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Kakutani's Splitting Procedure in Higher Dimension

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Dedicated to the memory of Fabio Rossi

SUMMARY. - *In this paper we will generalize to higher dimension the splitting procedure introduced by Kakutani for* $[0,1]$. *This method will provide a sequence of nodes belonging to* $[0,1]^d$ *which is uniformly distributed. The advantage of this approach is that it is intrinsecally d-dimensional.*

1. Introduction

A partition π of $I = [0,1]$ is a finite covering of I by a family of intervals $[t_{i-1}, t_i]$, with $1 \leq i \leq k$ and $t_{i-1} < t_i$, with pairwise disjoint interiors. In 1976 Kakutani introduced the very interesting notion of *uniformly distributed sequence of partitions* of the interval $[0,1]$.

DEFINITION 1.1. *If* π *is any partition of* $[0,1]$, *and* $\alpha \in]0,1[$, *its Kakutani's* α*-refinement* $\alpha\pi$ *is obtained by splitting all the intervals of* π *having maximal length in two parts, having lengths (left and right) proportional to* α *and* $\beta = 1 - \alpha$, *respectively.*

Kakutani's sequence of partitions $\{\kappa_n\}$ is obtained by successive α-refinements of the trivial partition $\omega = \{[0,1]\}$. For example, if $\alpha < \beta$, $\kappa_1 = \{[0,\alpha], [\alpha,1]\}$, $\kappa_2 = \{[0,\alpha], [\alpha, \alpha + \alpha\beta], [\alpha + \alpha\beta, 1]\}$, and so on.

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Definition 1.2. *Given a sequence of partitions* $\{\pi_n\}$, *with*

$$\pi_n = \{[t_{i-1}^n, t_i^n], 1 \le i \le k(n)\}\,,$$

we say that it is uniformly distributed, if for any continuous function f *on* $[0,1]$ *we have*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(t_i^n) = \int_0^1 f(t)\,dt\,.$$

We denote, as usual, by δ_t the Dirac measure concentrated in t.

Remark 1.3. *It follows from the definition that uniform distribution of the sequence* $\{\pi_n\}$ *is equivalent to the weak convergence of the sequence of measures*

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \delta_{t_i^n}$$

to the Lebesgue measure λ *on* $[0,1]$.

Remark 1.4. *It is obvious that the uniform distribution of the sequence of partitions* $\{\pi_n\}$ *is equivalent to each of the following two conditions:*

1. *For any choice of points* $\tau_i \in [t_{i-1}^n, t_i^n]$ *we have*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(\tau_i^n) = \int_0^1 f(t)\,dt\,,$$

for any continuous function f *on* $[0,1]$.

2. *For any choice of points* $\tau_i \in [t_{i-1}^n, t_i^n]$ *the sequence of measures*

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \delta_{\tau_i^n}$$

converges weakly to the Lebesgue measure λ *on* $[0,1]$.

The following beautiful theorem is the main result of [5]:

THEOREM 1.5. *For any $\alpha \in]0,1[$ the sequence of partitions $\{\kappa_n\}$ is uniformly distributed.*

This result got a considerable attention in the late seventies, when other authors provided different proofs of Kakutani's theorem and also proved its stochastic version [8]. The paper [1] extended the notion to compact metric spaces, and put in connection to a question rised by De Bruijn and Post, which has been addressed also in [7].

The aim of this paper is to extend Kakutani's splitting procedure to higher dimension.

It is convenient to introduce for later convenience the useful standard notation for the so called "α-dyadic" intervals. Let $I(\alpha) = [0, \alpha]$ and $I(\beta) = [\alpha, 1]$. If $I(\gamma_1 \dots \gamma_m) = [a, b]$ (with $\gamma_k \in \{\alpha, \beta\}$ for $1 \le k \le m$), then

$$I(\gamma_1 \dots \gamma_m \alpha) = [a, a + \alpha(b-a)]$$

and

$$I(\gamma_1 \dots \gamma_m \beta) = [a + \alpha(b-a), b]\,.$$

Naturally $\lambda(I(\gamma_1 \dots \gamma_m)) = \gamma_1 \dots \gamma_m = \alpha^p \beta^q$, where $p + q = m$ and p is the number of occurencies of α among the γ_k's, while q is the number of the occurencies of β.

2. Splitting the d-dimensional cube

By $I^d = [0,1]^d$ we denote the unit cube of $\mathbb{R}^d$. By a cartesian d-rectangle (or simply a rectangle) contained in I^d we always mean a set of the type $R = \prod_{j=1}^d [a_j, b_j]$. We denote by $v_i = (a_1, \dots, a_d)$ the *left endpoint* of R.

A *partition* of I^d will always mean in this paper a finite collection of rectangles $\{R_i, 1 \le i \le k\}$ as defined above, with disjoint interiors and which cover I^d.

The following definition is the natural extension of Kakutani's one-dimensional splitting procedure.

DEFINITION 2.1. *Fix $\alpha \in]0,1[$. If $\pi = \{R_i, 1 \le i \le k\}$ is any partition of $[0,1]^d$, its Kakutani's α-refinement $\alpha\pi$ is obtained by splitting all the rectangles of π having maximal d-dimensional measure λ_d in* *two rectangles, dividing in two segments the longest side such that the lower and upper part have length proportional to* α *and* $\beta = 1-\alpha$*, respectively. If the rectangle* R *has several sides with the same length, we split the side with the smallest coordinate index* j*.*

We define now the generalized Kakutani sequence of partitions $\{\kappa_n^d\}$ of I^d as the successive α-refinements of the trivial partition $\omega = \{I^d\}$.

The definition of uniformly distributed sequence of partitions extends naturally to higher dimension.

Definition 2.2. *Given a sequence of partitions* $\{\pi_n\}$*, with* $\pi_n = \{R_i^n, 1 \le i \le k(n)\}$*, we say that it is uniformly distributed if for any continuous function* f *on* I^d*, we have*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(v_i^n) = \int_{I^d} f(t)\, dt\,.$$

As in the previous section, it is possible to allow, in the above expression, other choices of the points $\sigma_i^n \in R_i^n$ and to express uniform distribution as the weak convergence of

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \delta_{\sigma_i^n}\,,$$

for any choice of $\sigma_i \in R_i^n$, to the d-dimensional Lebesgue measure λ_d on $[0,1]$.

Our aim is to prove that the d-dimensional Kakutani's sequence of partions $\{\kappa_n^d\}$ is uniformly distributed. This will be obtained introducing a convenient notation and proving two preparatory lemmas.

Let us begin with the following notation. By $R(\alpha)$ and $R(\beta)$ we denote the rectangles $[0,\alpha] \times [0,1]^{d-1}$ and $[\beta,1] \times [0,1]^{d-1}$, respectively. If $R(\gamma_1,\ldots,\gamma_m) = \prod_{i=1}^d [a_i,b_i]$ (with $\gamma_k \in \{\alpha,\beta\}$ for $1 \le k \le m$), then we define

$$R(\gamma_1,\ldots,\gamma_m\alpha) = \prod_{i=1}^{j-1}[a_i,b_i] \times [a_j, a_j + \alpha(b_j - a_j)] \times \prod_{i=j+1}^{d} [a_i,b_i]$$

and

$$R(\gamma_1, \ldots, \gamma_m\beta) = \prod_{i=1}^{j-1} [a_i, b_i] \times [a_j + \alpha(b_j - a_j), b_j] \times \prod_{i=j+1}^{d} [a_i, b_i]\,,$$

if

$$b_j - a_j > b_k - a_k$$

for all $1 \leq k < j$ and

$$b_j - a_j \geq b_h - a_h$$

for all $j \leq h \leq d$.

LEMMA 2.3. *The diameter of the Kakutani partition κ_n^d tends to zero, when n tends to infinity.*

Proof. As in the one-dimensional case, every rectangle of κ_n^d is eventually subdivided in two parts, therefore given any $m \in \mathbb{N}$ there exists n_0 such that for $n \geq n_0$ every R_i^n in κ_n^d results from at least md splittings. This implies that each side of R_i^n has length at most L^m, where $L = \max\{\alpha, \beta\} < 1$, and therefore its diameter is smaller than $L^m\sqrt{d}$. □

We have to introduce now in this context a notion which is widely used in the theory of uniformly distributed sequences of points (compare for instance Chapter 3 of [6] or Chapter 1 of [2]).

DEFINITION 2.4. *We say that a class of functions $\mathcal{F}$ is* **determining** *for the uniform convergence of partitions whenever, for a given sequence of partitions $\{\pi_n\}$ ($\pi_n = \{R_i^n, 1 \leq i \leq k(n)\}$), from*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(v_i^n) = \int_0^1 f(t)\,dt$$

for any $f \in \mathcal{F}$, it follows that $\{\pi_n\}$ is uniformly distributed.

By χ_C we will denote the characteristic function of C.

LEMMA 2.5. *Assume $\{\mathcal{C}_n\}$ is a sequence of finite partitions of I^d whose elements C_i^n, $1 \leq i \leq k(n)$, are rectangles and diam $\mathcal{C}_n$ tends to zero. Suppose moreover that for each C_j^m we have*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{C_j^m}(v_i^n) = \lambda_d(C_j^m)\,, \tag{1}$$

where v_i^n is the left endpoint of C_i^n. Then the family $\mathcal{F}$ of the characteristic functions of the C_i^n's is determining.

Proof. It is well known that the family of the characteristic functions of all the rectangles $R = \prod_{j=1}^d [a_j, b_j]$ is determining. So let $R \subset I^d$ be a (non degenerate) rectangle and denote by B the unit ball of $\mathbb{R}^d$. Fix $\varepsilon \in]0,1[$ and let us denote by $R_\varepsilon = (\cup_{z\in R}(z+\varepsilon B)) \cap I^d$.

Let $n_0 \in \mathbb{N}$ be such that for $n \geq n_0$, *diam* $\mathcal{C}_n < \varepsilon$. For such an n, let $\mathcal{C}_n(R)$ be the collection of all the sets in $\mathcal{C}_n$ intersecting R, and let us denote by C_R their union. Then we have $R \subset C_R \subset R_\varepsilon$ and therefore

$$\lambda_d(R) \leq \lambda_d(C_R) \leq \lambda(R_\varepsilon) \leq \lambda_d(R) + c\varepsilon\,, \tag{2}$$

where c is an appropriate constant.

The same inclusions imply that, for arbitrarily small ε,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v_i^n) &\leq \lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{C_R}(v_i^n) = \lambda_d(C_R) \\
&\leq \liminf_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v_i^n) + c\varepsilon \\
&\leq \limsup_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v_i^n) + c\varepsilon\,. \qquad (3)
\end{aligned}$$

The equality in the first line follows from (1). It follows now from (2) and (3) that

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v_i^n) = \lambda_d(R)\,,$$

for any rectangle $R \subset I^d$, and the conclusion follows. □

We are now in position to prove the main result of this paper.

THEOREM 2.6. *The sequence of partitions $\{\kappa_n^d\}$ introduced in Definition 2.1 is uniformly distributed.*

Proof. We apply the previous lemma to the sequence of partitions $\{\kappa_n^d\}$. Since by Lemma 2.3 its diameter tends to zero, we only have to prove that given any $s \in \mathbb{N}$ and any rectangle $R = R_j^s$ belonging to κ_s^d, we have that

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v_i^n) = \lambda_d(R)\,. \tag{4}$$

But from the previous discussion we know that $R_j^s = R(\gamma_1 \dots \gamma_m)$ for appropriate values $\gamma_k \in \{\alpha, \beta\}$. On the other hand there is a one to one correspondence between the rectangles $R(\gamma_1 \dots \gamma_m)$ showing up in the partitions κ_n^d and the intervals $I(\gamma_1 \dots \gamma_m)$ appearing in the one-dimensional partitions κ_n. Since

$$\lambda_d(R(\gamma_1 \dots \gamma_m)) = \gamma_1 \dots \gamma_m = \lambda(I(\gamma_1 \dots \gamma_m))\,,$$

the rectangle $R(\gamma_1 \dots \gamma_m)$ is split into $R(\gamma_1 \dots \gamma_m \alpha)$ and $R(\gamma_1 \dots \gamma_m \beta)$ exactly when the interval $I(\gamma_1 \dots \gamma_m)$ undergoes the same procedure. Now Kakutani's theorem says that $I = I(\gamma_1 \dots \gamma_m)$ is subdivided the right number of times, so that

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_I(t_i^n) = \lambda(I)\,,$$

and therefore the analogous identity (4) holds for $R = R(\gamma_1 \dots \gamma_m)$. $\square$

3. Conclusions

The interest of this result is that it is intrinsecally d-dimensional and this may be useful in applications to integration in higher dimension, where it is important (and not very easy) to find good sets of nodes.

Given κ_n^d, the centers of gravity of the rectangles R_i^n seem to be a convenient choice of nodes.

In a subsequent paper we will compare our results, and other intrinsecally multidimensional methods we are developing, with methods which are based on the subdivision of the one-dimensional factors of I^d as proposed in [3] and [4].

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