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Minimal Primes over $P_3(M)$

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Dedicated to the memory of Fabio Rossi

Summary. - *We provide the minimal primes over the ideal generated by 3×3 subpermanents of an $m \times n$ Hankel matrix.*

1. Introduction and Preliminars

Definitions. The name permanent has been introduced the first time in 1812 by Cauchy and Binet, separately. Cauchy [3] introduced it while he studied a special type of symmetric alternating function, that Muir [11] later called permanent. In the same period Binet [1] did that too and he provided also formulas to compute permanents. Later, Schur studied permanents as a specific type of generalized function: the Schur function. Now permanents have applications in many fields of Applied Mathematics, like Combinatorics, Probability, Invariant Theory, Physics and so on.

Specifically, given an $m \times n$ matrix $M = (m_{ij})$ in a commutative ring R, with $m \leq n$, the permanent of M is defined by

$$Per(M) = \sum_{\sigma} m_{1\sigma(1)} \cdots m_{m\sigma(m)}$$

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where σ is an injective function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$.

If $m = n$ then the permanent is the same thing as the determinant except in the lack of minus signs in the expansion. In this case we denote it with $per(M)$ instead of $Per(M)$. We are interesting in the study of properties of permanental ideals.

Given an $m \times n$ matrix $M = (m_{ij})$ in a commutative ring R, with $m \leq n$, and $t \in \{1, \ldots, m\}$, let $P_t(M)$ denote the ideal generated by all the $t \times t$ subpermanents of M.

We are interested in the case when M is a matrix of variables in a polynomial ring in several variables over a field K. We note that if the characteristic of the field is equal to 2 then the permanental ideal is the same thing of the determinantal ideal. So, we suppose that the field has characteristic different from 2. In particular we set $char(K) = 0$.

Backgrounds. Determinantal ideals have been widely studied and the results on primary decomposition of determinantal ideals are well-known. The first step in this direction is the study of the bounds on the height of minimal prime ideals over a determinantal ideal. The properties of the ideal generated by minors of generic matrices, generic symmetric matrices and generic antisymmetric matrices have been studied in works of De Concini, Eisenbud and Procesi [5], Bruns, Vetter [2] and many others. Recently, Watanabe [13] before and Conca [4] later, studied Hankel matrices. In particular, it is known that the ideals generated by minors of generic matrices (see [5]), and of Hankel matrices (see [4]) are prime.

Permanental ideals have not received the same attention, maybe because permanental theory is more connected to Combinatorics than Geometry. The natural curiosity for permanental ideals is to understand up to which degree one may expect a behavior similar to that of determinantal ideals. The first authors interested in permanental ideals are Eisenbud and Sturmfels [6], Niermann [12], Laubenbacher and Swanson [10], Kirkup [9] Grieco, Guerrieri and Swanson [8].

The Ph.D. Thesis of Niermann [12] in 1997 and a paper of Laubenbacher and Swanson [10] in 2000 are on the ideals generated by 2×2 subpermanents of a generic matrix. Niermann computed the radical of this ideals, while Laubenbacher and Swanson [10] found the irredundant primary decomposition and a Gröbner basis. Recently Kirkup [9] gave some indications on associated primes of the ideal generated by 3×3 permanents of a generic matrix (not complete list). In our recent work [8] we studied the ideal generated by 2×2 subpermanents of an Hankel matrix, in particular we provided the irredundant primary decomposition and a Gröbner basis.

We recall briefly the results that we are interested in.

Laubenbacher and Swanson [10]. In this work the authors studied the properties of the ideal generated by 2×2 subpermanents of a generic matrix M, whose entries are different variables in a polynomial ring over a field. In particular they provided an irredundant primary decomposition of $P_2(M)$ and a reduced Gröbner basis.

The first results concern the monomials appearing in $P_2(M)$, in particular they proved that the ideal $P_2(M)$ contains many monomials. We underline that the monomials in $P_2(M)$ are the products of three variables, two of which lie in the same row (resp. column) and all three lie in distinct column (resp. row) (see Lemmas 1.1 of [10]).

The existence of many monomials into an ideal plays an important role for the research of minimal primes, so that these results are on the basis of all works on permanental ideals.

They were able to provide the minimal primes over $P_2(M)$, proving that their number and structure change with respect the size of the matrix (see Theorem 4.1 and Corollary 4.3 of [10]). They also proved that the primary components of $P_2(M)$ corresponding to the minimal primes over $P_2(M)$ are exactly the minimal primes themselves (see Proposition 5.1 of [10]).

Moreover, they showed that $P_2(M)$ has a unique embedded component if and only if $m, n \geq 3$ (see Corollary 5.6 of [10]).

Related to the Gröbner basis for the ideal $P_2(M)$, Laubenbacher and Swanson proved that there is a unique pattern of Gröbner basis for $P_2(M)$ for any size of a generic matrix (see Theorem 3.1 of [10]).

Kirkup [9]. One of the goals of this work is to consider the minimal primes over the ideal generated by 3×3 subpermanents of generic matrices. The author conjectured that in a field K of characteristic 0 or strictly greater than t, the minimal primes over the ideal generated by $t \times t$ subpermanents of an $m \times n$ generic matrix must either contain a column of the generic matrix or the $t-1 \times t-1$ subpermanents of some $m-1$ rows (see Conjecture 1 of [9]). In particular he proved the conjecture in the case $t = 3$ (see Theorem 8 of [9]).

Grieco, Guerrieri and Swanson [8]. In our recent work, we analyzed the properties of the ideal generated by 2×2 subpermanents of an Hankel matrix. Specifically, let $m \leq n$ be positive integers and $R = K[x_1, x_2, \ldots, x_{m+n-1}]$ the polynomial ring in $m+n-1$ variables over a field K. An $m \times n$ Hankel matrix is

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_3 & x_4 & \cdots & x_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x_m & x_{m+1} & x_{m+2} & \cdots & x_{m+n-1} \end{bmatrix}.$$

The first step was to verify that also in case of Hankel matrices the ideal $P_2(M)$ contains many monomials (see Lemmas 2.1 and 2.2 of [8]). Thanks to that, we provided the minimal primes over the ideal $P_2(M)$ showing also that their number is equal to 2 and their structure is invariant with respect to the changing of the size and the shape of the matrix M (see Proposition 3.1 [8]). This represents the first relevant difference between the results on generic matrices. We provided also the minimal components showing that their structure is independent from the size and the shape of the matrix, too (see Proposition 3.2 [8]). The embedded component appears in many but not in every case, showing that it depends on the shape of the matrices (see Sections 5 and 6 of [8]). This work permitted us to give an irredundant primary decomposition of the ideal $P_2(M)$ (see Theorem 4.3 of [8]).

Related to the Gröbner basis for $P_2(M)$ we provided it showing that it changes with respect the size and the shape of the matrix (see Theorem 2.7, Proposition 2.8, 2.9 and 2.10 of [8]). This is different from the results of [10].

Overview. We are motivated by the results of [10], [9] on generic matrices and by the different results explained in [8].

We discuss the minimal primes over the ideal generated by 3×3 subpermanents of Hankel matrices.

The technique used are different from those used in [8]. The basic idea lies in a classical result of Linear Algebra (see Section 2). To compute the minimal primes over $P_3(M)$, we analyze case by case with respect to the size and the shape of the matrix. In particular we observe that

- $P_3(M)$ for a 3×3 Hankel matrix is a prime ideal;
- if M is a 3×4 Hankel matrix, SINGULAR [7] provides a primary decomposition of $P_3(M)$ with 8 minimal primes that are not easily controllable;
- if M is a 4×4 Hankel matrix, SINGULAR [7] does not finish the computation of $P_3(M)$.

So we consider only the cases of $m \times n$ Hankel matrices having at least 3 rows and 5 columns. We give the final result in Theorem 3.6.

We conjecture the possible structure of the minimal primes over the ideal generated by $t \times t$ subpermanents of Hankel matrices.

2. Some Linear Algebra

The key-point of our computation is Determinant Trick whose proof follows by a classical result of Linear Algebra, namely the Cramer's rule. We recall them but omit the proofs.

LEMMA 2.1 (Cramer's Rule). *Let $M = (m_{ij})$ be an $n \times n$ matrix in any commutative ring R. Let $adj(M)$ be the adjoint of M, i.e. the $n \times n$ matrix whose (j, i)-th entry is $((-1)^{i+j}\Delta_{ij})$ where Δ_{ij} is the determinant of the submatrix obtained from M after deleting the i-th row and the j-th column. Then*

$$(M \cdot adj(M))_{ij} = (adj(M) \cdot M)_{ij} = \begin{cases} det(M) & \textit{if } i = j, \\ 0 & \textit{otherwise.} \end{cases}$$

Fact 1 (Determinant trick). *Let M be a finitely generated R-module and I an ideal of R. Assume $M = IM$. Then there exists $x \in I$ such that $(1-x)M = 0$.*

Notation 1. For all matrices N we set $I_3(N)$ to be the ideal generated by the 3×3 minors of N.

Lemma 2.2. *Let M be an $m \times n$ Hankel matrix with $m \geq 3$ and $n \geq 5$. Let P be a prime over $P_3(M)$. Then, either $I_3(M) \subseteq P$ or there exists a $3 \times (n-3)$ submatrix N of M such that $P_2(N) \subseteq P$.*

Proof. We observe that with

$$A = \begin{bmatrix} x_3 & x_4 & x_5 \\ x_4 & x_5 & x_6 \\ x_5 & x_6 & x_7 \end{bmatrix} \text{ and } v = \begin{bmatrix} per \begin{bmatrix} x_2 & x_3 \\ x_3 & x_4 \end{bmatrix} \\ per \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \\ per \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \end{bmatrix}$$

we have that the entries of $A \cdot v$ are in $P_3(M)$. Multiplying by $adj(A)$ gives that the entries of $detA \cdot v$ are in $P_3(M)$. If P is a prime ideal over $P_3(M)$, we have $det(A) \in P$ or $v \in P$. As we can repeat this argument for any 3×3 minor of M then either $I_3(M) \subseteq P$ or if one 3×3 minor is not in P, set N the 3×2 submatrix of the 2 complementary columns, the ideal $P_2(N) \subseteq P$, so we are done. □

3. Trying the minimal primes over $P_3(M)$

Lemma 3.1. *Let M be an $m \times n$ Hankel matrix. Let P be a minimal prime over $P_3(M)$. If there exists an index $i \in \{1, \ldots, m+n-2\}$ such that x_i and x_{i+1} are in P, then P is one of the following ideals:*

$$(x_1, \ldots, x_{m+n-3}), \quad (x_2, \ldots, x_{m+n-2}), \quad (x_3 \ldots, x_{m+n-1}).$$

Proof. Suppose that x_i and x_{i+1} are in P. Consider the submatrix

$$\begin{bmatrix} x_j & x_{j+1} & x_{j+2} \\ x_{j+1} & x_{j+2} & x_{j+3} \\ x_{j+2} & x_{j+3} & x_{j+4} \end{bmatrix}$$

for all $j = 1, \dots, m+n-5$, so that $x_j x_{j+2} x_{j+4} + x_j x_{j+3}^2 + x_{j+1}^2 x_{j+4} + 2x_{j+1}x_{j+2}x_{j+3} + x_{j+2}^3 \in P$. By the way we have $x_j \in P$ for all $j = 3, \dots, m+n-3$. Now we see that $x_1x_3x_{m+n-1} + x_1x_4x_{m+n-2} + x_2^2 x_{m+n-1} + x_2x_3x_{m+n-2} + x_2x_4x_{m+n-3} + x_3^2 x_{m+n-3} \in P$ implies $x_2^2 x_{m+n-1} \in P$. As P is prime, we have $x_2 \in P$ or $x_{m+n-1} \in P$. Suppose $x_2 \in P$. By $x_1x_{m+n-3}x_{m+n-1} + x_1x_{m+n-2}^2 + x_2x_{m+n-4}x_{m+n-1} + x_2x_{m+n-3}x_{m+n-2} + x_3x_{m+n-4}x_{m+n-2} + x_3x_{m+n-3}^2 \in P$ we have $x_1x_{m+n-2}^2 \in P$, so $x_1 \in P$ or $x_{m+n-2} \in P$. Thus $(x_1, \dots, x_{m+n-3}) \subseteq P$ or $(x_2, \dots, x_{m+n-2}) \subseteq P$ and by minimality of P we have

$$(x_1, \dots, x_{m+n-3}) = P \quad \text{or} \quad (x_2, \dots, x_{m+n-2}) = P.$$

Symmetry shows the rest. □

Lemma 3.2. *Let M be an $m \times n$ Hankel matrix. Let P be a minimal prime over $P_3(M)$. Suppose $x_1, x_3 \in P$ or $x_{m+n-3}, x_{m+n-1} \in P$.*

1. *If $m = 3, n = 5$ then P is one of the following ideals*

$$(x_1, \dots, x_5), \quad (x_1, x_3, x_5, x_7, x_2x_6 + x_4^2), \quad (x_3, \dots, x_7).$$

2. *If $m \geq 3, n \geq 5$, $m + n \geq 9$ then P is one of the following ideals*

$$(x_1, \dots, x_{m+n-3}), \quad (x_3, \dots, x_{m+n-1}).$$

Proof. Suppose $x_1, x_3 \in P$. Since $x_1x_3x_5 + x_1x_4^2 + x_2^2x_5 + 2x_2x_3x_4 + x_3^3 \in P$ we have $x_2^2x_5 \in P$. By primality of P we have $x_2 \in P$ or $x_5 \in P$. Suppose $x_2 \in P$. By Lemma 3.1 we have

$$P = (x_1, \dots, x_{m+n-3}).$$

If $x_2 \notin P$ then $x_5 \in P$. Since $x_2x_4x_7 + x_2x_5x_6 + x_3^2x_7 + x_3x_4x_6 + x_3x_5^2 + x_4^2x_5 \in P$, we have $x_2x_4x_7 \in P$. But $x_2 \notin P$, so $x_4x_7 \in P$. Suppose $x_4 \in P$. By Lemma 3.1 we have $P = (x_1, x_3, \dots, x_{m+n-1})$ but this ideal contains $(x_3, \dots, x_{m+n-1})$, hence it is not minimal. Suppose $x_4 \notin P$ then $x_7 \in P$. We see that $x_2x_4x_6 + x_2x_5^2 + x_3^2x_6 + 2x_3x_4x_5 + x_4^3 \in P$ so we have $x_4(x_2x_6 + x_4^2) \in P$. As $x_4 \notin P$ we have $x_2x_6 + x_4^2 \in P$. Now, we have to divide the proof in two different cases.

1. Suppose $m = 3, n = 5$. By minimality of P, we have
$$P = (x_1, x_3, x_5, x_7, x_2x_6 + x_4^2).$$
2. Suppose $m \geq 3, n \geq 5$, $m + n \geq 9$. We see that $x_1x_3x_8 + x_1x_4x_7 + x_2^2x_8 + x_2x_3x_7 + x_2x_4x_6 + x_3^2x_6 \in P$ implies $x_2(x_2x_8 + x_4x_6) \in P$; $x_3x_5x_8 + x_3x_6x_7 + x_4^2x_8 + x_4x_5x_7 + x_4x_6^2 + x_5^2x_6 \in P$ implies $x_4(x_4x_8 + x_6^2) \in P$. But $x_2, x_4 \notin P$ implies $(x_2x_8 + x_4x_6), (x_4x_8 + x_6^2) \in P$. It is clear that $x_8(x_2x_6 - x_4^2) = x_6(x_2x_8 + x_4x_6) - x_4(x_4x_8 + x_6^2) \in P$. If $x_8 \in P$ then by $x_7 \in P$ and Lemma 3.1 we have $x_4 \in P$, contradiction. So $x_8 \notin P$ and $(x_2x_6 - x_4^2) \in P$. As $(x_2x_6 + x_4^2) \in P$, we have $x_4 \in P$, contradiction.

By symmetry, if $x_{m+n-3}, x_{m+n-1} \in P$ we obtain the other ideals. $\square$

LEMMA 3.3. *Let M be an $m \times n$ Hankel matrix. Let P be a minimal prime over $P_3(M)$. If there exist an index $i \in \{2, \ldots, m + n - 4\}$ such that x_i and x_{i+2} are in P, then P is one of the following ideals:*
$$(x_1, \ldots, x_{m+n-3}), \quad (x_2, \ldots, x_{m+n-2}), \quad (x_3 \ldots, x_{m+n-1}).$$

Proof. We prove the assertion by iteration on the index i. First of all, suppose $i = 2$ so $x_2, x_4 \in P$. As $x_2x_4x_6 + x_2x_5^2 + 2x_3x_4x_5 + x_3^2x_6 + x_4^3 \in P$ we have $x_3^2x_6 \in P$. By primality of P we have $x_3 \in P$ or $x_6 \in P$. Suppose $x_3 \in P$, by Lemma 3.1 we are done. If $x_3 \notin P$ then $x_6 \in P$. We see that $x_3x_5x_7 + x_3x_6^2 + x_4^2x_7 + 2x_4x_5x_6 + x_5^3 \in P$ implies $x_5(x_3x_7 + x_5^2) \in P$. If $x_5 \in P$, by Lemma 3.1 $x_3 \in P$, contradiction. Then $x_5 \notin P$ and $x_3x_7 + x_5^2 \in P$. We see also that $x_1x_3x_5 + x_1x_4^2 + x_2^2x_5 + 2x_2x_3x_4 + x_3^3 \in P$ implies $x_3(x_1x_5 + x_3^2) \in P$ and $x_1x_3x_7 + x_1x_4x_6 + x_2^2x_7 + x_2x_3x_6 + x_2x_4x_5 + x_3^2x_5 \in P$ implies $x_3(x_1x_7 + x_3x_5) \in P$. As $x_3 \notin P$, by primality of P we have $(x_1x_5 + x_3^2), (x_1x_7 + x_3x_5) \in P$. It is clear that $x_3(x_1x_7 + x_3x_5) - x_1(x_3x_7 + x_5^2) = x_5(x_3^3 - x_1x_5) \in P$. As $x_5 \notin P$ then $(x_3^2 + x_1x_5) - (x_3^2 - x_1x_5) = 2x_3^2 \in P$, contradiction. Now we suppose $3 \leq i \leq m + n - 5$ and $x_i, x_{i+2} \in P$. Since $x_ix_{i+2}x_{i+4} + x_ix_{i+3}^2 + x_{i+1}^2x_{i+4} + 2x_{i+1}x_{i+2}x_{i+3} + x_{i+2}^3 \in P$, we have $x_{i+1}^2x_{i+4} \in P$. By primality of P we have $x_{i+1} \in P$ or $x_{i+4} \in P$. If $x_{i+1} \in P$, Lemma 3.1 we are done. If $x_{i+1} \notin P$ then $x_{i+4} \in P$. Since $x_{i+1}x_{i+3}x_{i+5} + x_{i+1}x_{i+4}^2 + x_{i+2}^2x_{i+5} + 2x_{i+2}x_{i+3}x_{i+4} +$

$x_{i+3}^3 \in P$, we have $x_{i+3}(x_{i+1}x_{i+5} + x_{i+3}^2) \in P$. If $x_{i+3} \in P$, by Lemma 3.1 we have $x_{i+1} \in P$, contradiction. So $x_{i+3} \notin P$ and $x_{i+1}x_{i+5}+x_{i+3}^2 \in P$. We see that $x_{i-1}x_{i+1}x_{i+3}+x_{i-1}x_{i+2}^2+x_i^2x_{i+3}+2x_ix_{i+1}x_{i+2} + x_{i+1}^3 \in P$, implies $x_{i+1}(x_{i-1}x_{i+3} + x_{i+1}^2) \in P$ and $x_{i-1}x_{i+1}x_{i+5} + x_{i-1}x_{i+2}x_{i+4} + x_i^2x_{i+5} + x_ix_{i+1}x_{i+4} + x_ix_{i+2}x_{i+3} + x_{i+1}^2x_{i+3} \in P$, implies $x_{i+1}(x_{i-1}x_{i+5} + x_{i+1}x_{i+3}) \in P$. As $x_{i+1} \notin P$ we obtain $x_{i-1}x_{i+3} + x_{i+1}^2$, $x_{i-1}x_{i+5} + x_{i+1}x_{i+3} \in P$. It is clear that $x_{i-1}(x_{i+1}x_{i+5}+x_{i+3}^2)-x_{i+1}(x_{i-1}x_{i+5}+x_{i+1}x_{i+3}) = x_{i+3}(x_{i-1}x_{i+3}-x_{i+1}^2) \in P$. As $x_{i+3} \notin P$ we have $(x_{i-1}x_{i+3} - x_{i+1}^2) \in P$, and so $2x_{i+1}^2 = (x_{i-1}x_{i+3} + x_{i+1}^2) - (x_{i-1}x_{i+3} - x_{i+1}^2) \in P$, contradiction. By symmetry, similar arguments show the assert in the case $i = m + n - 4$. □

LEMMA 3.4. *Let M be an $m \times n$ Hankel matrix, with $m \geq 3$ and $n \geq 5$. Let P be a minimal prime over $P_3(M)$. If there exist two indices i, j with $j \geq i + 3$ and either $i \geq 3$ or $j \leq m + n - 3$, such that x_i, x_j are in P, then P is one of the following ideals:*

$$(x_1, \ldots, x_{m+n-3}), \quad (x_2, \ldots, x_{m+n-2}), \quad (x_3, \ldots, x_{m+n-1}).$$

Proof. Without loss of generality we suppose $j \leq m+n-3$, (if $i \geq 3$ we are done by symmetry), and $x_i, x_j \in P$. As $m \geq 3$ and $n \geq 5$ we can consider the following submatrix of M

$$\begin{bmatrix} x_i & x_{j-2} & x_{j-1} & x_j \\ x_{i+1} & x_{j-1} & x_j & x_{j+1} \\ x_{i+2} & x_j & x_{j+1} & x_{j+2} \end{bmatrix}.$$

As $x_ix_jx_{j+2}+x_ix_{j+1}^2+x_{i+1}x_{j-1}x_{j+2}+x_{i+1}x_jx_{j+1}+x_{i+2}x_{j-1}x_{j+1}+x_{i+2}x_j^2 \in P$ we have $x_{j-1}(x_{i+1}x_{j+2} + x_{i+2}x_{j+1}) \in P$. If $x_{j-1} \in P$, by Lemma 3.1 we are done. Suppose $x_{j-1} \notin P$ then $x_{i+1}x_{j-2} + x_{i+2}x_{j+1} \in P$. We see that $x_ix_{j-1}x_{j+1} + x_ix_j^2 + x_{i+1}x_{j-2}x_{j+1} + x_{i+1}x_{j-1}x_j + x_{i+2}x_{j-2}x_j + x_{i+2}x_{j-1}^2 \in P$, so $x_{i+1}x_{j-2}x_{j+1}+x_{i+2}x_{j+1} \in P$ and $x_{j-2}x_jx_{j+2}+x_{j-2}x_{j+1}^2+x_{j-1}^2x_{j+2}+2x_{j-1}x_jx_{j+1} + x_j^3 \in P$ so $x_{j-2}x_{j+1}^2 + x_{j-1}^2x_{j+2} \in P$. It is clear that $x_{j+2}(x_{i+1}x_{j-2}x_{j+1} + x_{i+2}x_{j+1}) - x_{i+2}(x_{j-2}x_{j+1}^2 + x_{j-1}^2x_{j+2}) = x_{j-2}x_{j+1}(x_{i+1}x_{j+2} - x_{i+2}x_{j-1}) \in P$. If $x_{j-2} \in P$, by Lemma 3.3 we have $x_{j-1} \in P$, contradiction. If $x_{j+1} \in P$, by Lemma 3.1 we have $x_{j-1} \in P$, contradiction. So $x_{i+1}x_{j+2} - x_{i+2}x_{j-1} \in P$. As

$x_{i+1}x_{j+2} + x_{i+2}x_{j-1} \in P$ we obtain $x_{i+1}x_{j+2} \in P$. If $x_{i+1} \in P$, we have a contradiction by Lemma 3.1. If $x_{j+2} \in P$, we have a contradiction by Lemma 3.3. $\square$

LEMMA 3.5. *Let M be an $m \times n$ Hankel matrix. Let P be a minimal prime over $P_3(M)$. Let N be a 3×2 submatrix of M such that $P_2(N) \subseteq P$. Then P contains at least two distinct entries of N.*

Proof. If N is a 3×2 submatrix of M then it is one of the following

1. a generic matrix
$$N_1 = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix};$$
2. an Hankel matrix
$$N_2 = \begin{bmatrix} a & b \\ b & c \\ c & d \end{bmatrix};$$
3. a partial Hankel matrix
$$N_3 = \begin{bmatrix} a & b \\ b & c \\ d & e \end{bmatrix}, \quad \text{or symmetrically} \quad N_4 = \begin{bmatrix} a & b \\ c & d \\ d & e \end{bmatrix};$$
4. a matrix with a jump
$$N_5 = \begin{bmatrix} a & b \\ c & d \\ b & e \end{bmatrix}.$$

1. Let P be a minimal prime over $P_3(M)$ such that $P_2(N_1) \subseteq P$. By Lemma 1.1 of [10] we have $acf, ade, adf, bcf, bce, bde \in P$. By primality of P and by structure of $P_2(N_1)$, we can conclude that P contains one of the following sets of elements $\{a, c, e\}$, $\{a, b, cf + de\}$, $\{b, d, f\}$, $\{c, d, af + be\}$, $\{e, f, ad + bc\}$.
2. By Lemma 2.1 of [8] (i.e. the Hankel version of Lemma 1.1 of [10]) and a similar argument of case 1 we show the assert in the case of Hankel matrices.

The cases 3 and 4 are analogous to case 1. $\square$

THEOREM 3.6. *Let M be an $m\times n$ Hankel matrix. Let P be a minimal prime over $P_3(M)$, then P is one of the following ideals*

1. *if $m=3$ and $n=5$,*

$$(x_1,\dots,x_5),\quad (x_2,\dots,x_6),\quad (x_3,\dots,x_7),$$

$$(x_1,x_3,x_5,x_7,x_2x_6+x_4^2);$$

2. *if $m\geq 3$ and $n\geq 5$, $m+n\geq 9$,*

$$(x_1,\dots,x_{m+n-3}),\quad (x_2,\dots,x_{m+n-2}),\quad (x_3,\dots,x_{m+n-1}).$$

Proof. Let P be a minimal prime over $P_3(M)$. By Lemma 2.2 if $I_3(M)\not\subseteq P$ then there exists a 3×2 submatrix N of M such that $P_2(N)\subseteq P$. Suppose $I_3(M)\subseteq P$. Then for all $i=3,\dots,m+n-3$

$$x_i(x_{i-2}x_{i+2}+2x_{i-1}x_{i+1})+(x_{i-2}x_{i+1}^2+x_{i-1}x_{i+2}+x_i^3),$$

$$x_i(x_{i-2}x_{i+2}+2x_{i-1}x_{i+1})-(x_{i-2}x_{i+1}^2+x_{i-1}x_{i+2}+x_i^3)$$

are in P. Thus $x_i(x_{i-2}x_{i+2}+x_{i-1}x_{i+1})\in P$. In particular $x_3(x_1x_5+2x_2x_4)\in P$, so $x_3\in P$ or $x_1x_5+2x_2x_4\in P$. Suppose $x_3\in P$. By $x_1x_3x_6+x_2x_3x_5+x_2x_4^2\in P$ we have $x_2x_4^2\in P$, so $x_2\in P$ or $x_4\in P$. By Lemma 3.1, we obtain one of the following ideals

$$(x_1,\dots,x_{m+n-3}),\quad (x_2,\dots,x_{m+n-2}),\quad (x_3,\dots,x_{m+n-1}).$$

If $x_3\notin P$, then $x_1x_5+2x_2x_4\in P$. We consider $x_4(x_2x_6+2x_3x_5)\in P$. If $x_4\in P$ then $x_1x_5\in P$. If $x_1\in P$ then $x_2x_3x_5\in P$ so $x_2\in P$ or if not, $x_5\in P$. In both cases, by $x_3^3+x_2^2x_5\in P$ we have $x_3\in P$, contradiction. Suppose that there exists an index $j\in\{4,\dots,m+n-3\}$ such that $x_j\in P$ and $x_i\notin P$ for all $i\leq j-1$. By the way $x_{j-3}x_{j+1}+2x_{j-2}x_j\in P$ so $x_{j-3}x_{j+1}\in P$, but $x_{j-3}\notin P$ thus $x_{j+1}\in P$. By $x_{j-1}^3+x_{j-3}x_j^2+x_{j-2}^2x_{j+1}\in P$, we get $x_{j-1}\in P$, contradiction. So $x_i\notin P$ for all $i=3,\dots,m+n-3$, and $x_{i-2}x_{i+2}+2x_{i-1}x_{i+1}\in P$ for all $i=3,\dots,m+n-3$. We observe that $x_5(x_1x_4^2+x_2^2x_5+x_3^3)-x_2(x_2x_5^2+x_3^2x_6+x_4^3)-x_4^2(x_1x_5+2x_2x_4)-x_3^2(x_2x_6+2x_3x_5)=-3x_2(x_3^2x_6+2x_4^3)\in P$. Since P is prime we have

$x_2 \in P$ or $x_3^2x_6 + 2x_4^3 \in P$. Suppose $x_2 \in P$. By $x_1x_5 + 2x_2x_4 \in P$ we have $x_1x_5 \in P$ so $x_1 \in P$. By $x_3^3 + x_1x_4^2 + x_2^2x_5 \in P$ we have $x_3 \in P$, contradiction. Then $x_2 \notin P$, and $x_3^2x_6 + 2x_4^3 \in P$. We see that $2x_3(x_3x_6^2 + x_4^2x_7 + x_5^3) - 2x_6(x_3^2x_6 + 2x_4^3) - x_4^2(x_3x_7 + 2x_4x_6) - x_5^2(x_2x_6 + 2x_3x_5) = -x_6(x_2x_5^2 + 8x_4^3) \in P$.

1. If $m = 3$ and $n = 5$, then $m + n - 3 = 5$ so $x_6 \in P$ or $x_6 \notin P$ and $x_2x_5^2 + 8x_4^3 \in P$. If $x_6 \in P$, by $x_3^2x_6 + 2x_4^3 \in P$ we have $x_4 \in P$, contradiction. So $x_6 \notin P$ and $(x_2x_5^2 + 8x_4^3) \in P$. Now, $(x_3^2x_6 + x_2x_4^3) + (x_2x_5^2 + 8x_4^3) - (x_2x_5^2 + x_3^2x_6 + x_4^3) = 9x_4^3 \in P$ implies $x_4 \in P$, contradiction.

2. If $m \geq 3$ and $n \geq 5$, $m + n \geq 9$ then $m + n - 3 \geq 6$ and by induction assumption $x_6 \notin P$, so $(x_2x_5^2 + 8x_4^3) \in P$. Now, $(x_3^2x_6 + x_2x_4^3) + (x_2x_5^2 + 8x_4^3) - (x_2x_5^2 + x_3^2x_6 + x_4^3) = 9x_4^3 \in P$ implies $x_4 \in P$, contradiction.

Now we suppose $I_3(M) \not\subseteq P$ then there exists a 3×2 submatrix N of M such that $P_2(N) \subseteq P$. By Lemma 3.5, P contains at least two distinct entries of N, and by Lemmas 3.1, 3.2, 3.3 and 3.4, we are done. □

4. Future work

It is clear that to work with $I_t(M)$ for $t \geq 4$ is a very difficult problem. However, the results obtained on minimal primes over $P_2(M)$ and $P_3(M)$ give some indications on what will be the minimal primes over $P_t(M)$ for matrices with large size.

CONJECTURE 4.1. *Let $P_t(M)$ be the ideal generated by $t \times t$ subpermanents of an $m \times n$ Hankel matrix M, with large m, n. Then the number of minimal primes over $P_t(M)$ is t and precisely we have:*

$$P_j = (x_j, \ldots, x_{m+n-t+j-1}) \text{ for all } j = 1, \ldots, t.$$

References

[1] G.P.M. BINET, *Mémoire sur un systme de formules analytiques, et leur application à des considrations géométriques*, Journal de l'École Polytechnique, Paris **9** (1812) 280–302.

[2] W. Bruns and U. Vetter, *Determinantal rings*, Lect. Notes Math. **1327**, Springer (1988).

[3] L.A. Cauchy, *Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment*, Journal de l'École Polytechnique, Paris **10** (1812), 29–112.

[4] A. Conca, *Straightening law and powers of determinantal ideals of Hankel matrices*, Adv. Math. **138**, no. 2 (1998), 236–292.

[5] C. De Concini, D. Eisenbud and C. Procesi, *Young diagram and determinantal varieties*, Invent. Math. **56** (1980), 129–165.

[6] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. **84**, no. 1 (1996), 1–45.

[7] G.M. Greuel, G. Pfister and H. Schönemann, *Singular 3.0: a computer algebra system for polynomial computations*, Centre for Computer Algebra, University of Kaiserslautern (2005), online at `http://www.singular.uni-kl.de`.

[8] E. Grieco, A. Guerrieri and I. Swanson, *Permanental ideals of Hankel matrices*, Abhandlungen aus dem Mathematischen Seminar der Universitaet Hamburg **77** (2007), 30–58.

[9] G.A. Kirkup, *Minimal primes over permanental ideals*, (2005), online at `http://arxiv.org/abs/math.AC/0510025` to appear in Trans. Amer. Math. Soc.

[10] R.C. Laubenbacher and I. Swanson, *Permanental ideals*, J. of Symb. Comp. **30**, no. 2 (2000), 195–205.

[11] T. Muir, *On a class of permanent symmetric functions*, Proc. Roy. Soc. Edinburgh **11** (1882), 409–418.

[12] M. Niermann, *Beiträgezur Konstruktiven Idealtheorie*, PhD thesis, University of Dortmund (1997).

[13] J. Watanabe, *Hankel matrices and Hankel ideals*, Proc. school. SCI Tokay Uni **32** (1997), 11–21.

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