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Global Generation

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Per Fabio che ci manca tanto

1. Introduction

We work over an algebraically closed field of characteristic zero and a curve $C \subset \mathbb{P}^3$ is a closed subscheme of (pure) dimension one, locally Cohen-Macaulay and generically a local complete intersection. For such a curve we introduce the following invariants:

$$\begin{aligned} \epsilon(C) &= \max\{k \,|\, \omega_C(-k) \text{ has a section generating} \\ &\qquad\qquad \text{almost everywhere}\} \\ m(C) &= \min\{k \,|\, \mathcal{I}_C(k) \text{ is generated by global sections}\}. \end{aligned}$$

We prove:

Theorem 1.1. *With notations as above:*

1. $m \geq \frac{\epsilon+4}{2}$ *with equality if and only if* C *is a complete intersection* (a,a)
2. $m = [\frac{\epsilon+4}{2}] + 1$ *if and only if* C *is one of the following:*
   - *a section of a null-correlation bundle*
   - C *is a.C.M. and one of the following:*
     (a) *a complete intersection* $(b, b-1)$, $(b, b-2)$
     (b) C *is linked to a plane curve of degree* $m-1$ *by a complete intersection* (m,m)

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(c) *C is bilinked to Y by complete intersections $(2,m),(m,m)$ where Y is either: a complete intersection $(2,2)$, a "twisted cubic", a plane curve of degree ≤ 2.*

Taking hyperplane sections we get:

COROLLARY 1.2. *Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a smooth codimension two subvariety which is not a.C.M. If $\mathcal{I}_X(m)$ is generated by global sections, then $m > [\frac{e(X)+n+1}{2}] + 1$.*

2. Global generation for curves in $\mathbb{P}^3$

DEFINITION 2.1. *In this note a curve $C \subset \mathbb{P}^3$ is a one-dimensional closed subscheme, which is locally Cohen-Macaulay and generically a local complete intersection. These are the curves associated to rank two reflexive sheaves (see [3]).*

We associate to such a curve the following numerical invariants:

$$\epsilon(C) := \max\{k \,|\, \omega_C(-k) \text{ has a non-zero section which generates almost everywhere}\}$$

and:

$$m(C) := \min\{k \,|\, \mathcal{I}_C(k) \text{ is generated by global sections}\}.$$

If no confusion can arise we will simply write ϵ, m. Of course we will also consider the degree (d) and the arithmetic genus (p_a) of a curve.

We observe that if C is integral, then $\epsilon = e := \max\{k \,|\, h^1(\mathcal{O}_C(k)) \neq 0\}$, ($e$ is the index of speciality of C); but in general we only have: $\epsilon \leq e$.

A general section of $\omega_C(-\epsilon)$ yields an exact sequence:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_C(\epsilon+4) \to 0 \quad (+)$$

where F is a rank two reflexif sheaf with Chern classes: $c_1 = \epsilon + 4$, $c_2 = d$, $c_3 = 2p_a - 2 - d\epsilon$. By abuse of language we will say that F is the rank reflexif sheaf associated to C.

LEMMA 2.2. *Let* $C \subset \mathbb{P}^3$ *be a curve. Then* $m \geq \frac{\epsilon+4}{2}$.

Furthermore $m = \frac{\epsilon+4}{2}$ *if and only if* C *is a complete intersection of type* (a, a).

Proof. It is clear that for a complete intersection of type (a, b), $a \leq b$, we have: $\frac{\epsilon+4}{2} = \frac{a+b}{2}$ and $m = b$. So $m \geq \frac{\epsilon+4}{2}$ with equality if and only if $a = b$. So we may assume, from now on, that C is not a complete intersection. Consider the exact sequence (+) twisted by $m - \epsilon - 4$:

$$0 \to \mathcal{O}(m - \epsilon - 4) \to F(m - \epsilon - 4) \to \mathcal{I}_C(m) \to 0$$

Since C is not a complete intersection and since $\mathcal{I}_C(m)$ is globally generated, we have: $h^0(F(m - \epsilon - 4)) \geq h^0(\mathcal{I}_C(m)) \geq 3$. Moreover a general section of $F(m - \epsilon - 4)$ vanishes in codimension two. So we have:

$$0 \to \mathcal{O} \to F(m - \epsilon - 4) \to \mathcal{I}_Y(2m - \epsilon - 4) \to 0$$

where Y is a (non empty) curve. Since $h^0(\mathcal{I}_Y(2m - \epsilon - 4)) > 0$, we get: $2m - \epsilon - 4 > 0$. □

REMARK 2.3. *In case* C *is a* smooth, subcanonical *curve (i.e.* $\omega_C(-e) \simeq \mathcal{O}_C$*), the result can be proved by completely different arguments.*

From the exact sequence:

$$0 \to \mathcal{I}_C^2(m) \to \mathcal{I}_C(m) \to N_C^*(m) \to 0$$

It follows that $N_C^*(m)$ *is generated by global sections. Since* N_C *has rank two and* $det(N_C) = \omega_C(4) = \mathcal{O}_C(e + 4)$*, we get:* $N_C^*(m) = N_C(-e - 4 + m)$*. A general section yields:*

$$0 \to \mathcal{O}_C \to N_C(-e - 4 + m) \to \mathcal{O}_C(-e - 4 + 2m) \to 0 \quad (*)$$

and $\mathcal{O}_C(-e - 4 + 2m)$ *is globally generated, hence* $deg(\mathcal{O}_C(-e - 4 + 2m) \geq 0$ *and this implies:* $m \geq \frac{e+4}{2}$*. If* $m = \frac{e+4}{2}$*, then* $(*)$ *becomes:*

$$0 \to \mathcal{O}_C \to N_C(-e - 4 + m) \to \mathcal{O}_C \to 0 \quad (**)$$

it follows that $h^0(N_C(-e - 4 + m)) \leq 2$*, since we have a surjection:* $H^0(N_C(-e-4+m)) \otimes \mathcal{O}_C \to N_C(-e-4+m)$*, we conclude that:* $N_C \simeq$

$2.\mathcal{O}_C(m)$. Now $C \subset F_m$ where F_m is a smooth surface of degree m (because $\mathcal{I}_C(m)$ is globally generated) and the exact sequence of normal bundles:

$$0 \to N_{C,F_m} \to N_C \to N_{F_m} \to 0 \ \ (+)$$

reads like:

$$0 \to \mathcal{O}_C(m) \to 2.\mathcal{O}_C(m) \to \mathcal{O}_C(m) \to 0$$

Hence $(+)$ splits and by [2], C is a complete intersection.

Now we try to investigate further. As already noticed the case of complete intersection curves is clear, hence from now on we will assume C is not a complete intersection.

Lemma 2.4. *Let $C \subset \mathbb{P}^3$ be a non-complete intersection curve. If ϵ is odd, then $m = [\frac{\epsilon+4}{2}] + 1$ if and only if C is linked to a plane curve of degree $m-1$ by a complete intersection (m,m).*

Proof. We set $\epsilon = 2t+1$ so $m = t+3$ and the associated exact sequence is:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_C(2t+5) \to 0$$

Since $\mathcal{I}_C(t+3)$ is generated by global sections, we have $h^0(F(-t-2)) \geq 3$ and a general section of $F(-t-2)$ vanishes in codimension two:

$$0 \to \mathcal{O} \to F(-t-2) \to \mathcal{I}_Y(1) \to 0$$

It follows that $h^0(\mathcal{I}_Y(1)) \geq 2$, hence Y is a line. Now, by construction (being sections of the same reflexive sheaf), C is bilinked to Y; more precisely this is achieved by complete intersections $(1, t+3), (t+3, t+3)$. The first linkage links Y to a plane curve, P, of degree $t+2$. Then P is linked to C by a complete intersection $(t+3, t+3)$.

Finally it is easy to check that such a C satisfies $m = [\frac{\epsilon+4}{2}]+1$. □

The case ϵ even is a little bit more tricky. Let us begin with:

Lemma 2.5. *Let $C \subset \mathbb{P}^3$ be a non-complete intersection curve. If ϵ is even and if $m = \frac{\epsilon+4}{2} + 1$, then C is bilinked by complete intersections $(2,m), (m,m)$ to one of the following curves:*

1. *a curve, Y, of degree ≤ 4, contained in a complete intersection* $(2,2)$
2. *a plane curve*
3. *the (scheme theoretical) union of a plane curve, P, with a line L.*

Proof. We set $\epsilon = 2t$, so $m = t+3$ and proceed like in the proof of Lemma 2.4, this time we get:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_C(2t+4) \to 0$$

and

$$0 \to \mathcal{O} \to F(-t-1) \to \mathcal{I}_Y(2) \to 0$$

and we conclude that $h^0(\mathcal{I}_Y(2)) \geq 2$. If there are two quadrics containing Y without a common component, we are in case (1). Assume now that all the quadrics in $H^0(\mathcal{I}_Y(2))$ share a common plane H_0, so $H^0(\mathcal{I}_Y(2)) \simeq \{H_0 \cup H_t\}$, where the H_t build an ∞^r linear system of planes. If $r > 1$, the base locus of this system has dimension ≤ 0 and Y is a plane curve: this is case (2). If $r = 1$, the base locus is a line L and we are in case (3). □

Now we have to see if these cases are indeed effective. There are many possibilities, for instance in case (3) we have: (a) $L \cap P = \emptyset$, (b) $L \cap P =$ one point, (c) $L \subset H_0$ but L is multiple. To make things more manageable we will first assume that C is not arithmetically Cohen-Macaulay (a.C.M.). Also observe that in this case we don't know the degree of Y, we just have $d(Y) \leq m$.

LEMMA 2.6. *Let $C \subset \mathbb{P}^3$ be a non a.C.M. curve. If ϵ is even, then $m = \frac{\epsilon+4}{2} + 1$ if and only if C is a section of a null-correlation bundle.*

Proof. We examine the various cases of Lemma 2.5.

1. Since C is not a.C.M., Y has necessarily degree two and is a double line of genus $-p$, $p \geq 1$ or the union of two skew lines. (Indeed a curve of degree three contained in a complete intersection $(2,2)$ is linked to a line, hence is a.C.M.). Now the extension: $0 \to \mathcal{O} \to F(-t-1) \to \mathcal{I}_Y(2) \to 0$ (see proof of

Lemma 2.5) corresponds to a section of $\omega_Y(2)$, hence $c_3(F) = 2p_a(Y) - 2 + 2d(Y)$. If Y is a double line of genus $-p$, we get: $c_3(F) = -2p + 2$. Since $c_3(F) \geq 0$, $p \leq 1$. If $p = 0$, Y is a.C.M. and this is excluded. So $p = 1$, $c_3(F) = 0$ and F is a null-correlation bundle. This is a fortiori true if Y is the union of two skew lines.

2. This case doesn't occur (Y is a.C.M.).

3. Here $Y = P \cup L$ and we have three cases: a) $Y \cap L = \emptyset$ b) $Y \cap L = \{p\}$ c) $L \subset H_0 =< P >$ but L is multiple.

In case b), Y is a.C.M. Indeed we have an exact sequence:

$$0 \to \mathcal{I}_Y \to \mathcal{I}_P \to \mathcal{O}_L(-1) \to 0$$

which induces $f_m : H^0(\mathcal{I}_P(m)) \to H^0(\mathcal{O}_L(m-1))$. We have $f_m(H_0F_{m-1}) = F_{m-1}|L$, so f_m is surjective for $m \geq 1$ and $H^1_*(\mathcal{I}_Y) = 0$.

For the other two cases we begin with a general remark. By Lemma 2.5 C is bilinked to Y by complete intersections $(2, m), (m, m)$. More precisely: $Y \cup Z$ is a complete intersection, U, of type $(2, m)$ and $Z \cup C$ a complete intersection, V, of type (m, m). The exact sequences of liaison yield:

$$0 \to \mathcal{I}_V(m) \to \mathcal{I}_C(m) \to \omega_Z(4-m) \to 0$$

$$0 \to \mathcal{I}_U(2) \to \mathcal{I}_Y(2) \to \omega_Z(4-m) \to 0$$

It follows that $\omega_Z(4-m)$ is globally generated and $h^0(\omega_Z(4-m)) = h^0(\mathcal{I}_Y(2)) - h^0(\mathcal{I}_U(2))$. If this number is $= 1$, then $\omega_Z(4-m) \simeq \mathcal{O}_Z$. It follows that $p_a(Z) = 1 + (m-2)z$ ($z = d(Z)$). On the other hand, by liaison, $p_a(Z) = p_a(Y) + (z-m)(m-2)$, hence: $p_a(Y) = 1 + m(m-2)$ (+).

Case a): If Y is the disjoint union of a plane curve of degree p and a line, then a direct computation yields: $p_a(Y) = \frac{(p-1)(p-2)}{2} - 1$. Combining with (+), we get: $2(m-1)^2 = p(p-3)$. Since $m > p$, this cannot hold. We conclude that

$h^0(\mathcal{I}_Y(2)) > 2$. This implies that P is a line, i.e. Y is the disjoint union of two lines and C is a section of a null-correlation bundle.

Case c): This time $L \subset H_0$ is a component of P but L carries a multiple structure which sticks out of the plane. We have the residual exact sequence with respect to H_0 ([1], proof of Thm. 8):

$$0 \to \mathcal{I}_L(-1) \to \mathcal{I}_Y \to \mathcal{I}_{Y\cap H_0, H_0} \to 0$$

here $Y \cap H_0$ is the union of P with a zero-dimensional subscheme, A, with support on L. If R is the residual scheme of A with respect to P, then we have:

$$0 \to \mathcal{O}_R(-p) \to \mathcal{O}_{Y\cap H_0} \to \mathcal{O}_P \to 0$$

and $p_a(Y) = \frac{(p-1)(p-2)}{2} - r$, where $r = length(R)$ ([1], proof of Thm. 8). Arguing as above, we get: $2(m-1)^2 + 2(r-1) = p(p-3)$. But this cannot hold, so $h^0(\mathcal{I}_Y(2)) > 2$. This implies that Y is a double line indeed Y has support on L and is contained in the first infinitesimal neighborhood of L, moreover Y is generically a local complete intersection. As in (1), we conclude that C is a section of a null-correlation bundle. Finally it is easy to check that sections of a null-correlation bundle satisfy $m = \frac{\epsilon+4}{2} + 1$.

□

To conclude we have:

LEMMA 2.7. *Let $C \subset \mathbb{P}^3$ an a.C.M. curve. If ϵ is even, then $m = \frac{\epsilon+4}{2} + 1$ if and only if C is one of the following:*

- *a complete intersection of type $(b, b-2)$ or:*
- *C is bilinked by complete intersections $(m,m), (2,m)$ to Y where Y is one of the following:*
  - *(a) a complete intersection $(2,2)$;*
  - *(b) a "twisted cubic" (i.e. Y has minimal free resolution: $0 \to 2\mathcal{O}(-3) \to 3\mathcal{O}(-2) \to \mathcal{I}_Y \to 0$);*

(c) a plane curve of degree ≤ 2.

Proof. By Lemma 2.5 we have to check the cases where Y is contained in a complete intersection $(2,2)$, where Y is a plane curve or the union of a plane curve with a line meeting it at one point. Let's start with this last case. If $Y = P \cup L$ where $L \cap P = \{x\}$, then Y is linked to a plane curve of degree $p-1$ by a complete intersection $(2,p)$. Indeed let $Q = H_0 \cup H$ where $H_0 =< P >$ and where H contains L, then take K a cone of base P, vertex a point of L, then $Q \cap K$ makes the job. From the resolution of a plane curve of degree $p-1$, we get, by mapping cone:

$$0 \to \mathcal{O}(-1-p) \oplus \mathcal{O}(-3) \to 2.\mathcal{O}(-2) \oplus \mathcal{O}(-p) \to \mathcal{I}_Y \to 0$$

Now we perform the liaisons $(2,m)$, (m,m) and by mapping cone we get:

$$\begin{aligned} 0 \quad &\to \quad \mathcal{O}(-2m+2) \oplus \mathcal{O}(-m-1) \oplus \mathcal{O}(-m+1-p) \\ &\to \quad 3.\mathcal{O}(-m) \oplus \mathcal{O}(-m+2-p) \to \mathcal{I}_C \to 0 \end{aligned} \tag{1}$$

Clearly if $p \leq 2$ then $\mathcal{I}_C(m)$ is generated by global sections and $e(C) = 2m-6$. If $p > 2$ and if $\mathcal{I}_C(m)$ is globally generated, then we have:

$$0 \to E \to 3.\mathcal{O}(m) \to \mathcal{I}_C \to 0$$

where E is a rank two vector bundle. Since C is a.C.M., $H^2_*(E) = 0$, by Serre duality and the isomorphism $E^* \simeq E(-c_1)$, also $H^1_*(E) = 0$, by Horrocks theorem, E splits, a contradiction (look at the minimal free resolution). So $p \leq 2$ and Y is either a (degenerated) conic or twisted cubic.

A similar phenomenon occurs when Y is a degree p plane curve. Performing the first liaison $(2,m)$ we link P to a curve Z and, as already noticed, if $\mathcal{I}_C(m)$ is globally generated, then $\omega_Z(4-m)$ is also. Let's see that this is not the case if $p > 2$. Let's consider the general case: the quadric is the union of two distinct planes, H, H', and Z is the union of a plane curve, X, of degree m with a plane curve, T, of degree $m-p$, X and T not containing $H \cap H'$. The genericity assumption is not a problem because the Hilbert scheme parametrizing the curves Z is irreducible and being globally generated is an open condition. Now since Z is a.C.M. $X \cap T = T\cap < X >$ and: $\omega_Z|T \simeq \omega_T(1)$. It follows that $\omega_Z(4-m)|T \simeq \omega_T(5-m) \simeq \mathcal{O}_T(-p+2)$ which is globally generated only if $p \leq 2$. Finally observe that if $m = p$, then C is a complete intersection $(m, m-1)$ (ϵ odd).

In the remaining cases (complete intersection $(2,2)$, twisted cubic) one checks directly that the required conditions are satisfied. □

This concludes the proof of Theorem 1.1.

References

[1] Ph. Ellia, *On the cohomology of projective space curves*, Boll. U.M.I. **7**, no. 9-A (1995), 593–607.
[2] Ellingsrud, G.-Peskine, Ch.-Gruson, L.-Strømme, S.A, *On the normal bundle of curves on smooth projective surfaces*, Invent. Math. **80** (1985), 181–184.
[3] R. Hartshorne, *Stable reflexive sheaves*, Math. Ann. **254** (1980), 121–176.

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