $\mathbb{R} \times \mathbb{R}^{N-1}$, that is, for mappings which are expansive along one direction [30]. On the other hand, with respect to some covering relations associated to Markov partitions [10, 39, 41], which imply the existence of fixed points and periodic points, our stretching assumptions appear as more general, at least in the simplified setting of sets homeomorphic to $B_u \times B_s$ with $u = 1$.

In view of the previous considerations, the aim of this paper is that of discussing some definitions and examples which are related to both the approaches described above, in order to possibly clarify the mutual relationships among our results and some of the main theorems developed by other authors. For further applications and remarks, see also the recent article [27].

The work is organized as follows. In Section 2 we introduce the definitions and the main results on maps that expand the paths or the continua. In Section 3 we compare our theory to that of Kennedy and Yorke and to the one by Zgliczyński and Gidea, through some examples and counterexamples. We provide a visual representation of them in Section 4 (Appendix).

2. Definitions and main results

Let X be a metric space. By a *path* γ in X we mean a continuous mapping $\gamma : \mathbb{R} \supseteq [a, b] \to X$. We also set $\bar{\gamma} := \gamma([a, b])$. Clearly, there is no loss of generality in assuming $[a, b] = [0, 1]$. A *sub–path* σ of γ is the restriction of γ to a compact sub–interval of its domain. An *arc* is the homeomorphic image of the compact interval $[0, 1]$.

DEFINITION 2.1. *Let X be a metric space. By an* oriented rectangle $\widetilde{\mathcal{R}}$ *we mean a pair*

$$\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-),$$

where $\mathcal{R} = h(\mathcal{Q}) \subseteq X$ *is the homeomorphic image of the unit square* $\mathcal{Q} := [0,1]^2 \subseteq \mathbb{R}^2$ *through a homeomorphism* h *and*

$$\mathcal{R}^- := \mathcal{R}^-_{left} \cup \mathcal{R}^-_{right}$$

is the disjoint union of two arcs $\mathcal{R}^-_{left}$ *and* $\mathcal{R}^-_{right}$ *(the "left" and the "right" components of* $\mathcal{R}^-$*) contained in the* contour $\vartheta\mathcal{R} := h(\partial\mathcal{Q})$ *of* $\mathcal{R}$*. The set*

$$\mathcal{R}^+ := \overline{\vartheta\mathcal{R} \setminus \mathcal{R}^-}$$

is the disjoint union of two arcs that we denote by $\mathcal{R}^+_d$ *and* $\mathcal{R}^+_u$ *(the components "down" and "up" of* $\mathcal{R}^+$*).*

We remark that, given an oriented rectangle $\widetilde{\mathcal{R}}$ in a metric space X with $\mathcal{R}$ defined by a homeomorphism h, it is always possible to find a homeomorphism $g : \mathcal{Q} \to \mathcal{R} = g(\mathcal{Q}) \subseteq X$, such that

$$\begin{array}{rclrcl} g(\{0\}\times[0,1]) & = & \mathcal{R}^-_{left}\,, & g(\{1\}\times[0,1]) & = & \mathcal{R}^-_{right}\,, \\ g([0,1]\times\{0\}) & = & \mathcal{R}^+_d\,, & g([0,1]\times\{1\}) & = & \mathcal{R}^+_u\,. \end{array} \tag{2}$$

For all the next definitions, the basic setting concerns the following situation: *Let* X *and* Y *be metric spaces and* $\phi : X \supseteq D_\phi \to Y$ *be a map (not necessarily continuous on its whole domain* D_ϕ *). Let* $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^-)$ *and* $\widetilde{\mathcal{B}} = (\mathcal{B}, \mathcal{B}^-)$ *be oriented rectangles with* $\mathcal{A} \subseteq X$ *and* $\mathcal{B} \subseteq Y$*, respectively. Let also*

$$\mathcal{D} \subseteq \mathcal{A} \cap D_\phi\,.$$

Definition 2.2. *We say that* $(\mathcal{D}, \phi)$ *stretches* $\widetilde{\mathcal{A}}$ *to* $\widetilde{\mathcal{B}}$ *along the paths and write*

$$(\mathcal{D}, \phi) : \widetilde{\mathcal{A}} \;\frown\!\!\!\!\!\longrightarrow\; \widetilde{\mathcal{B}},$$

if there exists a compact set $\mathcal{H} \subseteq \mathcal{D}$ *such that*

- ϕ *is continuous on* $\mathcal{H}$,
- *for every continuous map* $\gamma : [0,1] \to \mathcal{A}$ *with* $\gamma(0) \in \mathcal{A}^-_{left}$ *and* $\gamma(1) \in \mathcal{A}^-_{right}$*, there exists* $[t_0, t_1] \subseteq [0,1]$ *such that*

$$\gamma(t) \in \mathcal{H}, \quad \phi(\gamma(t)) \in \mathcal{B}, \quad \forall\, t \in [t_0, t_1]$$

and, either

$$\phi(\gamma(t_0)) \in \mathcal{B}^-_{left}\,, \quad \phi(\gamma(t_1)) \in \mathcal{B}^-_{right}$$

or

$$\phi(\gamma(t_0)) \in \mathcal{B}^-_{right}\,, \quad \phi(\gamma(t_1)) \in \mathcal{B}^-_{left}$$

(in each of such cases we also say that $\phi(\gamma(t_0))$ and $\phi(\gamma(t_1))$ belong to different components of $\mathcal{B}^-$).

Remark 2.3. *As we'll see, the role of the compact set $\mathcal{H}$ is crucial in the results which use Definition 2.2. For instance, in Theorem 2.9 and Theorem 2.10 we are able to prove the existence of a fixed point in the set $\mathcal{H}$. For this reason, when we need to specify the set $\mathcal{H}$, we write*

$$(\mathcal{D}, \mathcal{H}, \phi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow} \widetilde{\mathcal{B}}, \tag{3}$$

instead of $(\mathcal{D}, \phi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow} \widetilde{\mathcal{B}}$. We also note that when (3) holds, the stretching condition

$$(\mathcal{D}, \mathcal{H}', \phi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow} \widetilde{\mathcal{B}}$$

is satisfied for any compact set $\mathcal{H}'$, with $\mathcal{H} \subseteq \mathcal{H}' \subseteq \mathcal{D}$, on which ϕ is continuous.
On the other hand, in the special case when we can take $\mathcal{H} = \mathcal{D} = \mathcal{A}$, we use the simplified notation

$$\phi : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow} \widetilde{\mathcal{B}}$$

to express the fact that $(\mathcal{A}, \phi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow} \widetilde{\mathcal{B}}$.

Definition 2.4. *Let $m \geq 1$ be an integer. We say that $(\mathcal{D}, \phi)$* stretches $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$ along the paths with crossing number m, *and write*

$$(\mathcal{D}, \phi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow}{}^{m} \widetilde{\mathcal{B}},$$

if there exist m pairwise disjoint compact sets $\mathcal{H}_1, \dots, \mathcal{H}_m \subseteq \mathcal{D}$ such that

$$(\mathcal{D}, \mathcal{H}_i, \phi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow} \widetilde{\mathcal{B}}, \quad \forall\, i = 1, \dots, m.$$

When $\mathcal{D} = \mathcal{A}$ we simply write

$$\phi : \widetilde{\mathcal{A}} \stackrel{\frown}{\longrightarrow}{}^{m} \widetilde{\mathcal{B}}.$$

Remark 2.5. *Clearly, Definition 2.4 says something new with respect to Definition 2.2 only in the case $m \geq 2$ (indeed, for $m = 1$ they coincide).*

Definition 2.6. *We say that $(\mathcal{D}, \phi)$ stretches $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$ along the continua and write*

$$(\mathcal{D}, \phi) : \widetilde{\mathcal{A}} \rightleftharpoons\!\!\!\rightarrow \widetilde{\mathcal{B}},$$

if there exists a compact set $\mathcal{H} \subseteq \mathcal{D}$ such that

- *ϕ is continuous on $\mathcal{H}$,*
- *for every continuum (that is, a compact connected set) $\Gamma \subseteq \mathcal{A}$ with $\Gamma \cap \mathcal{A}^-_{left} \neq \emptyset$ and $\Gamma \cap \mathcal{A}^-_{right} \neq \emptyset$, there exists a continuum $\Gamma' \subseteq \Gamma \cap \mathcal{H}$ such that $\phi(\Gamma') \subseteq \mathcal{B}$ and*

$$\phi(\Gamma') \cap \mathcal{B}^-_{right} \neq \emptyset, \quad \phi(\Gamma') \cap \mathcal{B}^-_{left} \neq \emptyset.$$

While the above definition is based on the one given by Kennedy and Yorke in [14], the next one takes inspiration (with some differences) from that of "family of expanders" considered in [11].

Definition 2.7. *We say that $(\mathcal{D}, \phi)$ expands $\widetilde{\mathcal{A}}$ across $\widetilde{\mathcal{B}}$ and write*

$$(\mathcal{D}, \phi) : \widetilde{\mathcal{A}} \ \rightleftharpoons\!\!\!\rightarrow \widetilde{\mathcal{B}},$$

if there exists a compact set $\mathcal{H} \subseteq \mathcal{D}$ such that

- *ϕ is continuous on $\mathcal{H}$,*
- *for every continuum $\Gamma \subseteq \mathcal{A}$ with $\Gamma \cap \mathcal{A}^-_{left} \neq \emptyset$ and $\Gamma \cap \mathcal{A}^-_{right} \neq \emptyset$, there exists a nonempty compact set $P \subseteq \Gamma \cap \mathcal{H}$ such that $\phi(P)$ is a continuum contained in $\mathcal{B}$ and*

$$\phi(P) \cap \mathcal{B}^-_{right} \neq \emptyset, \quad \phi(P) \cap \mathcal{B}^-_{left} \neq \emptyset.$$

Remark 2.8. *Similarly as in Remark 2.3 we can define*

$$(\mathcal{D}, \mathcal{H}, \phi) : \widetilde{\mathcal{A}} \rightleftharpoons\!\!\!\rightarrow \widetilde{\mathcal{B}}$$

and

$$(\mathcal{D}, \mathcal{H}, \phi) : \widetilde{\mathcal{A}} \ \rightleftharpoons\!\!\!\rightarrow \widetilde{\mathcal{B}},$$

when we wish to put in evidence the role of the set $\mathcal{H}$*. The simplified notation* $\phi : \widetilde{\mathcal{A}} \overset{\frown}{\underset{\smile}{\longrightarrow}} \widetilde{\mathcal{B}}$ *and* $\phi : \widetilde{\mathcal{A}} \;\; \overset{\frown}{\underset{\smile}{\rightarrow}}\!\!\rightarrow \widetilde{\mathcal{B}}$ *is used when* $\mathcal{H} = \mathcal{D} = \mathcal{A}$*. Like in Definition 2.4, we can also set (with an obvious meaning)*

$$(\mathcal{D}, \phi) : \widetilde{\mathcal{A}} \overset{\frown}{\underset{\smile}{\longrightarrow}}{}^{m} \widetilde{\mathcal{B}}$$

and

$$(\mathcal{D}, \phi) : \widetilde{\mathcal{A}} \;\; \overset{\frown}{\underset{\smile}{\rightarrow}}\!\!\rightarrow^{m} \widetilde{\mathcal{B}},$$

for some integer $m \geq 1$*. When* $\mathcal{D} = \mathcal{A}$ *we simply write* $\phi : \widetilde{\mathcal{A}} \overset{\frown}{\underset{\smile}{\longrightarrow}}{}^{m} \widetilde{\mathcal{B}}$ *and* $\phi : \widetilde{\mathcal{A}} \;\; \overset{\frown}{\underset{\smile}{\rightarrow}}\!\!\rightarrow^{m} \widetilde{\mathcal{B}}$*.*

For convenience of the reader who is interested in comparing the present work with some previous ones, we observe that in [27, 30] the symbol $\Leftrightarrow\!\!\rightarrow$ was employed with the same meaning of $\overset{\frown}{\longrightarrow}$. Here we have preferred to introduce some new symbols in the effort of providing a more uniform set of notations.

Our first results guarantee the existence and the localization of fixed points in oriented rectangles for some classes of stretching/expansive maps. Indeed, we have:

THEOREM 2.9. *Let* X *be a metric space and* $\phi : X \supseteq D_{\phi} \to X$ *be a map. Let* $\widetilde{\mathcal{A}} = (\mathcal{A}, \mathcal{A}^{-})$ *be an oriented rectangle in* X*. Suppose also that* $\mathcal{D} \subseteq \mathcal{A} \cap D_{\phi}$ *and*

$$(\mathcal{D}, \mathcal{H}, \phi) : \widetilde{\mathcal{A}} \overset{\frown}{\longrightarrow} \widetilde{\mathcal{A}},$$

with $\mathcal{H} \subseteq \mathcal{D}$ *a compact set. Then there exists* $\bar{x} \in \mathcal{H}$ *such that* $\phi(\bar{x}) = \bar{x}$*.*

THEOREM 2.10. *For* X*,* ϕ*,* $\widetilde{\mathcal{A}}$ *and* $\mathcal{D}$ *like in Theorem 2.9, assume that*

$$(\mathcal{D}, \mathcal{H}, \phi) : \widetilde{\mathcal{A}} \overset{\frown}{\underset{\smile}{\longrightarrow}} \widetilde{\mathcal{A}},$$

with $\mathcal{H} \subseteq \mathcal{D}$ *a compact set. Then there exists* $\bar{x} \in \mathcal{H}$ *such that* $\phi(\bar{x}) = \bar{x}$*.*

A preliminary version of these fixed point theorems was obtained in [23, 24, 25]. Actually, in such articles, the authors dealt with a concept of stretching which is intermediate between Definition 2.2 and Definition 2.6 ; namely, instead of continua or paths joining the two components of the $[\cdot]^-$–set, the images of the paths (which are particular kind of continua) were considered. In fact, we warn the reader that in [23, 24, 25] the term "path" was used to indicate the set $\bar{\gamma}$, where $\gamma : [0,1] \to X$ is a continuous map, and the stretching condition (denoted therein by the symbol $\lhd\!\!\rightsquigarrow$) was defined in terms of $\bar{\gamma}$, while, according to the convention of the present paper, it is γ to be a path.

Proofs of Theorem 2.9 can be found in [26] and in [30], where a generalization of it to N–dimensional oriented rectangles was also obtained. On the other hand, we prefer to present here a proof of Theorem 2.10 (since in [24, Table 3.1, p. 123] the validity of such a result was only suggested). To this end, we first recall the following result from plane topology, which was already employed and proved in the previously quoted works.

Lemma 2.11 (Crossing Lemma). *Let $\widetilde{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ be an oriented rectangle in a metric space X and suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a compact set such that*

$$\mathcal{S} \cap \bar{\gamma} \neq \emptyset,$$

for each path $\gamma : [0,1] \to \mathcal{R}$ satisfying $\gamma(0) \in \mathcal{R}^-_{left}$ and $\gamma(1) \in \mathcal{R}^-_{right}$. Then there exists a compact connected set $\mathcal{C} \subseteq \mathcal{S}$ such that

$$\mathcal{C} \cap \mathcal{R}^+_d \neq \emptyset, \quad \mathcal{C} \cap \mathcal{R}^+_u \neq \emptyset.$$

Proof of Theorem 2.10. From Definition 2.1 there exists a homeomorphism $h : \mathbb{R}^2 \supseteq \mathcal{Q} \to h(\mathcal{Q}) = \mathcal{A} \subseteq X$, mapping in a correct way (i.e. as in (2)) the sides of $\mathcal{Q} = [0,1]^2$ into the arcs that compose the sets $\mathcal{A}^-$ and $\mathcal{A}^+$. Then, passing to the planar map $\psi := h^{-1} \circ \phi \circ h$ defined on $D_\psi := h^{-1}(D_\phi) \subseteq \mathcal{Q}$, we can reduce our proof to the search of a fixed point for ψ in the compact set $\mathcal{K} := h^{-1}(\mathcal{H}) \subseteq \mathcal{Q}$. The stretching assumption on ϕ is now translated to

$$(h^{-1}(\mathcal{D}), \mathcal{K}, \psi) : \widetilde{\mathcal{Q}} \stackrel{\frown}{\underset{\smile}{\longrightarrow}} \widetilde{\mathcal{Q}}.$$

On $\widetilde{\mathcal{Q}}$ we consider the natural "left–down–right–up" orientation. For $\psi = (\psi_1, \psi_2)$ and $x = (x_1, x_2)$, we define the compact set

$$\mathcal{S} := \{x \in \mathcal{K} : 0 \leq \psi_2(x) \leq 1,\ x_1 - \psi_1(x) = 0\} \subseteq \mathcal{K}.$$

Consider a path $\gamma = (\gamma_1, \gamma_2) : [0,1] \to \mathcal{Q}$ with $\gamma_1(0) = 0$ and $\gamma_1(1) = 1$. Clearly, the set $\bar{\gamma} \subseteq \mathcal{Q}$ is a continuum having nonempty intersection with the two components of $\mathcal{Q}^-$. By the stretching assumption, there exists a sub–continuum $\Gamma' \subseteq \bar{\gamma}$ such that $\Gamma' \subseteq \mathcal{K}$, $\psi(\Gamma') \subseteq \mathcal{Q}$ and there are $p, q \in \Gamma'$ such that $\psi_1(p) = 0 \leq p_1$ and $\psi_1(q) = 1 \geq q_1$, for $p = (p_1, p_2)$, $q = (q_1, q_2)$. Bolzano's theorem ensures the existence of $w = (w_1, w_2) \in \Gamma'$ such that $w_1 - \psi_1(w) = 0$. Since $\psi_2(\Gamma') \subseteq [0,1]$, we conclude that $w \in \mathcal{S}$ and therefore $\mathcal{S} \cap \bar{\gamma} \neq \emptyset$.

According to Lemma 2.11, there exists a compact connected set $\mathcal{C} \subseteq \mathcal{S}$ such that $\mathcal{C} \cap \mathcal{Q}_d^+ \neq \emptyset$ and $\mathcal{C} \cap \mathcal{Q}_u^+ \neq \emptyset$.

By definition of $\mathcal{S}$ we know that $\psi_2(x) \in [0,1], \forall\, x \in \mathcal{C}$. Hence, for every $p = (p_1, p_2) \in \mathcal{C} \cap \mathcal{Q}_d^+$ we have $p_2 - \psi_2(p) \leq 0$ and, similarly, $q_2 - \psi_2(q) \geq 0$ for every $q = (q_1, q_2) \in \mathcal{C} \cap \mathcal{Q}_u^+$. We can apply again Bolzano's theorem in order to find a point $z = (z_1, z_2) \in \mathcal{C}$ such that $z_2 - \psi_2(z) = 0$. From $\mathcal{C} \subseteq \mathcal{S} \subseteq \mathcal{K}$, we have also that $z_1 - \psi_1(z) = 0$ with $z \in \mathcal{K}$ and therefore $h(z)$ is a fixed point for ϕ in $\mathcal{H} \subseteq \mathcal{A}$. □

Remark 2.12. *As already mentioned, we point out that Theorem 2.9 and Theorem 2.10 provide not only the existence of fixed points, but also their localization. This fact turns out to be useful when the stretching condition* $\stackrel{\frown}{\longrightarrow}^{m}$ *(or, respectively,* $\stackrel{\frown}{\underset{\smile}{\longrightarrow}}^{m}$ *) is satisfied with a crossing number* $m \geq 2$*. In this case, we have the existence of a fixed point in each of the sets* $\mathcal{H}_i$*'s from Definition 2.4 and therefore there are at least* m *fixed points for* ϕ *in* $\mathcal{D}$*.*

3. Comparison with other approaches and some remarks

A natural question which arises now is whether a version of Theorem 2.9 and Theorem 2.10 holds with respect to the stretching condition $\stackrel{\frown}{\underset{\smile}{\rightsquigarrow}}$. The answer, in general, is negative both as regards the existence of fixed points and also with respect to their localization (in case that fixed points do exist).

Concerning the existence of fixed points, a possible counterexample is described in Fig. 1 in the Appendix and is inspired to the bulging horseshoe in [11, Fig. 4]. For sake of conciseness, we prefer to present it by means of a series of graphical illustrations in Section 4 (Figg. 1–6): we point out, however, that it is based on a concrete

definition of a planar map (whose form, although complicated, can be explicitly given in analytical terms).
With respect to the localization of fixed points, a possible counterexample can be obtained by suitably adapting a one–dimensional map to the planar case. Indeed, if $f : \mathbb{R} \supseteq [0,1] \to \mathbb{R}$ is any continuous function, we can set

$$\phi(x_1, x_2) := (f(x_1), x_2) \tag{4}$$

and have a continuous planar map defined on the unit square $[0,1]^2$ of $\mathbb{R}^2$, inheriting all the interesting properties of f. Note that, in this special case, any fixed point x^* for f generates a vertical line (x^*, s) (with $s \in [0,1]$) of fixed points for ϕ. The more general situation of a map ϕ defined as

$$\phi(x_1, x_2) := (f(x_1), g(x_2)),$$

for $g : [0,1] \to [0,1]$ a continuous function, could be considered as well.

In view of the above discussion, we define now a continuous function $f : [0,1] \to [0,1]$ of the form

$$f(s) := \begin{cases} \frac{1-c}{a}\, s + c, & 0 \le s < a \\ \frac{1}{a-b}(s-b), & a \le s \le b \\ \frac{d}{1-b}(s-b), & b < s \le 1 \end{cases} \tag{5}$$

where a, b, c, d are fixed constants such that $0 < a < b < 1$ and $0 < c < d < 1$.

For $\widetilde{\mathcal{A}}$ the unit square oriented in the standard left–right manner, $\mathcal{D} = \mathcal{A}$ and ϕ as in (4), it holds that $\phi : \widetilde{\mathcal{A}} \Bumpeq\!\!\!\longrightarrow \widetilde{\mathcal{A}}$. In particular, we can write both

$$(\mathcal{D}, \mathcal{H}_1, \phi) : \widetilde{\mathcal{A}} \Bumpeq\!\!\!\longrightarrow \widetilde{\mathcal{A}}, \quad \text{for } \mathcal{H}_1 = [a,b]$$

and

$$(\mathcal{D}, \mathcal{H}_2, \phi) : \widetilde{\mathcal{A}} \Bumpeq\!\!\!\longrightarrow \widetilde{\mathcal{A}}, \quad \text{for } \mathcal{H}_2 = [0,a] \cup [b,1].$$

In the former case, we also have

$$(\mathcal{D}, \mathcal{H}_1, \phi) : \widetilde{\mathcal{A}} \Bumpeq\!\!\!\longrightarrow \widetilde{\mathcal{A}}$$

and therefore, consistently with Theorem 2.10, there exist fixed points in $\mathcal{H}_1$. On the other hand, there are no fixed points for ϕ in $\mathcal{H}_2$. Thus, the localization of the fixed points is not guaranteed when only the relation $\stackrel{\frown}{\hookrightarrow}$ is satisfied. The same example could be slightly modified in order to have that $\phi : \widetilde{\mathcal{A}} \stackrel{\frown}{\hookrightarrow}^{m} \widetilde{\mathcal{A}}$ for an arbitrary $m \geq 2$, but only one fixed point does exists.

We finally observe that, playing with the coefficients a, b, c, d and choosing a suitable compact set $\mathcal{H}_3 \subset [0, 1]$, it is possible to have

$$(\mathcal{D}, \mathcal{H}_3, \phi) : \widetilde{\mathcal{A}} \stackrel{\frown}{\hookrightarrow} \widetilde{\mathcal{A}},$$

for a case in which neither ϕ, nor ϕ^2 possess fixed points in $\mathcal{H}_3$. The set $\mathcal{H}_3$ will consist of the union of some compact sub–intervals of $\mathcal{H}_1$ and $\mathcal{H}_2$. A graphical representation of f and f^2 is given in Fig. 7 of the Appendix.

So far we have compared our approach to some aspects coming from the theory of topological horseshoes according to Kennedy and Yorke. We consider now a splitting of the ambient space into compressive and expansive directions in order to show the relationship among our theorems and some covering relations for Markov–type partitions (as already discussed in the Introduction). Due to the broadness of the literature in this area, we focus our attention only on a few papers where such kind of results are presented with great generality and, at the same time, in a form which is easily comparable to ours.

First of all, we define a (u, s)*–cell* as a quadruple $\widehat{\mathcal{N}} = (\mathcal{N}, u, s, c_{\mathcal{N}})$, where $\mathcal{N} \subseteq \mathbb{R}^N$ is a compact set, $u + s = N$ with $u, s \geq 1$ and $c_{\mathcal{N}} : \mathbb{R}^N \to \mathbb{R}^N = \mathbb{R}^u \times \mathbb{R}^s$ is a homeomorphism with $c_{\mathcal{N}}(\mathcal{N}) = B_u \times B_s$. Our definition of (u, s)–cell coincides with that of h-set given by Zgliczyński and Gidea [41, Definition 1]. As in [41] we also introduce the boundary sets

$$\mathcal{N}^- := {c_{\mathcal{N}}}^{-1}(\,(\partial B_u) \times B_s\,), \quad \mathcal{N}^+ := {c_{\mathcal{N}}}^{-1}(\,B_u \times \partial B_s\,),$$

and the strips

$$\mathcal{S}_u(\mathcal{N}) := {c_{\mathcal{N}}}^{-1}(\,B_u \times \mathbb{R}^s\,), \quad \mathcal{S}_s(\mathcal{N}) := {c_{\mathcal{N}}}^{-1}(\,\mathbb{R}^u \times B_s\,).$$

Then, the following fixed point theorem holds for a continuous map

$$\psi : \mathbb{R}^N \supseteq D_\psi \supseteq \mathcal{R} \to \mathbb{R}^N.$$

Theorem 3.1 (Pireddu & Zanolin [29, Corollary 3.1]). *Let* $\widehat{\mathcal{R}} = (\mathcal{R}, u, s, c_{\mathcal{R}})$ *be a* (u,s)*–cell with* $\mathcal{R} \subseteq D_\psi$. *Suppose that* $u = 1$ *and define*

$$\mathcal{R}^-_{le} := {c_{\mathcal{R}}}^{-1}((-\infty,-1] \times \mathbb{R}^{N-1}), \quad \mathcal{R}^-_{re} := {c_{\mathcal{R}}}^{-1}([1,+\infty) \times \mathbb{R}^{N-1}),$$

as well as

$$\mathcal{R}^-_{l} := {c_{\mathcal{R}}}^{-1}(\{-1\} \times B_{N-1}), \quad \mathcal{R}^-_{r} := {c_{\mathcal{R}}}^{-1}(\{1\} \times B_{N-1}).$$

Assume also

$$\psi(\mathcal{R}) \subseteq \mathcal{S}_s(\mathcal{R}) \tag{6}$$

and, either

$$\psi(\mathcal{R}^-_l) \subseteq \mathcal{R}^-_{le} \text{ and } \psi(\mathcal{R}^-_r) \subseteq \mathcal{R}^-_{re},$$

or

$$\psi(\mathcal{R}^-_l) \subseteq \mathcal{R}^-_{re} \text{ and } \psi(\mathcal{R}^-_r) \subseteq \mathcal{R}^-_{le}.$$

Then ψ *has at least a fixed point in* $\mathcal{R}$.

The situation expressed by the assumptions of Theorem 3.1 was summarized in [29] by the symbol $\psi : \widehat{\mathcal{R}} \Leftrightarrow \widehat{\mathcal{R}}$. We also recall that a more general version of Theorem 3.1 was proved in [29, Corollary 2.1 and Remark 2.5], by assuming

$$\psi(\mathcal{R}^+) \subseteq \mathcal{S}_s(\mathcal{R})$$

in place of (6). From some point of view, however, condition (6) is more convenient as it fits well with respect to the composition of maps.

Theorem 3.1 is related to preceding results by Arioli and Zgliczyński [3], Pokrovskii, Szybka and McInerney [32], Zgliczyński and Gidea [41]. All of them are obtained via topological degree tools and, if applied to the setting of Theorem 3.1, they would need strict inclusions in all the assumptions.

The hypothesis $u = 1$ in our result is rather restrictive in contrast to the generality of the statements in [29, 32, 41] (where u, s with $u + s = N$ are arbitrary), but it allows to express the theorem in a simpler form. With this respect, we recall that in [41] the authors define, for two h-sets $\widehat{\mathcal{N}}$ and $\widehat{\mathcal{M}}$ (still denoted by $\mathcal{N}$ and $\mathcal{M}$) and for a continuous map $f : \mathcal{N} \to \mathbb{R}^N$, the *covering relation*

$$\mathcal{N} \stackrel{f}{\Longrightarrow} \mathcal{M},$$

which requires the existence of a continuous homotopy $h : [0,1] \times B_u \times B_s \to \mathbb{R}^u \times \mathbb{R}^s$, with

$$h_0 = c_{\mathcal{M}} \circ f \circ c_{\mathcal{N}}{}^{-1}$$

and such that

$$\begin{aligned} h([0,1], (\partial B_u) \times B_s) \cap (B_u \times B_s) &= \emptyset, \\ h([0,1], B_u \times B_s) \cap (B_u \times \partial B_s) &= \emptyset \end{aligned}$$

(see [41, p. 36]). As to h_1, it is assumed that

$$h_1(p,q) = (A(p), 0), \qquad \text{for } (p,q) \in B_u \times B_s\,,$$

with $A : \mathbb{R}^u \to \mathbb{R}^u$ a linear map satisfying

$$A(\partial B_u) \subseteq \mathbb{R}^u \setminus B_u$$

(a more general function A with nonzero degree could be considered as well). Then, in the special case $\mathcal{M} = \mathcal{N}$, the assumption $\mathcal{N} \stackrel{f}{\Longrightarrow} \mathcal{M}$ implies the existence of a fixed point for f in $\mathcal{N}$. Results in this direction can be also found in [1] with f the Poincaré map of a differential system which is expansive along some components and compressive with respect to the remaining ones (in that case h is the homotopy along the trajectories and A the identity map) and in [2] for multivalued maps.

Now we restrict the previous results (in particular, Theorem 3.1) to the planar case in order to enter a common framework for a comparison with the theory exposed in Section 2. Hence, from now on, we consider only the case

$$u = s = 1.$$

Arguing like in [29, Remark 3.4], one can verify that the assumptions of Theorem 3.1 imply

$$(\mathcal{R}, \psi) : \widetilde{\mathcal{R}} \stackrel{\frown}{\longrightarrow} \widetilde{\mathcal{R}},$$

for $\widetilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$ defined by setting $\mathcal{R}^- := \mathcal{R}_l^- \cup \mathcal{R}_r^-$, with $\mathcal{R}_l^-$ and $\mathcal{R}_r^-$ as in Theorem 3.1. Therefore Theorem 3.1 can be obtained as a consequence of Theorem 2.9 or Theorem 2.10. On the other hand, an example like the one described in Fig. 8 shows a case in which the assumptions of Theorem 3.1 are not satisfied for a given rectangle $\mathcal{R}$, while those of Theorem 2.9 (as well as Theorem 2.10) still hold (see the Appendix for a more detailed discussion about this point).

As a final remark we observe that Theorem 3.1 (and a fortiori Theorem 2.9) is sharp. To explain more precisely what we mean, we refer to the example illustrated in Figg. 10–11.

Conclusions. In this paper we have introduced some new stretching definitions, in order to compare our approach in [27, 30] to some results related to the horseshoe hypotheses by Kennedy and Yorke [13, 14] and Kennedy, Koçak and Yorke [11]. Subsequently, we have also shown the connection among our point of view and some theorems related to the concept of covering relations for Markov–type partitions considered by Zgliczyński in [40] and further developed in some other papers [10, 32, 41]. For sake of brevity, we have confined ourselves to the discussion of the aspects related to the existence, multiplicity and localization of fixed points. Hence, our contribution is just a first step toward an attempt of unifying some features common to all these different but linked theories. Actually they are far richer, as they include also the appropriate tools for the detection of complex dynamics (meant as the presence of periodic orbits, coin–flipping itineraries, semiconjugation or conjugation to the Bernoulli shift). Investigations in such directions will be pursued elsewhere.

4. Appendix

In this last section we provide a visual representation of the examples and counterexamples about the existence (Figg. 1–6) and the localization of fixed points (Fig. 7) which have been discussed in Section 3. In Figg. 8–11 we describe situations in which the approaches related to Theorem 2.9 and Theorem 3.1 are compared.

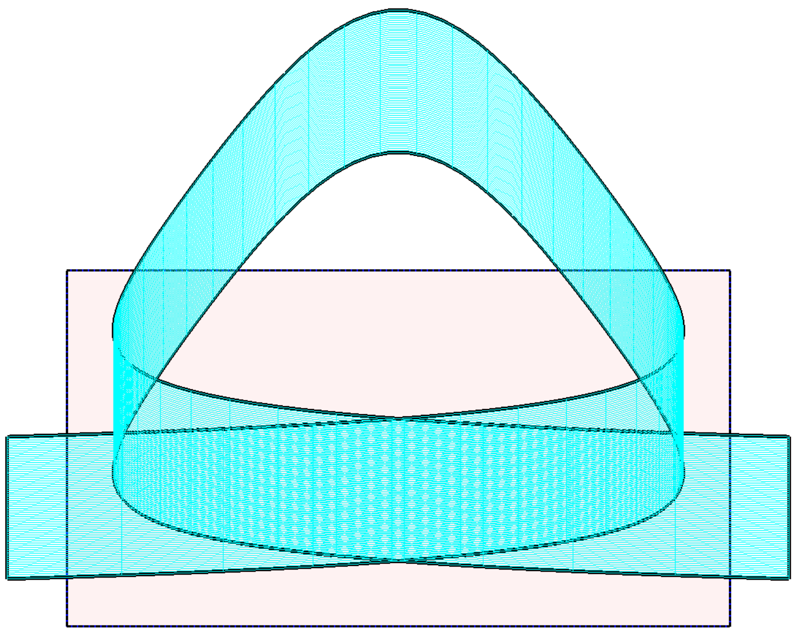

Figure 1: A rectangle (which is oriented by taking as $[\cdot]^-$–set the union of its left and right vertical segments) is deformed onto a ribbon bent across the rectangle itself. The left and right sides of the domain are homeomorphically transformed onto the two endings of the ribbon. This is an example of the case $\phi : \widetilde{\mathcal{A}} \;\; \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{A}}$ with $\mathcal{D} = D_\phi = \mathcal{A}$.

Figure 2: With respect to the example depicted in Fig. 1, we have put in evidence the compact set $\mathcal{H}$ from Definition 2.7, which consists of the union of the two darker regions close to the left and the right edges of the rectangle.

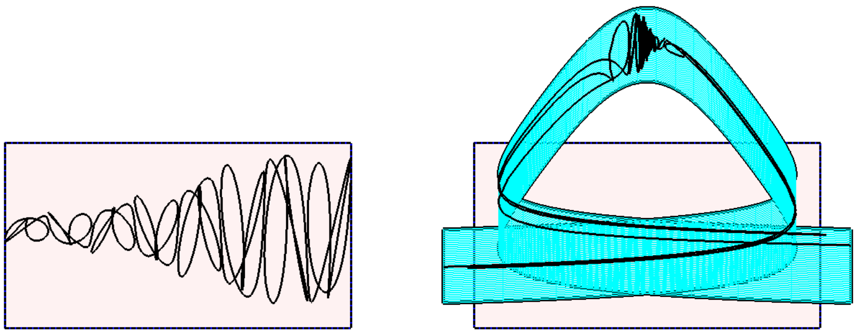

Figure 3: In the present picture we show how a continuum Γ (here we have chosen the image of a continuous curve) crossing the rectangle $\mathcal{A}$ and joining the two components of $\mathcal{A}^-$ (which are the left and the right sides of the boundary) is transformed by ϕ onto a continuum $\phi(\Gamma)$ rolling up along the bent ribbon. The intersection between $\phi(\Gamma)$ and the rectangle $\mathcal{A}$ is a continuum which joins again the two components of $\mathcal{A}^-$. More precisely it is the image of the compact set $P = \Gamma \cap \mathcal{H}$ (where $\mathcal{H}$ is the darker set in Fig. 2). Actually, for the validity of the stretching relation, any sub–continuum contained in $\phi(\Gamma) \cap \mathcal{A}$ and joining the left and the right sides of $\mathcal{A}$ would be enough.

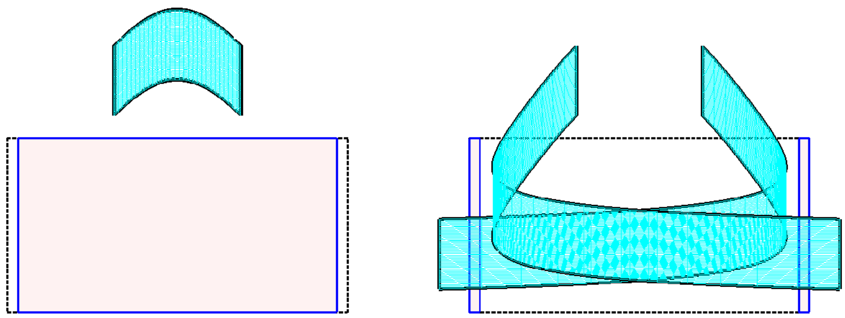

Figure 4: In regard to the example portrayed in the previous figures, we have emphasized how some sub–rectangles are transformed by the map ϕ. In particular, the main central part of the rectangle is pushed out of $\mathcal{A}$ (picture at the left), while the two narrow rectangles near the left and right sides of it are stirred onto two overlapping bent strips (picture at the right). This can happen in two different ways, as shown in Figure 5. One can pass from a configuration to the other one by considering, instead of the map ϕ, the related map $(x_1, x_2) \mapsto \phi(-x_1, x_2)$.

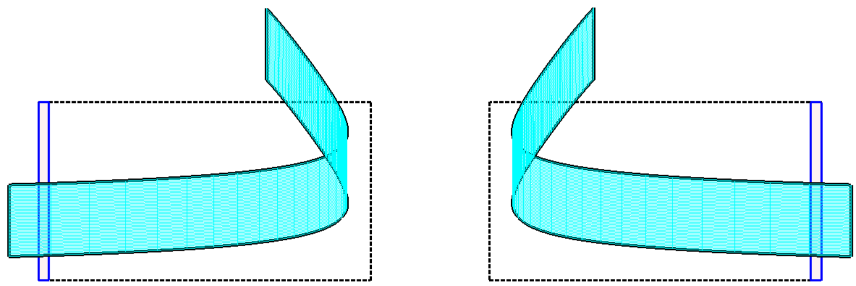

Figure 5: In this case, we enter in the setting of Theorem 2.10. By applying our result to each of the two narrow sub–rectangles of the domain $\mathcal{A}$ (provided they are suitably oriented in an obvious left–right manner), we can prove the existence of at least two fixed points for the planar map ϕ in $\mathcal{A}$ (each fixed point lies in one of the two narrow rectangles).

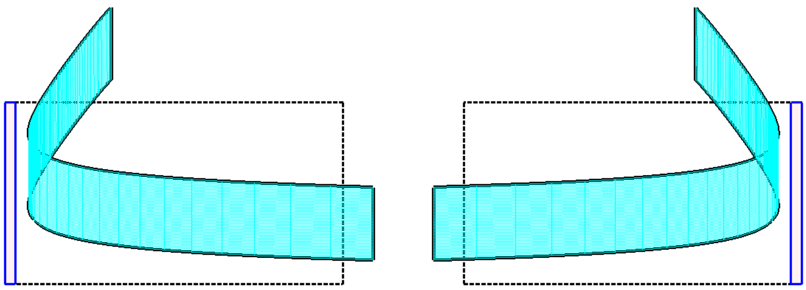

Figure 6: In the situation depicted above, we don't have any fixed point. In fact, the central main part of the rectangle is mapped into a hat–like figure outside the domain (see the left of Fig. 4) and, at the same time, each of the two narrow rectangles, which constitute the remaining part of the domain, is mapped onto a set which is disjoint from itself.

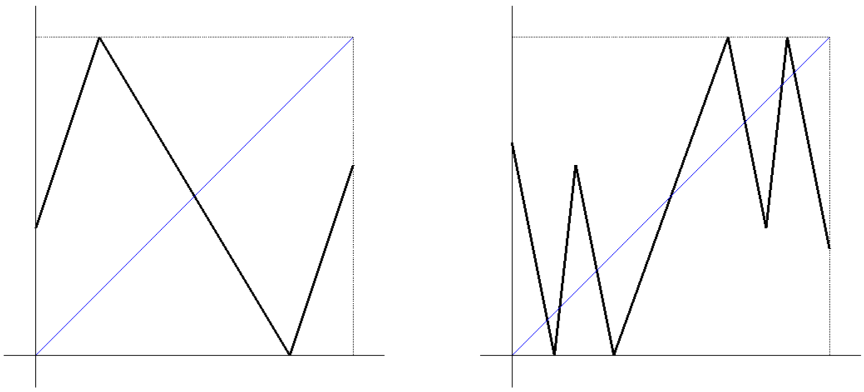

Figure 7: Example of the graph of a function f (left) and of its iterate f^2 (right), with f defined as in (5). We have put in evidence the line $y = x$ in order to show the fixed points and the points of period 2.

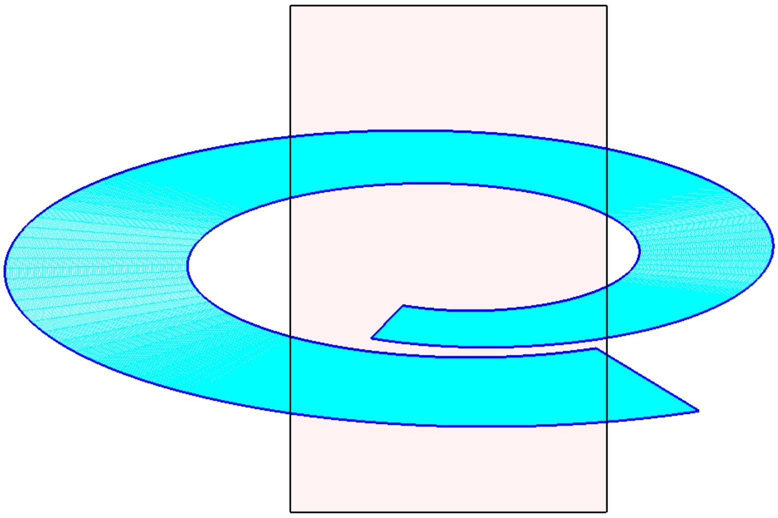

Figure 8: A rectangle $\mathcal{R}$, oriented in the usual left–right manner, is deformed into a spiral–like strip by a planar map ψ (in our example ψ is a homeomorphism). Each path γ, with range in $\mathcal{R}$ and such that $\gamma(0)$ and $\gamma(1)$ belong to the left and the right sides of $\mathcal{R}$, is transformed by ψ into a path which runs along the spiral–like strip and therefore it crosses $\mathcal{R}$ at least once (through the upper intersection between $\mathcal{R}$ and $\psi(\mathcal{R})$). Thus the existence of a fixed point for ψ in $\mathcal{R} \cap \psi(\mathcal{R})$ follows from Theorem 2.9. On the other hand, results based on the degree approach (like Theorem 3.1) cannot be directly applied since the endings of $\psi(\mathcal{R})$ do not lie outside $\mathcal{R}$ itself.

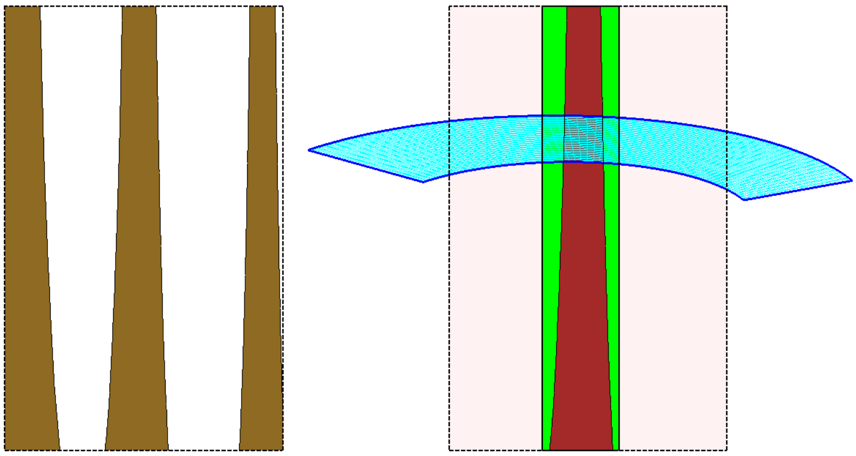

Figure 9: With respect to the example described in Fig. 8, we have drawn in a dark color the parts of $\mathcal{R}$ which are mapped into $\mathcal{R}$ itself by ψ, i.e. the set $\mathcal{R} \cap \psi^{-1}(\mathcal{R})$ (picture at the left). The central one among these three regions is also the set $\mathcal{H}$ such that $(\mathcal{R}, \mathcal{H}, \psi)$ stretches $\mathcal{R}$ to itself along the paths. This fact suggests to look for a suitable sub–rectangle $\mathcal{R}' \subseteq \mathcal{R}$ containing $\mathcal{H}$ and such that $\psi(\mathcal{R}')$ crosses $\mathcal{R}'$ in the right manner (as we did in the picture at the right). It is easy to see that both Theorem 2.9 and Theorem 3.1 can be applied for ψ restricted to $\mathcal{R}'$.

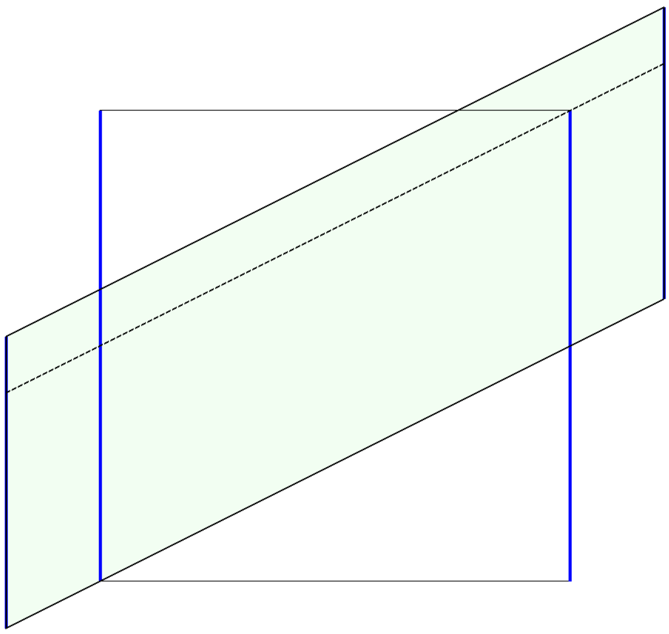

Figure 10: The unit square $\mathcal{Q} = [0,1]^2$ with the canonical left–right orientation is deformed by a planar homeomorphism onto a parallelogram across it. In particular, we choose an orientation–preserving homeomorphism which maps $\{0\}\times[0,1]$ to the left edge of the image (and, similarly, $\{1\}\times[0,1]$ gets mapped to the right edge). The dashed line represents the upper limit for the applicability of Theorem 3.1. Indeed, as soon as the upper side of the parallelogram goes beyond the dashed line (like in the present figure), we can define a suitable homeomorphism of $\mathcal{Q}$ onto the parallelogram without fixed points, as shown in the next picture.

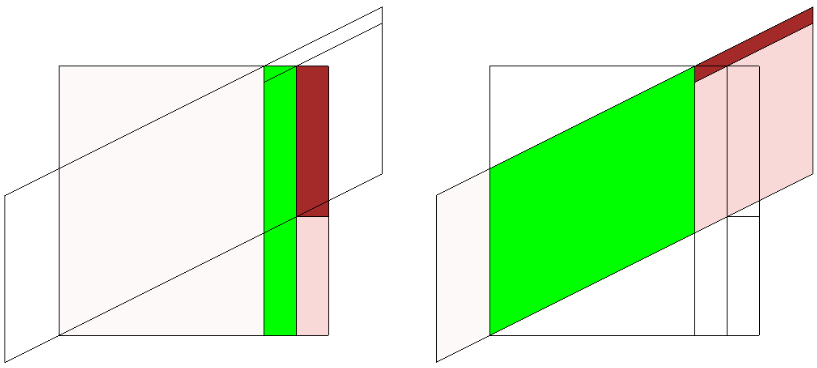

Figure 11: We describe a possible fixed point free homeomorphism ψ of $\mathcal{Q}$ onto the parallelogram. We have split the domain in four regions drawn in different colors (left figure). The images through ψ of the four zones (right figure) are depicted by using the same color as the corresponding starting region. It is easy to define ψ so that there are no fixed points in any of these sub–domains.

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