Rend. Istit. Mat. Univ. Trieste
Vol. XXXIX, 141–175 (2007)

Klein-Gordon Type Equations with a Singular Time-dependent Potential

D. Del Santo, T. Kinoshita and M. Reissig (*)

Dedicated to the memory of Fabio Rossi

Summary. - *In this note we study Klein-Gordon type Cauchy problems with a time-dependent singular potential. We ask for the influence of the sign and the singularity order of the potential on the regularity of solutions with respect to time.*

1. Introduction

The present paper is devoted to the study of the Cauchy problem for the following Klein-Gordon type equation with unbounded time-dependent potential

$$u_{tt} - \triangle u + \frac{a(T-t)}{(T-t)^\beta} u = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \quad (1)$$

$(t,x) \in [0,T) \times \mathbb{R}^n$. Here $\beta > 0$ and a is a continuous function defined on $[0,T]$. We will set $a(0) = a_0$ and it will be useful to introduce a

(*) Authors' addresses: Daniele Del Santo, Dipartimento di Matematica e Informatica, Università di Trieste, Via A.Valerio 12/1, 34127 Trieste, Italy; E-mail: delsanto@univ.trieste.it
Tamotu Kinoshita, Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan, e-mail: kinosita@math.tsukuba.ac.jp
Michael Reissig, Institut für Angewandte Analysis, Fakultät für Mathematik und Informatik, TU Bergakademie Freiberg, Prüferstraße 9, 09596 Freiberg, Germany, e-mail: reissig@math.tu-freiberg.de
Keywords: Klein-Gordon Equation, Singular Potential, Cauchy Problem, Well-Posedness.
AMS Subject Classification: 35L15.

reference function $\mu = \mu(t)$ characterizing the behavior of $a(T-t)$ near T in the following way:

$$|a(s) - a_0| \leq \mu(s) \quad \text{for all} \quad s \in [0, T]. \tag{2}$$

It is well-known that (1) is H^s-well-posed on each time interval $[0, T_0]$ with $T_0 < T$, that is, to given data $u_0 \in H^s$ and $u_1 \in H^{s-1}$ there exists a unique solution $u \in C([0, T_0], H^s) \cap C^1([0, T_0], H^{s-1})$ depending continuously on the data. Thus $u(t, \cdot) \in H^s$ for all $t \in [0, T)$ and the following questions are of interest:

what about the properties of the solution u with respect to t on the whole interval $[0, T]$? What are the correct function spaces B_0 and B_1 such that the solution u belongs to $B_0([0, T], H^s) \cap B_1([0, T], H^{s-1})$?

With a transformation in time we shift the singularity into $t = 0$, that is, we consider the following backward Cauchy problem

$$u_{tt} - \triangle u + \frac{a(t)}{t^\beta} u = 0, \;\; u(T, x) = u_0(x), \;\; u_t(T, x) = u_1(x), \tag{3}$$

$(t, x) \in (0, T] \times \mathbb{R}^n$ and we will study the regularity of the solution with respect to t up to $t = 0$. We will distinguish essentially between three different situations. The potential in the equation from (3) will be called *singular* if $\beta = 2$ and $a_0 \neq 0$; the potential will be called *sub-singular* if $\beta = 2$ and $a_0 = 0$; finally the potential will be denoted as *super-singular* if $\beta > 2$ and $a_0 \neq 0$.

Before giving a description of the content of the paper, it is interesting to recall some problems connected with the one studied here and in which singular time-dependent masses in the linear Klein-Gordon problem are of importance.

The first one concerns the semi-linear wave equation $u_{tt} - \triangle u - u^5 = 0$, which has, as remarked in [13], the solution $u(t, x) = u(t) = (3/4)^{1/4}(T - t)^{-1/2}$. This fact has the considerable consequence that the Cauchy problem

$$u_{tt} - \triangle u - \frac{3}{4(T-t)^2} u = 0, \;\; u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x), \tag{4}$$

has in general no solution $u \in L^2((0, T), H^s)$, i.e. no *energy solution* to given data $u_0 \in H^s$ and $u_1 \in H^{s-1}$ with $s \geq 1$, as localization in space and finite speed of propagation show. We will see that the example (4) is not by chance. Moreover, let us consider the equation $u_{tt} - \triangle u + u^5 = 0$. If we are interested in real time-dependent solutions $u \in L^4(0,T)$, the non-linear potential u^5 can be written in the form $|u|^4 u$, this means, the "coefficient" $|u|^4$ is integrable. A Gronwall type argument enables us to show that $u \in L^\infty(0,T)$. Therefore, in the study of semi-linear equations it is sufficient to prove $u \in L^4(0,T)$ instead of $u \in L^\infty(0,T)$ (see [13]). But, when $u \in L^2(0,T)$, then such a reduction to $L^\infty(0,T)$ does not work. The coefficient is in general not integrable. Thus, singular masses in linear problems are of importance.

The second one concerns the global existence of large data solutions to some semi-linear weakly hyperbolic Cauchy problems as presented in [10]. If the principal part of the operator studied in [10] coincides with the Grushin operator $\partial_{tt} - t^\lambda \triangle$, then a standard transformation leads to the study of

$$\Big(\partial_{tt} - \triangle + \frac{a_0}{t^2}\Big)u = 0 \quad \text{with} \quad a_0 = \frac{\lambda^2 + 4\lambda}{4(\lambda+2)^2} \in (0, 1/4). \tag{5}$$

There exist at least two different strategies to treat the corresponding semi-linear model. The first one is to manipulate the operator by the fundamental solution to the classical wave operator. The second one is to manipulate the semi-linear operator by the fundamental solution of the operator (5). This assumes a precise knowledge of this Klein-Gordon operator with singular potential (cf. with open problems from [10]). The second strategy is used in [18] and [19] to prove the global existence of small data solutions for semi-linear Tricomi type equations. The construction of the fundamental solution of the operator $\partial_t^2 - t^m \triangle$ bases on theory of special functions, namely, on the use of *hypergeometric functions* (see [4]).

The content of the present paper is the following: in Section 2 we study singular operators with $a(t) \equiv a_0 \neq 0$. This scale invariant model case is studied by the theory of special functions. The results are optimal and hint to effects we have to expect for more general models which will be treated in the Sections 3 to 5. Such a strategy is already used for example

- in the theory of weakly hyperbolic equations (see [14] and [2]),

$$\text{model cases:} \quad u_{tt} - t^{2l} u_{xx} + a t^{l-1} u_x = 0,$$
$$u_{tt} - \frac{1}{t^4} \exp(-\frac{2}{t}) u_{xx} + a \frac{1}{t^4} \exp(-\frac{1}{t}) u_x = 0;$$

- in the theory of $L^p - L^q$ decay estimates for wave equations with time-dependent propagation speed (see [11] and [6]),

$$\text{model cases:} \quad u_{tt} - t^{2l} \triangle u = 0,$$
$$u_{tt} - \exp(2t) \triangle u = 0;$$

- in the theory of wave equations with weak dissipation (see [16]),

$$\text{model case:} \quad u_{tt} - \triangle + \frac{\mu}{1+t} u_t = 0.$$

Section 3 explains the influence of special classes of super-singular potentials. Here the sign of the potential plays an important role. In Section 4 we discuss general sub-singular potentials. Finally, Section 5 is devoted to the general case of a singular potential. In both Sections 4 and 5 we introduce an auxiliary function $\mu = \mu(t)$ as a *reference function* which measures the asymptotic behavior of $a = a(t)$ at $t = 0$ (see (2)).

2. Scale invariant model case

Let us devote to the backward Cauchy problem

$$u_{tt} - \triangle u + \frac{a_0}{t^2} u = 0, \quad u(T,x) = u_0(x), \quad u_t(T,x) = u_1(x), \tag{6}$$

with a constant $a_0 \neq 0$. Setting $x := yt$, $\tau = \ln t^{-1}$, $w(\tau, y) := \exp(\frac{1}{2}\tau) u(\tau, y)$, straight-forward calculations lead to the Klein-Gordon model

$$w_{\tau\tau} - \triangle_y w + \Big(a_0 - \frac{1}{4}\Big) w = 0 \quad \text{with Cauchy data.} \tag{7}$$

The model (7) explains our approach. If $a_0 > 1/4$ we have a positive mass which brings a stabilizing effect (hyperbolic WKB-analysis in the phase space). The case $a_0 = 1/4$ leads to the wave case, this should be considered as an exceptional case. If $a_0 < 1/4$ we have a negative mass which should bring some instability into the model (as well as hyperbolic and elliptic WKB-analysis in the phase space). But we have to exclude $a_0 = 0$, thus it is reasonable to distinguish between $a_0 \in (0, 1/4)$ and $a_0 < 0$. This explains the following four cases. The *scaling property* of the operator from (6) hints to application of the *theory of special functions*. This will be done in the next subsections.

2.1. The case $a_0 > 1/4$

Let us start with $\hat{u}_{tt} + |\xi|^2\hat{u} + \frac{a_0}{t^2}\hat{u} = 0$. Setting $\hat{u} = \tau^\rho v$, $2\rho = 1 + i\sqrt{4a_0 - 1}$, $\tau = t|\xi|$, yields $\tau v_{\tau\tau} + 2\rho v_\tau + \tau v = 0$. A second transformation $z = 2i\tau$, $w(z) = e^{i\tau}v(\tau)$ leads to

$$zw_{zz} + (2\rho - z)w_z - \rho w = 0, \;\; 2\rho = 1 + i\sqrt{4a_0 - 1}.$$

The equation $zw_{zz} + (\gamma - z)w_z - \alpha w = 0$ is called *Kummer's equation or confluent hypergeometric equation*. Following [4] we know that $\Phi(\alpha, \gamma; z)$ and $z^{1-\gamma}\Phi(1 + \alpha - \gamma, 2 - \gamma; z)$ form a fundamental system of solutions if γ is not an integer as in our case. Transforming back gives

$$\begin{aligned} u_1(t, \xi) &= (t|\xi|)^\rho e^{-it|\xi|}\Phi(\rho, 2\rho; 2it|\xi|), \\ u_2(t, \xi) &= (t|\xi|)^\rho e^{-it|\xi|}(2it|\xi|)^{1-2\rho}\Phi(1 - \rho, 2 - 2\rho; 2it|\xi|). \end{aligned}$$

Both solutions are continuous at $\xi = 0$, thus frequencies localized near $\xi = 0$ imply smooth properties of the solution in the physical space. Consequently in the following we are allowed to restrict ourselves to large frequencies. The initial conditions are $\hat{u}(T, \xi) = \hat{u}_0(\xi)$, $\hat{u}_t(T, \xi) = \hat{u}_1(\xi)$. We have $\hat{u}(t, \xi) = V_1(t, \xi)\hat{u}_0(\xi) + V_2(t, \xi)\hat{u}_1(\xi)$, where we have set

$$\begin{aligned} V_1(t, \xi) &:= \frac{u_1(t, \xi)u_{2,t}(T, \xi) - u_2(t, \xi)u_{1,t}(T, \xi)}{u_1(T, \xi)u_{2,t}(T, \xi) - u_{1,t}(T, \xi)u_2(T, \xi)}; \\ V_2(t, \xi) &:= \frac{u_2(t, \xi)u_1(T, \xi) - u_1(t, \xi)u_2(T, \xi)}{u_1(T, \xi)u_{2,t}(T, \xi) - u_{1,t}(T, \xi)u_2(T, \xi)}. \end{aligned}$$

We have to determine the asymptotic behavior with respect to ξ of V_1, V_2 and its first derivatives in t. Therefore we use the following properties of $\Phi(\alpha,\gamma;z)$:

- Φ is entire in z, $\quad \Phi(\alpha,\gamma;0)=1$;
- $|\Phi(\alpha,\gamma;z)| \le C_{\alpha\gamma}|z|^{\max(\operatorname{Re}(\alpha-\gamma),-\operatorname{Re}\alpha)}$ for large $|z|$ under the restriction $0<\arg z<\pi$;
- $d_z\Phi(\alpha,\gamma;z)=\frac{\alpha}{\gamma}\Phi(\alpha+1,\gamma+1;z)$.

To determine the *asymptotic behavior* we divide the *extended phase space* into two zones: the *pseudo-differential zone* i. e. the set $\{(t,\xi): t|\xi|\le N, |\xi|\ge M\}$ and the *hyperbolic zone* i. e. the set $\{(t,\xi): t|\xi|\ge N, |\xi|\ge M\}$. In the *pseudo-differential zone* we have

$$|u_1(t,\xi)|\le C(t|\xi|)^{1/2},\quad |u_2(t,\xi)|\le C(t|\xi|)^{1/2},$$
$$|u_{1,t}(t,\xi)|\le Ct^{-1}(t|\xi|)^{1/2},\quad |u_{2,t}(t,\xi)|\le Ct^{-1}(t|\xi|)^{1/2};$$

while in the *hyperbolic zone* we have

$$|u_1(t,\xi)|\sim C,\ \ |u_2(t,\xi)|\sim C,\ \ |u_{1,t}(t,\xi)|\sim C|\xi|,\ \ |u_{2,t}(t,\xi)|\sim C|\xi|.$$

To estimate the denominator $u_1(T,\xi)u_{2,t}(T,\xi)-u_{1,t}(T,\xi)u_2(T,\xi)$ let us put $p(t,\xi)=(t|\xi|)^\rho e^{-it|\xi|}$ and write

$$u_1(t,\xi)=p(t,\xi)w_1(2it|\xi|),\quad u_2(t,\xi)=p(t,\xi)w_2(2it|\xi|),$$

where $w_1(z)=\Phi(\rho,2\rho;z)$ and $w_2(z)=z^{1-\rho}\Phi(1-\rho,2-2\rho;z)$ satisfy (see [4, page 253, formula (8)])

$$w_{1,z}(z)w_2(z)-w_1(z)w_{2,z}(z)=(2\rho-1)z^{-2\rho}e^z.$$

Hence,

$$\begin{aligned}&2i|\xi|p^2(w_1w_{2,z}-w_{1,z}w_2)\\ &\quad =2e^{i(-2\log 2\sqrt{a_0-1/4}+\pi/2)}\sqrt{a_0-1/4}\,e^{\pi\sqrt{a_0-1/4}}|\xi|.\end{aligned}$$

Thus,

$$|u_1(T,\xi)u_{2,t}(T,\xi)-u_{1,t}(T,\xi)u_2(T,\xi)|=2\sqrt{a_0-1/4}\,e^{\pi\sqrt{a_0-1/4}}|\xi|.$$

Summarizing we obtain in the *pseudo-differential zone* $\{(t,\xi) : t|\xi| \le N, |\xi| \ge M\}$:

$$|V_1(t,\xi)| \le C(t|\xi|)^{1/2}, \quad |V_2(t,\xi)| \le C(t|\xi|)^{1/2}|\xi|^{-1},$$
$$|V_{1,t}(t,\xi)| \le Ct^{-1/2}|\xi|^{1/2}, \quad |V_{2,t}(t,\xi)| \le Ct^{-1/2}|\xi|^{-1/2};$$

and in the *hyperbolic zone* $\{(t,\xi) : t|\xi| \ge N, |\xi| \ge M\}$:

$$|V_1(t,\xi)| \le C,\ |V_2(t,\xi)| \le C|\xi|^{-1},\ |V_{1,t}(t,\xi)| \le C|\xi|,\ |V_{2,t}(t,\xi)| \le C.$$

Using these estimates in

$$\hat{u}(t,\xi) = V_1(t,\xi)\hat{u}_0(\xi) + V_2(t,\xi)\hat{u}_1(\xi),$$
$$\hat{u}_t(t,\xi) = V_{1,t}(t,\xi)\hat{u}_0(\xi) + V_{2,t}(t,\xi)\hat{u}_1(\xi),$$

respectively, and taking into account the fact that $t \in (0,T]$, we obtain the following result:

THEOREM 2.1. *Let us assume $a_0 > 1/4$. Then the Cauchy problem*

$$u_{tt} - \triangle u + \frac{a_0}{(T-t)^2}u = 0\ ,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x)$$

with data u_0, u_1 belonging to H^s, H^{s-1} respectively has a uniquely determined solution $u \in C([0,T],H^s) \cap C^1([0,T),H^{s-1})$ with $(T-t)^{\frac{1}{2}}u_t \in L^\infty((0,T),H^{s-1})$.

2.2. The case $a_0 = 1/4$

In this case we obtain Kummer's equation $zw_{zz} + (1-z)w_z - \frac{1}{2}w = 0$. This is the so-called *logarithmic case* and as a fundamental system of solutions we get $\Psi(\frac{1}{2},1;z)$ and $e^z\Psi(\frac{1}{2},1;-z)$. For the transformed equation we obtain the fundamental system of solutions

$$u_1(t,\xi) = (t|\xi|)^{1/2}e^{-it|\xi|}\Psi\Big(\frac{1}{2},1;2it|\xi|\Big),$$
$$u_2(t,\xi) = (t|\xi|)^{1/2}e^{it|\xi|}\Psi\Big(\frac{1}{2},1;-2it|\xi|\Big).$$

From [17, page 103] we have

$$\Psi\Big(\frac{1}{2},1;2it|\xi|\Big) = H_-\Big(\frac{1}{2},1;2it|\xi|\Big)e^{-i\varepsilon\frac{1}{2}\pi},$$
$$\Psi\Big(\frac{1}{2},1;-2it|\xi|\Big) = H_+\Big(\frac{1}{2},1;2it|\xi|\Big)e^{i\varepsilon\frac{1}{2}\pi},$$

here ε is either 1 or -1.
Straight-forward calculations imply

$$\begin{aligned}
&u_{1,t}(t,\xi)u_2(t,\xi) - u_{2,t}(t,\xi)u_1(t,\xi) \\
&\quad = -i|\xi|^2 t\Big(2\Psi\Big(\frac{1}{2},1;2it|\xi|\Big)\Psi\Big(\frac{1}{2},1;-2it|\xi|\Big) \\
&\qquad +\Psi\Big(\frac{3}{2},2;-2it|\xi|\Big)\Psi\Big(\frac{1}{2},1;2it|\xi|\Big) \\
&\qquad +\Psi\Big(\frac{3}{2},2;2it|\xi|\Big)\Psi\Big(\frac{1}{2},1;-2it|\xi|\Big)\Big)
\end{aligned}$$

by using the rule

$$d_z\Psi(\alpha,\gamma;z) = -\alpha\Psi(\alpha+1,\gamma+1;z).$$

Taking into account of the formula given in [4, page 278] we have

$$\Psi(\alpha,\gamma;z) = \sum_{k=0}^{N}(-1)^k C_{\alpha,\gamma,k} z^{-\alpha-k} + O(|z|^{-\alpha-N-1}),$$
$$N = 0,1,2,\cdots, \quad z\to\infty, \quad -\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi,$$

for real α,γ. Consequently the Wronskian at $t=T$, $|\xi|$ large, can be estimated as follows:

$$\begin{aligned}
&|u_{1,t}(T,\xi)u_2(T,\xi) - u_{2,t}(T,\xi)u_1(T,\xi)| \\
&\qquad \geq |\xi|^2 T\Big|H_-\Big(\frac{1}{2},1;2iT|\xi|\Big)H_+\Big(\frac{1}{2},1;2iT|\xi|\Big)\Big|.
\end{aligned}$$

From [17, formula (2.1.23)] we have

$$H_+(\alpha,\gamma;z) \sim z^{\alpha-\gamma}(1+\sum_{k=1}^{\infty}C_{\alpha,\gamma,k}z^{-k}),$$
$$H_-(\alpha,\gamma;z) \sim (e^{-\pi i}z)^{-\alpha}(1+\sum_{k=1}^{\infty}C_{\alpha,\gamma,k}z^{-k}),$$
$$|H_+(\alpha,\gamma;z)| \leq |z|^{\alpha-\gamma}, \quad |H_-(\alpha,\gamma;z)| \leq |z|^{-\alpha}$$

for $0 < \arg z < \pi$ and large $|z|$. Then we obtain

$$\begin{aligned}
&|u_{1,t}(T,\xi)u_2(T,\xi) - u_{2,t}(T,\xi)u_1(T,\xi)| \\
&\qquad \geq C|\xi|^2 T\left(|\xi|^{-\frac{1}{2}}|\xi|^{-\frac{1}{2}}\right) \geq C|\xi|
\end{aligned}$$

for large frequencies. Using

$|\Psi(\alpha,\gamma;z)| \sim |\log\, z|$ for small $|z|$ we arrive at the following estimates:
in the *pseudo-differential zone* $\{(t,\xi): t|\xi| \le N, |\xi| \ge M\}$:

$$|V_1(t,\xi)| \le C(t|\xi|)^{\frac{1}{2}}|\log(t|\xi|)|, \quad |V_2(t,\xi)| \le C(t|\xi|)^{\frac{1}{2}}|\log(t|\xi|)||\xi|^{-1},$$
$$|V_{1,t}(t,\xi)| \le Ct^{-1/2}|\xi|^{\frac{1}{2}}|\log(t|\xi|)|,$$
$$|V_{2,t}(t,\xi)| \le Ct^{-1/2}|\xi|^{\frac{1}{2}}|\log(t|\xi|)||\xi|^{-1};$$

and in the *hyperbolic zone* $\{(t,\xi): t|\xi| \ge N, |\xi| \ge M\}$:

$$|V_1(t,\xi)| \le C,\ |V_2(t,\xi)| \le C|\xi|^{-1},\ |V_{1,t}(t,\xi)| \le C|\xi|,\ |V_{2,t}(t,\xi)| \le C.$$

We collect the results in the following:

THEOREM 2.2. *Let us assume $a_0 = 1/4$. Then the statement of Theorem 2.1 holds with $u \in C([0,T],H^s) \cap C^1([0,T),H^{s-1})$ and $(T-t)^{\frac{1}{2}}\Big(\log\frac{1}{T-t}\Big)^{-1}u_t \in L^\infty((0,T),H^{s-1})$.*

2.3. The case $a_0 \in (0,1/4)$

Now $2\rho = \gamma = 1+\sqrt{1-4a_0}$. To determine the *asymptotic behavior* we define the same zones as in the case $a_0 > 1/4$. We conclude in the *pseudo-differential zone*:

$$|u_1(t,\xi)| \le C(t|\xi|)^{\rho}, \quad |u_2(t,\xi)| \le C(t|\xi|)^{1-\rho},$$
$$|u_{1,t}(t,\xi)| \le C\,t^{-1}(t|\xi|)^{\rho}, \quad |u_{2,t}(t,\xi)| \le C\,t^{-1}(t|\xi|)^{1-\rho},$$

and in the *hyperbolic zone* we get the same estimates as in the case $a_0 > 1/4$. Summarizing gives
in the *pseudo-differential zone* $\{(t,\xi): t|\xi| \le N, |\xi| \ge M\}$:

$$|V_1(t,\xi)| \le C(t|\xi|)^{1-\rho}, \quad |V_2(t,\xi)| \le C(t|\xi|)^{1-\rho}|\xi|^{-1},$$
$$|V_{1,t}(t,\xi)| \le Ct^{-\rho}|\xi|^{1-\rho}, \ \ |V_{2,t}(t,\xi)| \le Ct^{-\rho}|\xi|^{-\rho};$$

and in the *hyperbolic zone* $\{(t,\xi): t|\xi| \ge N, |\xi| \ge M\}$:

$$|V_1(t,\xi)| \le C,\ |V_2(t,\xi)| \le C|\xi|^{-1},\ |V_{1,t}(t,\xi)| \le C|\xi|,\ |V_{2,t}(t,\xi)| \le C.$$

We have:

THEOREM 2.3. *Let us assume* $a_0 \in (0, 1/4)$. *Then the statement of Theorem 2.1 holds with* $u \in C([0,T], H^s) \cap C^1([0,T), H^{s-1})$ *and* $(T-t)^{\frac{1+\sqrt{1-4a_0}}{2}} u_t \in L^\infty((0,T), H^{s-1})$.

2.4. The case $a_0 < 0$

Looking at the results in the case of $a_0 \in (0, 1/4)$ one may expect a singular behavior of the solution itself in $t = T$. To study the present situation we will use some results of the theory of *Euler-Poisson-Darboux equation* (see [15] and [16]). Introducing $\hat{v}(t,\xi) = (\frac{t}{T})^{-\frac{d}{2}} \hat{u}(t,\xi)$ with $d = 1 - \sqrt{1-4a_0}$ we obtain

$$\hat{v}_{tt} + |\xi|^2 \hat{v} + \frac{d}{t} \hat{v}_t = 0, \ d \in (-\infty, 1),$$

and consequently we deduce the Euler-Poisson-Darboux equation

$$v_{tt} - \triangle v + \frac{d}{t} v_t = 0 \quad \text{for} \ \ d \in (-\infty, 1).$$

Setting $\rho = \frac{d-1}{2} = -\frac{\sqrt{1-4a_0}}{2}$ we have the following known representations:

- For non-integer ρ:

  $\hat{v}(t,\xi)(t|\xi|)^\rho = C_1(\xi) J_\rho(t|\xi|) + C_2(\xi) J_{-\rho}(t|\xi|),$

- for integer ρ:

  $\hat{v}(t,\xi)(t|\xi|)^\rho = C_1(\xi) J_\rho(t|\xi|) + C_2(\xi) Y_\rho(t|\xi|),$

where J_ρ, $J_{-\rho}$ denote the Bessel functions and Y_ρ the Weber function. Using the asymptotic behavior of J_ρ, $J_{-\rho}$ and Y_ρ and the value of J_ρ at $\tau = t|\xi| = 0$ and for $\tau = t|\xi| \to +\infty$, brings

$$v \in \bigcap_{j=0}^{k} C^j([0,T], H^{s-j}) \quad \text{for} \quad k \le \sqrt{1-4a_0}\,.$$

Taking into account the fact that $\lim_{\tau \to 0} \tau^\rho J_{-\rho}(\tau) = \text{const.} \neq 0$ we obtain:

THEOREM 2.4. *Let us assume $a_0 < 0$. Then the statement of Theorem 2.1 holds with $u \in C([0,T),H^s) \cap C^1([0,T),H^{s-1})$ such that*

$$(T-t)^{\frac{-1+\sqrt{1-4a_0}}{2}} u \in L^\infty((0,T),H^s)$$

and

$$(T-t)^{\frac{1+\sqrt{1-4a_0}}{2}} u_t \in L^\infty((0,T),H^{s-1}).$$

EXAMPLE 2.1. *The function $u(t) = (T-t)^{(1-\sqrt{1-4a_0})/2}$ satisfies the equation*

$$u_{tt} - \triangle u + \frac{a_0}{(T-t)^2} u = 0.$$

This shows the optimality of the statement from Theorem 2.4.

REMARK 2.1. *All the results of this section describe in an optimal way the influence of a singular potential with $a(t) \equiv a_0 \neq 0$.*

REMARK 2.2. *We can use the results of this section to obtain energy estimates or $L^p - L^q$ decay estimates for solutions to*

$$u_{tt} - \triangle u + \frac{a_0}{(1+t)^2} u = 0$$

(see [5]).

3. Super-singular potentials

3.1. Positive super-singular potentials

As a model case we consider the Cauchy problem

$$u_{tt} - \triangle u + \frac{a_0}{(T-t)^\beta} u = 0, \ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \quad (8)$$

with $a_0 > 0$, $\beta > 2$.

THEOREM 3.1. *Let us consider the Cauchy problem (8). If the data u_0, u_1 are supposed to belong to H^s, H^{s-1} respectively, then there exists a uniquely determined solution $u \in C([0,T),H^s) \cap C^1([0,T),H^{s-1})$ with $u \in L^\infty((0,T),H^s)$ and $(T-t)^{\frac{\beta}{4}} u_t \in L^\infty((0,T),H^{s-1})$.*

Proof. As usual we transform the above Cauchy problem into the following one

$$t^\beta \hat{u}_{tt} + t^\beta |\xi|^2 \hat{u} + a_0 \hat{u} = 0,\ \hat{u}(T,\xi) = \hat{u}_0(\xi),\ \hat{u}_t(T,\xi) = \hat{u}_1(\xi).$$

Now we introduce a new variable $\tau = \frac{2}{\beta-2} t^{-\frac{\beta-2}{2}}$ and the new function $v = v(\tau,\xi) = \hat{u}(t,\xi)$. Then the above Cauchy problem is transferred to

$$\begin{aligned} &v_{\tau\tau} + \frac{|\xi|^2}{C_\beta \tau^{\frac{2\beta}{\beta-2}}} v + a_0 v + \frac{\beta}{(\beta-2)\tau} v_\tau = 0, \\ &v(\tau_0,\xi) = v_0(\xi),\ \ v_\tau(\tau_0,\xi) = v_1(\xi) \end{aligned}$$

with $\tau_0 := \frac{2}{\beta-2} T^{-\frac{\beta-2}{2}}$, $v_0(\xi) := \hat{u}_0(\xi)$, $v_1(\xi) := -T^{\frac{\beta}{2}} \hat{u}_1(\xi)$ and with a positive constant C_β. Finally, we introduce the new function $w(\tau,\xi) = \tau^{\frac{\beta}{2(\beta-2)}} v(\tau,\xi)$. Then we get

$$\begin{aligned} &w_{\tau\tau} + \frac{|\xi|^2}{C_\beta \tau^{\frac{2\beta}{\beta-2}}} w + a_0 w + \frac{\beta^2 - 4\beta}{4(\beta-2)^2 \tau^2} w = 0, \\ &w(\tau_0,\xi) = w_0(\xi),\ w_\tau(\tau_0,\xi) = w_1(\xi) \end{aligned}$$

with $w_0(\xi) := \tau_0^{\frac{\beta}{2(\beta-2)}} v_0(\xi)$ and

$$w_1(\xi) := \tau_0^{\frac{\beta}{2(\beta-2)}} v_1(\xi) + \frac{\beta}{2(\beta-2)} \tau_0^{\frac{4-\beta}{2(\beta-2)}} v_0(\xi).$$

First we assume that (t,ξ), $|\xi| \geq M$, belongs to the *pseudo-differential zone* $Z_{pd}(N) = \{(t,\xi) : t^{\frac{\beta}{2}} |\xi| \leq N\}$ ($Z_{pd}(N) = \{(\tau,\xi) : \tau^{-\frac{\beta}{\beta-2}} |\xi| \leq N\}$). Here N denotes a universal large constant connected with the definition of zones. We introduce the functions t_ξ, τ_ξ respectively, as the solutions of $t_\xi^{\frac{\beta}{2}} |\xi| = N$, $\tau_\xi^{-\frac{\beta}{\beta-2}} |\xi| = N$. To study the above equation we introduce the energy

$$E^2(w)(\tau,\xi) := |w_\tau|^2 + \Big(\frac{|\xi|^2}{C_\beta \tau^{\frac{2\beta}{\beta-2}}} + a_0 + \frac{\beta^2 - 4\beta}{4(\beta-2)^2 \tau^2} \Big) |w|^2.$$

Here we take account that it is sufficient to consider a small time interval $[0,T]$. Instead of the Cauchy problem (1) we may prescribe

Cauchy data on $t = t_0$ with t_0 near to T because the potential is regular on $[0, t_0]$, that is, we have on this interval the typical regularity of the solution and its first derivatives in the evolution spaces. If we study the Cauchy problem on $[t_0, T)$, then shifting the singularity into $t = 0$ means, that the new interval $(0, T_1]$ is small. So T can be assumed to be small from the beginning. Thus to a given a_0 and $\beta > 2$ we can choose T small, thus τ_0 large, such that $E^2(w)(\tau,\xi) \geq 0$ on the set $\{(\tau,\xi) \in [\tau_0,\infty) \times \{|\xi| \geq M\}\}$. After differentiation with respect to τ we obtain

$$d_\tau E^2(w)(\tau,\xi) = -\frac{2\beta}{\beta-2}\frac{|\xi|^2}{C_\beta \tau^{\frac{2\beta}{\beta-2}+1}}|w|^2 - \frac{\beta^2-4\beta}{2(\beta-2)^2\tau^3}|w|^2.$$

If $\beta \geq 4$, then $d_\tau E^2(w)(\tau,\xi) \leq 0$. If $\beta \in (2,4)$, then $d_\tau E^2(w)(\tau,\xi) \leq \frac{4\beta-\beta^2}{2(\beta-2)^2\tau^3}E^2(w)(\tau,\xi)$. Due to the term τ^{-3} we may conclude in both cases $0 \leq E^2(w)(\tau,\xi) \leq CE^2(w)(\tau_\xi,\xi)$, where the constant C is independent of $\tau \in [\tau_\xi,\infty)$. This inequality gives us the following estimate for the solution in $Z_{pd}(N)$:

$$\begin{aligned}|w_\tau(\tau,\xi)|^2 &+ \frac{|\xi|^2}{\tau^{\frac{2\beta}{\beta-2}}}|w(\tau,\xi)|^2 + \frac{1}{\tau^2}|w(\tau,\xi)|^2 + |w(\tau,\xi)|^2 \\ &\leq C\Big(1 + \frac{1}{\tau_\xi^2} + \frac{|\xi|^2}{\tau_\xi^{\frac{2\beta}{\beta-2}}}\Big)|w(\tau_\xi,\xi)|^2 + |w_\tau(\tau_\xi,\xi)|^2.\end{aligned}$$

The first backward transformation gives

$$\begin{aligned}\tau^{\frac{\beta}{\beta-2}}|v_\tau(\tau,\xi)|^2 &+ \frac{|\xi|^2}{\tau^{\frac{\beta}{\beta-2}}}|v(\tau,\xi)|^2 + \tau^{\frac{\beta}{\beta-2}}|v(\tau,\xi)|^2 + \tau^{\frac{4-\beta}{\beta-2}}|v(\tau,\xi)|^2 \\ &\leq \tau_\xi^{\frac{\beta}{\beta-2}}|v_\tau(\tau_\xi,\xi)|^2 + \Big(\frac{|\xi|^2}{\tau_\xi^{\frac{\beta}{\beta-2}}} + \tau_\xi^{\frac{\beta}{\beta-2}} + \tau_\xi^{\frac{4-\beta}{\beta-2}}\Big)|v(\tau_\xi,\xi)|^2.\end{aligned}$$

The second backward transformation gives

$$\begin{aligned}&t^{\frac{\beta}{2}}(|\hat{u}_t(t,\xi)|^2 + |\xi|^2|\hat{u}(t,\xi)|^2) + t^{-\frac{\beta}{2}}|\hat{u}(t,\xi)|^2 + t^{-\frac{4-\beta}{2}}|\hat{u}(t,\xi)|^2 \\ &\quad\leq t_\xi^{\frac{\beta}{2}}(|\hat{u}_t(t_\xi,\xi)|^2 + |\xi|^2|\hat{u}(t_\xi,\xi)|^2) + \Big(t_\xi^{-\frac{\beta}{2}} + t_\xi^{-\frac{4-\beta}{2}}\Big)|\hat{u}(t_\xi,\xi)|^2.\end{aligned}$$

Now let us assume

$$|\hat{u}_t(t_\xi,\xi)|^2 + \langle\xi\rangle^2|\hat{u}(t_\xi,\xi)|^2 \le C(\langle\xi\rangle^2|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2). \tag{9}$$

Then from the second backward transformation we obtain for all $t \in (0, t_\xi]$

$$\begin{aligned} |\hat{u}(t,\xi)|^2 &\le (tt_\xi)^{\frac{\beta}{2}}(|\hat{u}_t(t_\xi,\xi)|^2 + |\xi|^2|\hat{u}(t_\xi,\xi)|^2) \\ &\qquad + \Big(\frac{t}{t_\xi}\Big)^{\frac{\beta}{2}}|\hat{u}(t_\xi,\xi)|^2 + \frac{t^{\frac{\beta}{2}}}{t_\xi^{\frac{4-\beta}{2}}}|\hat{u}(t_\xi,\xi)|^2 \qquad (10) \\ &\le \langle\xi\rangle^{-2}C(\langle\xi\rangle^2|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2). \end{aligned}$$

Here we used the definition of t_ξ, (9) and $\beta > 2$. Moreover, we conclude for all $t \in (0, t_\xi]$

$$t^{\frac{\beta}{2}}|\hat{u}_t(t,\xi)|^2 \le C(\langle\xi\rangle^2|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2). \tag{11}$$

Now let us devote to the *hyperbolic zone* $Z_{hyp}(N) = \{(t,\xi) : t^{\frac{\beta}{2}}|\xi| \ge N\}$. Our goal is to show

$$|\hat{u}_t(t,\xi)|^2 + \langle\xi\rangle^2|\hat{u}(t,\xi)|^2 \le C(\langle\xi\rangle^2|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2) \tag{12}$$

for all $t \in [t_\xi, T]$. Setting $t = t_\xi$ we obtain (9), and (12) together with (10) and (11) leads to the statements of our theorem.

To derive (12) we define $E^2(\hat{u})(t,\xi) = |\hat{u}_t(t,\xi)|^2 + \Big(|\xi|^2 + \frac{a_0}{t^\beta}\Big)|\hat{u}(t,\xi)|^2$. Thus we have

$$d_tE^2(\hat{u})(t,\xi) = \frac{-\beta a_0}{t^{\beta+1}}|\hat{u}(t,\xi)|^2 \ge -\frac{C}{t^{\beta+1}|\xi|^2}E^2(\hat{u})(t,\xi).$$

Hence,

$$E^2(\hat{u})(T,\xi) \ge E^2(\hat{u})(t,\xi)\exp\Big(-\int_t^T \frac{C}{\tau^{\beta+1}|\xi|^2}d\tau\Big).$$

Thus it follows that

$$E^2(\hat{u})(t,\xi) \le E^2(\hat{u})(T,\xi)\exp\Big(\frac{C}{t^\beta|\xi|^2}\Big) \le E^2(\hat{u})(T,\xi)\exp C_N.$$

But this leads to

$$(|\xi|^2 + \frac{a_0}{t^\beta})^{1/2}|\hat{u}(t,\xi)| + |D_t\hat{u}(t,\xi)| \le C(\langle\xi\rangle|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|).$$

The last inequality gives immediately (12). The theorem is proved. $\square$

Remark 3.1. *Theorem 3.1 is a reasonable continuation of Theorem 2.1 for $\beta = 2$ to $\beta > 2$.*

3.2. Negative super-singular potentials

As a model case we consider the Cauchy problem

$$\begin{aligned} &u_{tt} - \triangle u - \frac{a_0}{(T-t)^\beta}u = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \\ &\text{with} \quad a_0 > 0 \quad \text{and} \quad \beta > 2, \end{aligned} \tag{13}$$

where we suppose

$$u_0,\, u_1 \in C_0^\infty(\mathbb{R}^n) \text{ with } \int_{\mathbb{R}^n} u_0(x)dx > 0 \text{ and } \int_{\mathbb{R}^n} u_1(x)dx \ge 0. \tag{14}$$

Recalling the result of Theorem 2.4 one may suspect that the presence of the negative super-singular potential should have a deteriorating influence on the properties of the solution (with respect to t). The following theorem shows that it is really so.

Theorem 3.2. *Let us consider the Cauchy problem (13) under the assumption (14). Then there does not exist any positive constant α such that $(T-t)^\alpha u \in L^\infty((0,T), H^s)$.*

Proof. The proof is based on the so-called *functional method*: see for instance [3, Ch.2].

Obviously, $u \in C^\infty([0,T) \times \mathbb{R}^n)$ and if $\operatorname{supp} u_0$, $\operatorname{supp} u_1 \subseteq \{x \in \mathbb{R}^n : |x| \le R\}$, then $\operatorname{supp} u(t,\cdot) \subseteq \{x \in \mathbb{R}^n : |x| \le R + t\}$. We set

$$U(t) = \int_{\mathbb{R}^n} u(t,x)dx.$$

Then

$$U''(t) - \frac{a_0}{(T-t)^\beta}U(t) = 0 \qquad \text{in} \quad [0,T) \tag{15}$$

with $U(0) = \int_{\mathbb{R}^n} u_0(x)dx > 0$ and $U'(0) = \int_{\mathbb{R}^n} u_1(x)dx \geq 0$. We claim now that $U''(t) > 0$ for all $t \in [0,T)$ and, consequently,

$$U(t) \geq U(0) + tU'(0) \geq U(0) \tag{16}$$

for all $t \in [0,T)$. In fact from (15) we have $U''(0) = a_0 T^{-\beta} U(0) > 0$ and if we suppose that there exists $\bar{t} \in (0,T)$ such that $U''(\bar{t}) = 0$ and $U''(t) > 0$ for all $t \in [0,\bar{t})$, since we have, on one side, from the convexity of U on $[0,\bar{t}]$, $U(\bar{t}) \geq U(0) + tU(0) > 0$ and, on the other side, again from (15), $U(\bar{t}) = 0$, we obtain a contradiction. Consequently, $U''(t) > 0$ for all $t \in [0,T)$ and the convexity of U gives (16). From (15) and (16) we deduce that

$$U''(t) = \frac{a_0}{(T-t)^\beta} U(t) \geq \frac{a_0}{(T-t)^\beta} U(0) \tag{17}$$

for all $t \in [0,T)$. Integrating (17) and taking into account that $U'(0) \geq 0$ we have

$$U'(t) \geq -\frac{a_0 U(0)}{\beta - 1}\big(T^{1-\beta} - (T-t)^{1-\beta}\big)$$

for all $t \in [0,T)$. Again by integration, taking now into account that $U(0) > 0$, we obtain

$$U(t) \geq -C_1 + C_1'(T-t)^{-\gamma} \tag{18}$$

for all $t \in [0,T)$, where

$$C_1 = \frac{a_0 U(0)}{\beta - 2} T^{2-\beta} > 0, \quad C_1' = \frac{a_0 U(0)}{(\beta-1)(\beta-2)} > 0$$

and $\gamma = \beta - 2 > 0$. We claim now that for all integers $n \geq 1$ there exist positive constants C_n and C_n' such that

$$U(t) \geq (-C_n(T-t)^\gamma + C_n')(T-t)^{-n\gamma} \tag{19}$$

for all $t \in [0,T)$, where $\gamma = \beta - 2 > 0$.

We prove this claim by a recursive argument. The inequality (18) gives (19) in the case $n = 1$. Let (19) hold for a fixed n. From (15) we obtain

$$U''(t) = \frac{a_0}{(T-t)^\beta} U(t) \geq -a_0 C_n (T-t)^{-(n-1)\gamma - \beta} + a_0 C_n' (T-t)^{-n\gamma - \beta}$$

for all $t \in [0,T)$. Integrating the above inequality and taking into consideration that

$$U'(0) \geq 0, \quad \frac{a_0 C_n}{(n-1)\gamma+\beta-1} T^{-(n-1)\gamma-\beta+1} > 0,$$

we conclude

$$U'(t) \geq -\frac{a_0 C_n}{(n-1)\gamma+\beta-1}(T-t)^{-(n-1)\gamma-\beta+1} + \frac{a_0 C_n'}{n\gamma+\beta-1}\Big((T-t)^{n\gamma-\beta+1} - T^{n\gamma-\beta+1}\Big),$$

that is,

$$U'(t) \geq \frac{-a_0 C_n}{(n-1)\gamma+\beta-1}(1+f_n(t))(T-t)^{-(n-1)\gamma-\beta+1} + \frac{a_0 C_n'}{n\gamma+\beta-1}(T-t)^{-n\gamma-\beta+1} \tag{20}$$

for all $t \in [0,T)$, where

$$f_n(t) = \frac{C_n'\big((n-1)\gamma+\beta-1\big)}{C_n(n\gamma+\beta-1)}\,(T-t)^{(n-1)\gamma+\beta-1}\,T^{-n\gamma-\beta+1}.$$

We remark that f_n is a positive bounded function on $[0,T]$. So (20) implies that

$$U'(t) \geq -a_0\tilde{C}_n(T-t)^{-(n-1)\gamma-\beta+1} + a_0\tilde{C}_n'(T-t)^{-n\gamma-\beta+1} \tag{21}$$

for all $t \in [0,T)$, where

$$\tilde{C}_n = \frac{C_n}{(n-1)\gamma+\beta-1}\Big(1 + \frac{C_n'\big((n-1)\gamma+\beta-1\big)}{C_n\big(n\gamma+\beta-1\big)}\,T^{-\gamma}\Big)$$

and

$$\tilde{C}_n' = \frac{C_n'}{n\gamma+\beta-1}.$$

Integrating (21) and taking into account that $U(0) > 0$ we obtain

$$U(t) \geq -a_0\tilde{C}_n n\gamma(1+g_n(t))(T-t)^{-n\gamma} + \frac{a_0\tilde{C}_n'}{(n+1)\gamma}(T-t)^{-(n+1)\gamma} \tag{22}$$

for all $t \in [0,T)$, where

$$g_n(t) = \frac{n\tilde{C}'_n}{(n+1)\tilde{C}_n}\,(T-t)^{n\gamma}\,T^{-(n+1)\gamma}.$$

From (22) we immediately obtain that there exist positive constants C_{n+1} and C'_{n+1} such that

$$U(t) \ge \Big(-C_{n+1}(T-t)^{\gamma} + C'_{n+1}\Big)(T-t)^{-(n+1)\gamma}$$

for all $t \in [0,T)$ and the claim (19) is proved.

From the claim (19) we deduce that for all integers $n \ge 1$ there exist $t_n \in (0,T)$ and a positive constant D_n such that, for all $t \in (t_n, T)$,

$$U(t) \ge D_n(T-t)^{-n}.$$

Since by Schwarz's inequality we have that $U(t) \le C_{R,T}\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)}$ we obtain that for all integers $n \ge 1$ there exist $t_n \in (0,T)$ and a positive constant $\tilde{D}_n$ such that

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \ge \tilde{D}_n(T-t)^{-n}$$

for all $t \in (t_n,T)$. This completes the proof. □

Remark 3.2. *Theorem 3.2 is a reasonable continuation of Theorem 2.4 for $\beta = 2$ to $\beta > 2$.*

4. Sub-singular potentials

In this section we shall devote to the case when the coefficient $\frac{a(T-t)}{(T-t)^2}$ is not a quadratic singular potential, that is, the continuous function $a(t)$ on $[0,T]$ satisfies

$$|a(t)| \le \mu(t) \quad \text{for all} \quad t \in [0,T], \tag{23}$$

where the reference function

$$\mu \text{ is continuously differentiable, } \mu \ge 0 \text{ on } [0,T] \text{ and } \mu(0) = 0, \tag{24}$$

that is, with $a_0 = 0$ in (2). The case of sub-singular potentials can be regarded as a family of singular potentials near the classical wave case without any potential.

In this section we take into consideration reference functions satisfying conditions from one of the following two cases A or B:

Case A: $\displaystyle\int_0^T \int_0^s \frac{\mu(T-r)}{(T-r)^2} dr ds \equiv \alpha(T) < \infty$

(e.g. $\mu(t) = t^\beta, \beta \in (0,1]$, or $\mu(t) = (\log t^{-1})^{-1} \cdots (\log^{[n]} t^{-1})^{-\gamma}, \gamma > 1$ for $t \in (0,T]$),

Case B: $\displaystyle\mu'(t) \le \delta \frac{\mu(t)}{t}$ with $0 < \delta < 1$

(e.g. $\mu(t) = t^\beta, \beta \in (0,1)$, or $\mu(t) = (\log t^{-1})^{-1} \cdots (\log^{[n]} t^{-1})^{-1}$ for $t \in (0,T]$).

Remark 4.1. *Most of the reference functions μ satisfy both cases A and B. But, the function $\mu(t) = (\log t^{-1})^{-1} \cdots (\log^{[n]} t^{-1})^{-1}$ for $t \in (0,T]$ is excluded in case A and the functions $\mu(t) = t$ or $\mu(t) = t \log t^{-1}$ for $t \in (0,T]$ are excluded in case B.*

Theorem 4.1. *Assume that (23) and (24) hold. Moreover, we assume that the reference function μ satisfies the conditions from Case A or Case B. Let us consider the family of Cauchy problems*

$$u_{tt} - \triangle u + \frac{a(T-t)}{(T-t)^2} u = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x).$$

Then there exists a uniquely determined solution $u \in C([0,T), H^s) \cap C^1([0,T), H^{s-1})$ with

$$\Big(\exp\Big[K \int_0^t \frac{\mu(T-s)}{T-s} ds\Big]\Big)^{-1} u \in L^\infty((0,T), H^s), \tag{25}$$

$$\Big(\Big(1 + \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds\Big) \exp\Big[K \int_0^t \frac{\mu(T-s)}{T-s} ds\Big]\Big)^{-1} u_t \in L^\infty((0,T), H^{s-1}), \tag{26}$$

if the data u_0, u_1 are supposed to belong to H^s, H^{s-1}, respectively. Here K is a positive constant. In particular, if $\int_0^T \frac{\mu(T-s)}{T-s} ds < \infty$, *then*

$$
\begin{aligned}
& u \in L^\infty((0,T), H^s), \\
& \Big(1 + \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds\Big)^{-1} u_t \in L^\infty((0,T), H^{s-1}).
\end{aligned}
\tag{27}
$$

Proof. Let us consider the Klein-Gordon equation

$$
u_{tt} - \triangle u - (A'(t) + A(t)^2)u = 0, \tag{28}
$$

where $A(t)$ with $A(0) = 0$ will be defined later. A Gronwall type argument requires the integrability of $|A'(t) + A(t)^2|$ with the standard energy

$$
E(t) := \|(\hat{u}_t, \langle\xi\rangle\hat{u})(t,\cdot)\|_{L^2}.
$$

For the cancellation of the term $-(A'(t) + A(t)^2)u$, we shall define the following modified energy for the solutions to (28):

$$
\mathcal{E}(t) := \Big\| \Big(\hat{u}_t - A(t)\hat{u}, \langle\xi\rangle\hat{u}\Big)\Big\|_{L^2} = \Big\|\Big(\frac{\langle\xi\rangle}{\langle\xi\rangle_A}\hat{u}_t, \frac{A(t)}{\langle\xi\rangle_A}\hat{u}_t - \langle\xi\rangle_A\hat{u}\Big)\Big\|_{L^2}, \tag{29}
$$

where $\langle\xi\rangle_A = \sqrt{|\xi|^2 + A(t)^2 + 1}$. From the definition we find that

$$
\begin{aligned}
& \|\langle\xi\rangle\hat{u}(t,\cdot)\|_{L^2} \le \mathcal{E}(t), \quad \|\hat{u}_t(t,\cdot)\|_{L^2} \le (1 + |A(t)|)\mathcal{E}(t) \\
& \text{and} \quad E(0) = \mathcal{E}(0).
\end{aligned}
\tag{30}
$$

By (28) and (29) we deduce that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t)^2 &= 2\,\mathrm{Re}\int_{\mathbb{R}_\xi} \big(\hat{u}_{tt} - A'\hat{u} - A\hat{u}_t\big)\overline{(\hat{u}_t - A\hat{u})}\, d\xi \\
&\qquad\qquad\qquad + 2\,\mathrm{Re}\int_{\mathbb{R}_\xi} \langle\xi\rangle^2 \hat{u}_t \overline{\hat{u}}\, d\xi \\
&= 2\,\mathrm{Re}\int_{\mathbb{R}_\xi} \big(-\langle\xi\rangle^2\hat{u} + \hat{u}\big)\overline{(\hat{u}_t - A\hat{u})}\, d\xi \\
&\qquad + 2\,\mathrm{Re}\int_{\mathbb{R}_\xi} \langle\xi\rangle^2 \hat{u}_t\overline{\hat{u}}\, d\xi - 2A\|\hat{u}_t - A\hat{u}\|_{L^2}^2 \\
&= 2A\|\langle\xi\rangle\hat{u}\|_{L^2}^2 + 2\,\mathrm{Re}\int_{\mathbb{R}_\xi} (\hat{u}_t - A\hat{u})\overline{\hat{u}} d\xi - 2A\|\hat{u}_t - A\hat{u}\|_{L^2}^2 \\
&\le 2(|A| + 1)\mathcal{E}(t)^2.
\end{aligned}
$$

Hence, we have

$$\frac{d}{dt}\mathcal{E}(t) \le (|A(t)|+1)\mathcal{E}(t),$$

and by (30) Gronwall's inequality yields

$$\mathcal{E}(t) \le E(0)\exp\int_0^t \big(|A(s)|+1\big)ds. \tag{31}$$

If $A(t)$ is given by the following Ricatti equation:

$$A'(t) = \frac{-a(T-t)}{(T-t)^2} - A(t)^2, \tag{32}$$

then (31) is also the energy inequality for solutions to $u_{tt} - \triangle u + \frac{a(T-t)}{(T-t)^2} = 0$.

Now we define recursively the sequence $\{A_k(t)\}_{k\ge 0}$ for $t \in [0,T)$ by

$$A_0(t) = 0, \quad A_{k+1}(t) = \int_0^t \frac{-a(T-s)}{(T-s)^2}ds - \int_0^t A_k(s)^2 ds. \tag{33}$$

Noting that

$$\begin{aligned}\int_0^t \frac{\mu(T-s)}{(T-s)^2}ds &= \frac{\mu(T-t)}{T-t} - \frac{\mu(T)}{T} + \int_0^t \frac{\mu'(T-s)}{T-s}ds\\ &\le \frac{\mu(T-t)}{T-t} + \delta\int_0^t \frac{\mu(T-s)}{(T-s)^2}ds,\end{aligned}$$

we obtain in the case B

$$\int_0^t \frac{\mu(T-s)}{(T-s)^2}ds \le \frac{1}{1-\delta}\ \frac{\mu(T-t)}{T-t}, \tag{34}$$

and in both cases A and B

$$\frac{\mu(T-t)}{T-t} \le \int_0^t \frac{\mu(T-s)}{(T-s)^2}ds + \frac{\mu(T)}{T}. \tag{35}$$

We assume that

$$|A_k(t)| \le \begin{cases} 2M\int_0^t \frac{\mu(T-s)}{(T-s)^2}ds \ \ (M>1) & \text{in the case A,}\\ 2M\frac{\mu(T-t)}{T-t} \ \ \Big(M = \frac{1}{1-\delta} > 1\Big) & \text{in the case B.}\end{cases}$$

Without loss of generality we may suppose that T is sufficiently small, that is,

$$T \leq \begin{cases} \alpha^{-1}(\frac{1}{4M^2}) & \text{in the case A,} \\ \mu^{-1}(\frac{1}{4M^2}) & \text{in the case B.} \end{cases}$$

Thus, by (23) and (34) we deduce that for $t \in [0, T)$ it holds in the case A

$$\begin{aligned}
&|A_{k+1}(t)| \\
&\leq \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds + 4M^2 \int_0^t \int_0^s \frac{\mu(T-r)}{(T-r)^2} dr \int_0^s \frac{\mu(T-r)}{(T-r)^2} dr ds \\
&= \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds + 4M^2 \Big(\int_0^t \int_0^s \frac{\mu(T-r)}{(T-r)^2} dr ds \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds \\
&\qquad\qquad - \int_0^t \int_0^s \int_0^r \frac{\mu(T-\tau)}{(T-\tau)^2} d\tau dr \frac{\mu(T-s)}{(T-s)^2} ds \Big) \\
&\leq \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds + 4M^2 \int_0^T \int_0^s \frac{\mu(T-r)}{(T-r)^2} dr ds \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds \\
&\leq (1 + 4M^2 \alpha(T)) \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds \leq 2M \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds.
\end{aligned}$$

Moreover, we conclude in the case B

$$\begin{aligned}
|A_{k+1}(t)| &\leq \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds + 4M^2 \int_0^t \frac{\mu(T-s)^2}{(T-s)^2} ds \\
&\leq (1 + 4M^2 \mu(T)) \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds \\
&\leq 2 \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds \leq 2M \frac{\mu(T-t)}{T-t}.
\end{aligned}$$

Consequently, the solution $A(t)$ to (32) satisfies for $t \in [0, T)$

$$|A(t)| \leq \begin{cases} 2M \int_0^t \frac{\mu(T-s)}{(T-s)^2} ds & \text{in the case A,} \\ 2M \frac{\mu(T-t)}{T-t} & \text{in the case B.} \end{cases} \tag{36}$$

Therefore, by (30), (31) and (36) we have

$$\mathcal{E}(t) \leq \begin{cases} E(0) \exp \Big[2M\alpha(T) + T \Big] & \text{in the case A,} \\ E(0) \exp \Big[2M \int_0^t \frac{\mu(T-s)}{T-s} ds + T \Big] & \text{in the case B.} \end{cases}$$

Hence, by (30) and (36) it follows that

$$\begin{aligned}\|u(t,\cdot)\|_{H^1} &\leq \begin{cases} E(0)\exp\Big[2M\alpha(T)+T\Big] & \text{in the case A}\\ E(0)\exp\Big[2M\int_0^t \frac{\mu(T-s)}{T-s}ds+T\Big] & \text{in the case B}\end{cases}\\ &\leq C_T\exp\Big[K\int_0^t \frac{\mu(T-s)}{T-s}ds\Big],\end{aligned}$$

and by (35)

$$\begin{aligned}&\|u_t(t,\cdot)\|_{L^2}\\ &\leq \begin{cases} E(0)(1+2M\int_0^t \frac{\mu(T-s)}{(T-s)^2}ds)\exp\Big[2M\alpha(T)+T\Big] & \text{in c. A,}\\ E(0)(1+2M\frac{\mu(T-t)}{T-t})\exp\Big[2M\int_0^t\frac{\mu(T-s)}{T-s}ds+T\Big] & \text{in c. B,}\end{cases}\\ &\leq \begin{cases} C_T(1+\int_0^t \frac{\mu(T-s)}{(T-s)^2}ds) & \text{in the case A,}\\ C_T(1+\frac{\mu(T-t)}{T-t})\exp\Big[K\int_0^t\frac{\mu(T-s)}{T-s}ds\Big] & \text{in the case B,}\end{cases}\\ &\leq C_T\Big(1+\int_0^t\frac{\mu(T-s)}{(T-s)^2}ds\Big)\exp\Big[K\int_0^t\frac{\mu(T-s)}{T-s}ds\Big].\end{aligned}$$

This implies (25) and (26). In this way all statements are proved. □

EXAMPLE 4.1. *If $a(t)=at^\beta$, $\beta\in(0,1)$ (resp. $a(t)=at$), then*

$$u\in L^\infty((0,T),H^s)\quad\text{and}\quad (T-t)^{1-\beta}u_t\in L^\infty((0,T),H^{s-1}),$$

$$\Big(\text{resp.}\quad u\in L^\infty((0,T),H^s)\quad\text{and}\quad \frac{u_t}{\log(T-t)}\in L^\infty((0,T),H^{s-1})\Big).$$

EXAMPLE 4.2. *If $a(t)=a(\log t^{-1})^{-1}\cdots(\log^{[n]}t^{-1})^{-\gamma},\gamma>1$ for $t\in(0,T]$, then by using (27) we get*

$$u\in L^\infty((0,T),H^s),$$
$$(T-t)(\log(T-t))\cdots(\log^{[n]}(T-t))^\gamma u_t\in L^\infty((0,T),H^{s-1}).$$

EXAMPLE 4.3. *If $a(t)=a(\log t^{-1})^{-1}\cdots(\log^{[n]}t^{-1})^{-1}$ for $t\in(0,T]$, then by using (25) and (26) we get*

$$\Big(\log^{[n]}(T-t)\Big)^{-aK}u\in L^\infty((0,T),H^s),$$
$$(T-t)\log(T-t)\Big(\log^{[n]}(T-t)\Big)^{-aK}u_t\in L^\infty((0,T),H^{s-1}).$$

Example 4.4. *Let us consider the model Cauchy problem*

$$u_{tt} - \triangle u + \frac{a}{(T-t)^{4/3}} u = 0, \;\; u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x),$$

with $a \in \mathbb{R}$. Using domain of dependence property it is sufficient to understand properties of solutions to

$$u_{tt} + \frac{a}{(T-t)^{4/3}} u = 0, \; u(0) = u_0, \; u_t(0) = u_1.$$

Its solution is given by

$$\begin{aligned} u(t) &= \Big(\frac{(T-t)^{1/3}u_0}{T^{1/3}} + \frac{\{(T-t)^{1/3} - T^{1/3}\}u_1}{3a}\Big) \\ &\qquad\qquad \cdot \cosh\Big(3\sqrt{-a}\big(T^{1/3} - (T-t)^{1/3}\big)\Big) \\ &\quad + \Big(\frac{u_0}{3\sqrt{-a}T^{1/3}} + \frac{(T-t)^{1/3}T^{1/3}u_1}{\sqrt{-a}} + \frac{u_1}{9a\sqrt{-a}}\Big) \\ &\qquad\qquad \cdot \sinh\Big(3\sqrt{-a}\big(T^{1/3} - (T-t)^{1/3}\big)\Big). \end{aligned}$$

Its derivative is given by

$$\begin{aligned} u'(t) &= \frac{T^{1/3}u_1}{(T-t)^{1/3}} \cosh\Big(3\sqrt{-a}\big(T^{1/3} - (T-t)^{1/3}\big)\Big) \\ &\quad + \frac{1}{(T-t)^{1/3}}\Big(\frac{\sqrt{-a}u_0}{T^{1/3}} - \frac{u_1}{3\sqrt{-a}}\Big) \\ &\qquad\qquad \cdot \sinh\Big(3\sqrt{-a}\big(T^{1/3} - (T-t)^{1/3}\big)\Big). \end{aligned}$$

In particular, for $u_0 = -\frac{1}{3a}$, $u_1 = \frac{1}{T^{1/3}}$, we obtain

$$u'(t) = \frac{1}{(T-t)^{1/3}} \cosh\Big(3\sqrt{-a}\big(T^{1/3} - (T-t)^{1/3}\big)\Big) \geq \frac{1}{(T-t)^{1/3}}.$$

In the case $a > 0$ we write the above representations in the form

$$\begin{aligned} u(t) &= \Big(\frac{(T-t)^{1/3}u_0}{T^{1/3}} + \frac{\big((T-t)^{1/3} - T^{1/3}\big)u_1}{3a}\Big) \\ &\qquad\qquad \cdot \cos\Big(3\sqrt{a}\big(T^{1/3} - (T-t)^{1/3}\big)\Big) \\ &\quad + \Big(\frac{u_0}{3\sqrt{a}T^{1/3}} + \frac{(T-t)^{1/3}T^{1/3}u_1}{\sqrt{a}} + \frac{u_1}{9a\sqrt{a}}\Big) \\ &\qquad\qquad \cdot \sin\Big(3\sqrt{a}\big(T^{1/3} - (T-t)^{1/3}\big)\Big), \end{aligned}$$

and

$$\begin{aligned} u'(t) \;=\; & \frac{T^{1/3}u_1}{(T-t)^{1/3}} \cos\Big(3\sqrt{a}\big(T^{1/3}-(T-t)^{1/3}\big)\Big) \\ & + \frac{1}{(T-t)^{1/3}}\Big(\frac{-\sqrt{a}u_0}{T^{1/3}} - \frac{u_1}{3\sqrt{a}}\Big)\sin\Big(3\sqrt{a}\big(T^{1/3}-(T-t)^{1/3}\big)\Big). \end{aligned}$$

In consequence this example shows that the statement of Theorem 4.1 is sharp.

5. General time-dependent singular potentials

Let us devote to the Cauchy problem

$$u_{tt} - \triangle u + \frac{a(t)}{t^2}u = 0, \;\; u(T,x) = u_0(x), \;\; u_t(T,x) = u_1(x), \tag{37}$$

where the coefficient $a(t)$ satisfies the condition (2). Our strategy is to write the Cauchy problem in the form

$$\begin{aligned} & u_{tt} - \triangle u + \frac{a_0}{t^2}u = \frac{a_0 - a(t)}{t^2}u, \\ & u(T,x) = u_0(x), \quad u_t(T,x) = u_1(x), \end{aligned} \tag{38}$$

and to interpret $\frac{a_0-a(t)}{t^2}u$ as a right-hand side. From the statements of Theorems 2.1 to 2.4 we have the optimal regularity of solutions for $a(t) \equiv a_0$. In this section we study the following question:

In which way does the right-hand side of (38) influence the regularity behavior of solutions and its derivatives up to $t = T$?

Using the successive approximation scheme

$$u_{tt}^{(k+1)} - \triangle u^{(k+1)} + \frac{a_0}{t^2}u^{(k+1)} = \frac{a_0 - a(t)}{t^2}u^{(k)}, \tag{39}$$

$$u^{(k+1)}(T,x) = u_0(x), \;\; u_t^{(k+1)}(T,x) = u_1(x), \tag{40}$$

we will determine the regularity of solutions $\{u^{(k+1)}\}$ tending to a limit element u having the same regularity and being the solution to (37).

Now we consider reference functions μ satisfying the following condition:

$$\text{if} \int_0^T \frac{\mu(s)}{s} ds = \infty, \ \text{then} \ \mu'(s) \leq \delta \frac{\mu(s)}{s} \ \text{with} \ \delta < 1/2 \tag{41}$$

for all $s \in (0, T]$.

THEOREM 5.1. *Let us consider the Cauchy problem*

$$u_{tt} - \triangle u + \frac{a(T-t)}{(T-t)^2} u = 0, \ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x),$$

where $a(t)$ satisfies the conditions (2) and (41) with $a_0 > 1/4$.

If the data u_0, u_1 belong to H^s, H^{s-1}, respectively, then there exists a unique solution $u \in C([0,T), H^s) \cap C^1([0,T), H^{s-1})$ such that

$$\Big(1 + \int_0^t \frac{\mu(T-s)}{T-s} ds\Big)^{-1} u \in L^\infty((0,T), H^s),$$

and

$$\Big((T-t)^{-\frac{1}{2}} + \frac{\mu(T-t)}{T-t} \int_0^t \frac{\mu(T-s)}{T-s} ds\Big)^{-1} u_t \in L^\infty((0,T), H^{s-1}).$$

REMARK 5.1. *If $\int_0^T \frac{\mu(s)}{s} ds < \infty$, then $u \in L^\infty((0,T), H^s)$.*

Proof. It is clear that a small T is sufficient to study. Let us recall the approximation scheme with $u^{(0)}$ being the solution to

$$u^{(0)}_{tt} - \triangle u^{(0)} + \frac{a_0}{t^2} u^{(0)} = 0, \quad u^{(0)}(T,x) = u_0(x), \ u^{(0)}_t(T,x) = u_1(x).$$

The statements of Theorems 2.1 to 2.3 explain us the regularity and the asymptotical behavior of the solution $u^{(0)}$ up to $t = 0$. Taking into consideration this regularity then we will show in the next step that a special regularity and asymptotical behavior of $u^{(k)}$

in $f_k(t,x) := \frac{a_0-a(t)}{t^2}u^{(k)}$ is transferred to the solution $u^{(k+1)}$ by (39), (40). For this reason we study in the phase space the Cauchy problem

$$v_{tt}^{(k+1)} + |\xi|^2 v^{(k+1)} + \frac{a_0}{t^2}v^{(k+1)} = g_k(t,\xi),$$
$$v^{(k+1)}(T,\xi) = \hat{u}_0(\xi),\ v_t^{(k+1)}(T,\xi) = \hat{u}_1(\xi),$$

with $g_k(t,\xi) := \dfrac{a_0 - a(t)}{t^2}v^{(k)}(t,\xi), \quad v^{(0)}(t,\xi) \equiv 0.$

The application of the principle of variation of constants gives us the representation of solution

$$\begin{aligned} v^{(k+1)}(t,\xi) = \quad & V_1(t,\xi)\Big(\hat{u}_0(\xi) + \int_T^t \frac{-V_2(s,\xi)g_k(s,\xi)}{D(s,\xi)}ds\Big) \\ & + V_2(t,\xi)\Big(\hat{u}_1(\xi) + \int_T^t \frac{V_1(s,\xi)g_k(s,\xi)}{D(s,\xi)}ds\Big) \end{aligned}$$

For the discriminant $D(s,\xi) = V_1(s,\xi)\partial_s V_2(s,\xi) - V_2(s,\xi)\partial_s V_1(s,\xi)$ we get $\partial_s D(s,\xi) = 0$ and setting $s = T$ we have $D(s,\xi)$=1.

In the *hyperbolic zone* $\{t_\xi \le t \le T\}$ we use the estimates for $V_1(t,\xi)$ and for $V_2(t,\xi)$ from Section 2.1 and proceed as follows: due to Section 2.1 we have the estimate $|v^{(1)}(t,\xi)| \le C(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|)$ for $|\xi|$ large. Let us assume

$$|v^{(k)}(t,\xi)| \le H\Big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|\Big)\Big(1 + \int_T^t -\frac{\mu(s)}{s}ds\Big) \tag{42}$$

for $k \ge 2$, where the constant H is independent of k. We will show that the same estimate holds for $v^{(k+1)}$. Taking into account the estimates for V_1 and V_2 from Section 2.1, then

$$\begin{aligned} & |v^{(k+1)}(t,\xi)| \le C(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|) \\ & \quad + CH(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|)\frac{1}{|\xi|}\Big(\int_T^t -\frac{\mu(s)}{s^2}\Big(1 + \int_T^s -\frac{\mu(r)}{r}dr\Big)ds\Big). \end{aligned}$$

By the definition of the hyperbolic zone it follows for the first integral

$$\frac{1}{|\xi|}\int_T^t -\frac{\mu(s)}{s^2}ds \le \frac{\max\limits_{t\in[0,T]}\mu(t)}{N}.$$

For the second integral we obtain

$$\begin{aligned}\frac{1}{|\xi|}\int_T^t -\frac{\mu(s)}{s^2}\int_T^s -\frac{\mu(r)}{r}drds = \frac{1}{|\xi|}\frac{\mu(t)}{t}\int_T^t -\frac{\mu(s)}{s}ds - \frac{1}{|\xi|}\int_T^t -\frac{\mu(s)^2}{s^2}ds\\ -\frac{1}{|\xi|}\int_T^t \frac{\mu'(s)}{s}\int_T^s -\frac{\mu(r)}{r}drds.\end{aligned}$$

If we use assumption (41) (where it is sufficient that $\delta < 1$), then the last integral can be included into the left-hand side. Hence, we have only to take into consideration the first integral. By using again the definition of the hyperbolic zone we arrive at

$$\begin{aligned}|v^{(k+1)}(t,\xi)| &\le \frac{C}{1-\delta}(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|)\\ &\quad + \frac{CH}{1-\delta}\frac{\max\limits_{t\in[0,T]}\mu(t)}{N}(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|)\Big(1 + \int_T^t -\frac{\mu(s)}{s}ds\Big)\\ &\le H(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|)\Big(1 + \int_T^t -\frac{\mu(s)}{s}ds\Big)\end{aligned}$$

if we choose N sufficiently large. But this is (42) for $v^{(k+1)}$. If $\int_0^T \frac{\mu(s)}{s}ds < \infty$, then (42) follows immediately from

$$\begin{aligned}|v^{(k+1)}(t,\xi)| &\le C(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|)(1 + \frac{1}{|\xi|}\int_T^t -\frac{\mu(s)}{s^2}ds)\\ &\le C_N(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|).\end{aligned}$$

Summary: *for all k and $t \in [t_\xi, T]$ the estimate (42) holds with a constant H independent of k.*

In the *pseudo-differential zone* $\{0 < t \le t_\xi\}$ we use the representation

$$v^{(k+1)}(t,\xi) = V_1(t,\xi)\hat{u}_0(\xi) + V_2(t,\xi)\hat{u}_1(\xi)$$
$$+\Big(\int_{t_\xi}^{t} -V_2(s,\xi)g_k(s,\xi)ds\Big)V_1(t,\xi) + \Big(\int_{t_\xi}^{t} V_1(s,\xi)g_k(s,\xi)ds\Big)V_2(t,\xi)$$
$$+\Big(\int_{T}^{t_\xi} -V_2(s,\xi)g_k(s,\xi)ds\Big)V_1(t,\xi) + \Big(\int_{T}^{t_\xi} V_1(s,\xi)g_k(s,\xi)ds\Big)V_2(t,\xi).$$

Using the asymptotic behavior of V_1 and V_2 as determined in Section 2.1 and arguing similarly as we did in the hyperbolic zone, we may conclude

$$\Big|\Big(\int_{T}^{t_\xi} -V_2(s,\xi)g_k(s,\xi)ds\Big)V_1(t,\xi) + \Big(\int_{T}^{t_\xi} V_1(s,\xi)g_k(s,\xi)ds\Big)V_2(t,\xi)\Big|$$
$$\le H\Big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|\Big)\Big(1 + \int_{T}^{t_\xi} -\frac{\mu(s)}{s}ds\Big), \tag{43}$$

where H is independent of k. To estimate the remaining integrals we proceed as follows: let us assume (42) for $v^{(k)}$, $k \ge 2$. We obtain

$$\Big|\Big(\int_{t_\xi}^{t} -V_2(s,\xi)g_k(s,\xi)ds\Big)V_1(t,\xi) + \Big(\int_{t_\xi}^{t} V_1(s,\xi)g_k(s,\xi)ds\Big)V_2(t,\xi)\Big|$$
$$\le CH(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|)\frac{1}{|\xi|}(t|\xi|)^{1/2}$$
$$\cdot\Big(\int_{t_\xi}^{t} -(s|\xi|)^{1/2}\frac{\mu(s)}{s^2}\Big(1 + \int_{T}^{s} -\frac{\mu(r)}{r}dr\Big)ds\Big).$$

It remains to compute

$$t^{1/2}\Big(\int_{t_\xi}^{t} -\frac{\mu(s)}{s^{3/2}}\Big(1 + \int_{T}^{s} -\frac{\mu(r)}{r}dr\Big)ds\Big).$$

For the first integral we only use $s \geq t$. For the second integral we have

$$t^{1/2}\Big(\int_{t_\xi}^{t} -\frac{\mu(s)}{s^{3/2}} \int_{T}^{s} -\frac{\mu(r)}{r} dr ds\Big) \leq$$
$$2\mu(t)\int_{t_\xi}^{t} -\frac{\mu(s)}{s} ds + 2t^{1/2}\int_{t_\xi}^{t} \frac{\mu(s)^2}{s^{3/2}} ds - 2t^{1/2}\int_{t_\xi}^{t} \frac{\mu'(s)}{s^{1/2}} \int_{T}^{s} -\frac{\mu(r)}{r} dr ds.$$

Again we can use (41), now with $\delta < 1/2$, and we include the third integral into the left-hand side. Hence,

$$\begin{aligned}\Big|\Big(\int_{t_\xi}^{t} -V_2(s,\xi)g_k(s,\xi)ds\Big)V_1(t,\xi) + \Big(\int_{t_\xi}^{t} V_1(s,\xi)g_k(s,\xi)ds\Big)V_2(t,\xi)\Big| \\ \leq H\Big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|\Big)\Big(1 + \int_{t_\xi}^{t} -\frac{\mu(s)}{s} ds\Big),\end{aligned} \tag{44}$$

where H is independent of k. From (43) and (44) it follows that (42) is also satisfied for $t \in (0, t_\xi]$. Finally, if $\int_0^T \frac{\mu(s)}{s} ds < \infty$, then (42) follows immediately from

$$|v^{(k+1)}(t,\xi)| \leq C\int_{0}^{T} \frac{\mu(s)}{s} ds (t|\xi|)^{1/2}(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|).$$

We can show that a sufficiently large N in the hyperbolic zone or a small T in the pseudo-differential zone guarantee the Cauchy sequence property of $v^{(k)}(t,\xi)$ for each fixed (t,ξ) from the extended phase space. Thus the limit element $v = v(t,\xi)$ fulfils

$$\begin{aligned}|v(t,\xi)| &\leq H\Big(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|/|\xi|\Big)\Big(1 + \int_{T}^{t} -\frac{\mu(s)}{s} ds\Big) \\ &\leq C\langle\xi\rangle^s\Big(1 + \int_{T}^{t} -\frac{\mu(s)}{s} ds\Big)\end{aligned}$$

together with the assumptions for the data. This proves the *first statement* of the theorem. To prove the second statement we have only to differentiate

$$
\begin{aligned}
v(t,\xi) = \quad & V_1(t,\xi)\Big(\hat{u}_0(\xi) + \int_T^t -V_2(s,\xi)\frac{a_0 - a(s)}{s^2} v(s,\xi) ds\Big) \\
& + V_2(t,\xi)\Big(\hat{u}_1(\xi) + \int_T^t V_1(s,\xi)\frac{a_0 - a(s)}{s^2} v(s,\xi) ds\Big)
\end{aligned} \tag{45}
$$

with respect to t and we have to take account of the asymptotic behavior of $V_1, V_2, V_{1,t}, V_{2,t}$ and the first statement of this theorem. This yields immediately the *second statement* if we remark that the asymptotic behavior of v_t or u_t respectively, will be determined by the asymptotic behavior of

$$
V_{1,t}(t,\xi)\hat{u}_0(\xi), \quad V_{2,t}(t,\xi)\hat{u}_1(\xi), \quad V_1(t,\xi)V_2(t,\xi)\frac{\mu(t)}{t^2} v(t,\xi),
$$

$$
\int_T^t V_2(s,\xi)\frac{\mu(s)}{s^2} v(s,\xi) ds V_{1,t}(t,\xi), \ \int_T^t V_1(s,\xi)\frac{\mu(s)}{s^2} v(s,\xi) ds V_{2,t}(t,\xi)
$$

from (45). This completes the proof. □

Following the same strategy we can prove the following results:

THEOREM 5.2. *Let us consider the Cauchy problem*

$$
u_{tt} - \triangle u + \frac{a(T-t)}{(T-t)^2} u = 0, \ \ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x),
$$

where $a(t)$ satisfies the conditions (2) and (41) with $a_0 = 1/4$. Moreover, we assume for the reference function μ the condition

$$
(\log t)^2 \mu(t) \to 0 \ \ \textit{for} \ \ t \to +0. \tag{46}
$$

If the data u_0, u_1 belong to H^s, H^{s-1} respectively, then there exists a unique solution $u \in C([0,T], H^s) \cap C^1([0,T), H^{s-1})$ such that

$$\Big((T-t)^{-\frac{1}{2}}\big(\log\tfrac{1}{T-t}\big)+\tfrac{\mu(T-t)}{T-t}\big(\log\tfrac{1}{T-t}\big)^2\Big)^{-1}u_t\in L^\infty((0,T),H^{s-1}).$$

THEOREM 5.3. *Let us consider the Cauchy problem*

$$u_{tt}-\triangle u+\frac{a(T-t)}{(T-t)^2}u=0,\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),$$

where $a(t)$ satisfies the conditions (2) and (41) with $a_0\in(0,1/4)$. Moreover, we assume for the reference function μ the condition

$$t^{-\sqrt{1-4a_0}}\mu(t)\le C\ \ for\ \ t\in(0,T]. \tag{47}$$

If the data u_0, u_1 belong to H^s, H^{s-1}, respectively, then there exists a unique solution $u\in C([0,T],H^s)\cap C^1([0,T),H^{s-1})$ such that

$$\Big(\frac{1}{(T-t)^{\frac{1+\sqrt{1-4a_0}}{2}}}+\frac{\mu(T-t)}{(T-t)^{1+\sqrt{1-4a_0}}}\Big)^{-1}u_t\in L^\infty((0,T),H^{s-1}).$$

REMARK 5.2. *The Theorems 5.1 to 5.3 are generalizations of the Theorems 2.1 to 2.3.*

REMARK 5.3. *Without the conditions (46) or (47) we are only able to apply Gronwall's lemma to (45). This gives the regularity*

$$\exp\Big(-\int_0^t\frac{\mu(T-s)}{T-s}\Big(\log\frac{1}{T-s}\Big)^2ds\Big)u\in L^\infty((0,T),H^s),$$
$$\exp\Big(-\int_0^t\frac{\mu(T-s)}{(T-s)^{1+\sqrt{1-4a_0}}}ds\Big)u\in L^\infty((0,T),H^s),$$

respectively, for general reference functions μ. Under the assumptions (46) or (47) these statements are weaker than those from Theorems 5.2 or 5.3.

6. Concluding remarks

1. In this paper we are not interested in Cauchy problems

$$u_{tt}-\triangle u+\frac{a(t)}{t^2}u=0,\ \ u(T,x)=u_0(x),\ \ u_t(T,x)=u_1(x), \tag{48}$$

with $a_0 < 0$. The main reason is that the regularity of solutions from Theorem 2.4 with respect to t is too bad. The next question for the regularity behavior of solutions with respect to t could be of interest.

Under which assumptions to the reference function μ does the regularity behavior of solutions to (48) coincide with the regularity behavior of solutions to

$$u_{tt} - \triangle u + \frac{a_0}{t^2}u = 0, \quad u(T,x) = u_0(x), \quad u_t(T,x) = u_1(x)?$$

2. If the corresponding reference function $\mu = \mu(t)$ does not satisfy (46) or (47), then we may introduce a regularization a^* of a with $\lim_{t\to+0} a(t) = \lim_{t\to+0} a^*(t) = a_0$. The regularization a^* belongs to $C^\infty[0,T]$. Thus the reference function μ^* satisfies (46) or (47). In consequence we have to study now

$$u_{tt} - \triangle u + \frac{a^*(T-t)}{(T-t)^2}u = \frac{a^*(T-t) - a(T-t)}{(T-t)^2}u.$$

The goal is to prove that the statements of Theorems 3.3 or 3.5 are applicable with H^s replaced by $\gamma(D_x)H^s$. Hence, we have a *loss of regularity with respect to the spatial variables.*

3. In analogy with the studies on the behavior of the solutions to nonlinear hyperbolic systems as presented in [7], [9] or [1] it should be of interest to consider the Cauchy problem

$$\begin{aligned}
&u_{tt} - \triangle u + \frac{a_1(T-t)}{(T-t)^{\beta_1}}v = 0, \quad v_{tt} - \triangle v + \frac{a_2(T-t)}{(T-t)^{\beta_2}}u = 0,\\
&u(0,x) = u_0(x),\ v(0,x) = v_0(x),\\
&u_t(0,x) = u_1(x),\ v_t(0,x) = v_1(x).
\end{aligned}$$

Here the regularity of solutions up to $t = T$ is of interest.

Acknowledgements. A part of this paper was written during three one week stays of the third author at the Department of Mathematics of University of Trieste in 2005-2007. He thanks the staff of the Department of Mathematics, in particular, Prof's D.Del Santo and B.Zimmermann for their hospitality.

References

[1] R. Agemi R., Y. Kurokawa and H. Takamura, *Critical curve for p-q systems of nonlinear wave equations in three space dimensions*, J.Differential Equations **167** (2000), 87–133.

[2] G. Alexandrian, *Parametrix and propagation of the wave front of a solution to a Cauchy problem for a model hyperbolic equation* (in Russian), Izv. Akad.Nauk Arm.SSR **19** (1984) 33–46.

[3] S. Alinhac, *Blowup for Nonlinear Hyperbolic Equations*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 17, Birkhäuser, Boston, 1995.

[4] H. Bateman and A. Erdelyi, *Higher transcendental functions*, vol.1, Mac Graw-Hill, New York, 1953.

[5] Ch. Böhme, *Generalized energy conservation for Klein-Gordon type models with time-dependent mass*, Master thesis (in preparation).

[6] A. Galstian, *$L_p - L_q$ decay estimates for the wave equations with exponentially growing speed of propagation*, Applicable Analysis **82** (2003) 3, 197–214.

[7] D. Del Santo, V. Georgiev and E. Mitidieri, *Global existence of the solutions and formation of singularities for a class of hyperbolic systems*, Geometrical optics and related topics (Cortona, 1996), 117–140, Progr. Nonlinear Differential Equations Appl. **32**, 117–140 Birkhäuser, Basel-Boston-MA (1997).

[8] F. Hirosawa and M. Reissig, *From wave to Klein-Gordon type decay rates*, Eds. S. Albeverio et.al. Operator Theory Advances and Applications **145**, 95–155, Birkhäuser, Basel-Boston-Berlin (2003).

[9] H. Kubo and M. Ohta, *On the global behavior of classical solutions to coupled systems of semilinear wave equations*, Eds. M. Reissig and B.-W. Schulze, New trends in the theory of hyperbolic equations, 113–211, Operator Theory Advances and Applications **159**, 113–211, Birkhäuser, Basel-Boston-Berlin (2005).

[10] S. Lucente, *On a class of semilinear weakly hyperbolic equations*, Annali dell' Universita di Ferrara **52** (2006) 2, 317–335.

[11] M. Reissig, *On $L_p - L_q$ estimates for solutions of a special weakly hyperbolic equation*, Ed. Li Ta-Tsien, Nonlinear Evolution Equations and Infinite-Dimensional Dynamical Systems, 153–164, World Scientific (1997).

[12] M. Reissig and K. Yagdjian, *Klein-Gordon type decay rates for wave equations with time-dependent coefficients*, Banach Center Publications **52** (2000) 189–212.

[13] C. Sogge, Lectures on nonlinear wave equations. Monographs in Analysis, II. International Press, Boston, MA, (1995).

[14] K. Taniguchi and Y. Tozaki, *A hyperbolic equation with double characteristics which has solution with branching singularities*, Math. Japonica **25** (1980) 279–300.

[15] A. Weinstein, *On the wave equation and the equation of Euler-Poisson*, Proceedings of Symposia in Applied Mathematics **5**, 137–147, McGraw-Hill, New York-Toronto-London (1954).

[16] J. Wirth, *Solution representations for a wave equation with weak dissipation*, Math. Meth. Appl. Sc. **27** (2004) 1, 101–124.

[17] K. Yagdjian, *The Cauchy problem for hyperbolic operators. Multiple characteristics.Micro-local approach*, Akademie-Verlag, Berlin, 1997.

[18] K. Yagdjian, *A note on the fundamental solution for the Tricomi-type equation in the hyperbolic domain*, J.Differential Equations **206** (2004), 227–252.

[19] K. Yagdjian, *Global existence for the n-dimensional semilinear Tricomi-type equations*, Comm. in PDE **31** (2006) 907–944.

Received November 19, 2007.