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Symmetry of Singular Solutions of Degenerate Quasilinear Elliptic Equations

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In ricordo di Fabio

Summary. - *We prove radial symmetry of singular solutions to an overdetermined boundary value problem for a class of degenerate quasilinear elliptic equations.*

1. Introduction

We consider solutions u to

$$\mathrm{div}(a(|\nabla u|)\nabla u) = 0, \qquad \text{in } \Omega \setminus \{O\}, \tag{1}$$

which vanish on $\partial\Omega$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{2}$$

and have a positive singularity at the origin O

$$\lim_{x \to O} u(x) = M \in (0, +\infty]. \tag{3}$$

We prove that if u satisfies the overdetermined boundary condition

$$\frac{\partial u}{\partial \nu} = -c, \qquad \text{on } \partial\Omega, \tag{4}$$

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with $c > 0$ constant, then Ω is a ball centered at O and u is radially symmetric.

To be more specific, we shall assume Ω to be a bounded connected open set in $\mathbb{R}^n$, $n \geq 2$, containing the origin O, and with $C^{2,\alpha}$-smooth boundary $\partial\Omega$. The nonlinearity a is assumed to be a C^1 function from $(0, +\infty)$ to $(0, +\infty)$ and satisfying the degenerate ellipticity condition

$$0 < \lambda \leq 1 + \frac{sa'(s)}{a(s)} \leq \Lambda, \quad \text{for every } s > 0, \tag{5}$$

for some positive constants λ, Λ.

Such a class of quasilinear degenerate elliptic equations, which strictly contains the one of p-Laplacian type equations, was introduced by Lieberman [5] and independently, in the two dimensional case, by Alessandrini, Lupo and Rosset [3].

With such assumptions the main result of this note is the following.

THEOREM 1.1. *Let $u \in C^{1,\alpha}(\overline{\Omega} \setminus \{O\})$ be a weak solution to* (1), *satisfying the conditions* (2), (3)*. If, in addition, u satisfies* (4) *then Ω is a ball centered at the origin O and u is radially symmetric.*

Let us observe that in view of the regularity results by Lieberman [5] it is reasonable to treat solutions in the $C^{1,\alpha}$-class.

Note also that for singular solutions satisfying (1)–(3) the limit M in (3) may be finite or infinite depending on the nonlinearity a. This fact is particularly evident in the special case when $a(t) = t^{p-2}$, $p > 1$. One readily sees that when $p \leq n$ then we have $M = +\infty$, whereas for $p > n$ one must have $M < +\infty$. See in this respect Kichenassamy and Veron [4] and, for a detailed study of singular solutions in two variables we refer to Rosset [6].

Our proof is based on an adaptation of the well-known method of moving planes by Alexandrov and Serrin. The adaptation of the method to degenerate equations was initiated in [2]. Here we refer mostly to arguments introduced in [1], however, the presence of the singularity adds a little further difficulty, since it may appear, at a first glance, that the method of moving planes cannot be used after the moving plane has crossed the singularity. We shall show that if this is the case for a certain direction ξ, then for the opposite direction $-\xi$ the problem of "hitting the singularity" cannot occur.

REMARK 1.2. *We take this opportunity to point out that an* erratum *is in order in [1]. In fact, it is improperly stated there that the nonlinearity a may depend on $|\nabla u|$ and* also on u. *This is not correct, in fact one should assume $a = a(|\nabla u|)$ and with this proviso all the statements there are correct.*

2. Proof of Theorem 1.1

We recall some basic properties of solutions to equation (1) that we shall use repeatedly in our arguments. Local solutions to the equation

$$\operatorname{div}(a(|\nabla u|)\nabla u) = 0, \tag{6}$$

are obtained as limits in $C^{1,\alpha}$ of solutions u_ϵ to regularized equations

$$\operatorname{div}(a_\epsilon(|\nabla u_\epsilon|)\nabla u_\epsilon) = 0, \tag{7}$$

where a_ϵ satisfies the same conditions as a and in addition is C^∞ and $a_\epsilon \geq \epsilon > 0$ everywhere. Consequently u_ϵ can also be seen as a strong solution to the non-divergence uniformly elliptic equation

$$\sum_{i,j}\left(\delta_{ij} + \frac{a'_\epsilon(|\nabla u_\epsilon|)}{|\nabla u_\epsilon|^2 a_\epsilon(|\nabla u_\epsilon|)}\frac{\partial u_\epsilon}{\partial x_i}\frac{\partial u_\epsilon}{\partial x_j}\right)\frac{\partial^2 u_\epsilon}{\partial x_i x_j} = 0. \tag{8}$$

Consequently solutions to (6) inherit some properties of strong solutions to uniformly elliptic equations.

We quote, in particular, the *Harnack inequality*, that is, there exists $C = C(\lambda, \Lambda)$ such that: if u solves (6) in $B_R(x_0)$ and $u \geq 0$ then

$$\max_{B_{R/2}(x_0)} u \leq C \min_{B_{R/2}(x_0)} u. \tag{9}$$

A further consequence of the use of the regularized solutions is that solutions to (6) satisfy the comparison principle in the weak form. That is, if v, u solve (6) in a domain G and $v \leq u$ on ∂G then $v \leq u$ also inside. In addition, if $|\nabla u| + |\nabla v| > 0$ in $\overline{G}$ then the strong version of the comparison principle holds, that is if $v \leq u$ on ∂G then either $v \equiv u$ or $v < u$ in G.

Let us also observe that, if u is a solution to (6), then for every constant C, also $C - u$ is a solution. Hence, by the Harnack inequality, one readily obtains that the solution to (1)–(3) satisfies

$$0 < u(x) < M, \quad \text{for every } x \in \Omega \setminus \{O\}. \tag{10}$$

Let us now introduce the moving plane apparatus. For any direction $\xi \in \mathbb{R}^n$, $|\xi| = 1$, and for any $t \in \mathbb{R}$, we define the hyperplane

$$\Pi_t^\xi = \{x \in \mathbb{R}^n \mid x \cdot \xi = t\}.$$

We denote by R_t^ξ the reflection in Π_t^ξ, that is

$$R_t^\xi x = 2(t - x \cdot \xi)\xi + x.$$

We shall denote

$$\left(R_t^\xi u\right)(x) = u\left(R_t^\xi x\right).$$

If we agree to say that if $x \cdot \xi < t$, x is on the *left hand side* of Π_t^ξ, and conversely x is on the *right hand side* of Π_t^ξ if $x \cdot \xi > t$, we denote by $R_t^\xi \Omega$ the reflection of the part of Ω which is on the left hand side of Π_t^ξ, that is

$$R_t^\xi \Omega = \{x \in \mathbb{R}^n \mid x \cdot \xi > t, R_t^\xi x \in \Omega\}.$$

Given ξ, we fix $\overline{t}$ such that $R_{\overline{t}}^\xi \Omega = \emptyset$. Letting $t > \overline{t}$ increase, we denote by $t(\xi)$ the largest number such that

$$R_t^\xi \Omega \subset \Omega, \quad \text{for every } t \in (\overline{t}, t(\xi)).$$

As is well-known since Serrin [7], when $t = t(\xi)$ one of the following two cases is satisfied

(i) $\partial(R_t^\xi \Omega)$ is tangent to $\partial\Omega$ at a point $P \notin \Pi_t^\xi$,

(ii) $\partial(R_t^\xi \Omega)$ is tangent to $\partial\Omega$ at a point $P \in \Pi_t^\xi$.

Let us consider the family of moving planes associated to the opposite direction $-\xi$ and the corresponding reflections. One can easily verify that

$$\Pi_t^\xi = \Pi_{-t}^{-\xi}, \quad \text{for every } t,$$

and also

$$R_t^\xi = R_{-t}^{-\xi}, \quad \text{for every } t.$$

Now we observe that for every $s < t(\xi)$ we also have $-s > t(-\xi)$. In fact

$$R_s^\xi \Omega \subsetneqq \Omega \cap \{x \cdot \xi > s\}$$

and therefore, applying $R_{-s}^{-\xi}$ to both sides,

$$\Omega \cap \{x \cdot \xi < s\} \subsetneqq R_{-s}^{-\xi}\Omega,$$

that is $R_{-s}^{-\xi}\Omega$ is *not* contained in Ω and hence $-s > t(-\xi)$.

Hence, letting s increase to $t(\xi)$, we obtain

$$t(\xi) + t(-\xi) \leq 0 \tag{11}$$

Consequently either $t(\xi) = t(-\xi) = 0$ or one of the two numbers $t(\xi)$, $t(-\xi)$ is strictly negative.

If $t(\xi) = t(-\xi) = 0$ then, obviously, Ω is symmetric in $\Pi_0^\xi = \Pi_0^{-\xi}$.

Assume now $t(\xi) < 0$ (the other case $t(-\xi) < 0$ being equivalent). We simplify our notation by posing

$$\Pi = \Pi_{t(\xi)}^\xi, \quad R = R_{t(\xi)}^\xi, \quad G = R_{t(\xi)}^\xi\Omega, \quad v = R_{t(\xi)}^\xi u.$$

Since the origin O is on the right hand side of Π, by (10), we have that there exists N, $0 < N < M$ such that the level set

$$E = \{x \in \Omega \setminus \{O\} \mid u(x) \geq N\}$$

is strictly on the right hand side of Π.

Consequently, on $\overline{G}$, $v = R_{t(\xi)}^\xi u < N$. Now we observe that on $\partial(G \setminus E)$ we have $v \leq u$, in fact $\partial(G \setminus E)$ can be decomposed as

$$\partial(G \setminus E) = (\partial G \cap \Pi) \cup (\partial G \setminus (E \cup \Pi)) \cup (\partial E \cap G)$$

and we have

$$\begin{aligned} v = u, &\qquad \text{on } \partial G \cap \Pi, \\ v = 0 \leq u, &\qquad \text{on } \partial G \setminus (E \cup \Pi), \\ v < N \leq u, &\qquad \text{on } \partial E \cap G. \end{aligned}$$

Hence, by the weak comparison principle,

$$v \leq u, \quad \text{in } G \setminus E,$$

and also, trivially,

$$v < N \leq u, \quad \text{in } G \cap E.$$

Consequently

$$v \leq u \quad \text{in } G. \tag{12}$$

From now on we rephrase arguments in [1] to prove that Π is a plane of symmetry for Ω.

Let U be an ϵ-neighborhood of $\partial\Omega$ in Ω, with ϵ small enough to have $|\nabla u| > 0$ in U and $O \notin U$. Let A be the connected component of $(RU) \cap U$ such that $P \in \partial A$. In view of the boundary conditions (2), (4), we have

$$(u - v)(P) = 0, \quad \nabla(u - v)(P) = 0.$$

Moreover, when case II) occurs, by applying the arguments in [7] to equation (6) we also have that

$$\frac{\partial^2}{\partial \eta^2}(u - v)(P) = 0,$$

for every direction η. Since $A \subset G \cap U$, $u - v$ is a non-negative solution of a uniformly elliptic equation in A. By using the Hopf lemma when case I) occurs and its variant due to Serrin [7, Lemma 2] when case II) occurs and by the strong comparison principle, it follows that

$$u = v, \quad \text{in } \overline{A}. \tag{13}$$

Now, let us prove that $R(\partial\Omega) \subset \partial\Omega$, obtaining that Π is a plane of symmetry of Ω. Assume, by contradiction, that there exists a point $Q \in R(\partial\Omega) \setminus \partial\Omega$. Let γ be an arc in $R(\partial\Omega)$ joining Q with P. One can find a subarc γ' in $\gamma \cap (U \cup \partial\Omega)$ having as endpoints P and a point $R \in U$. Since in any neighborhood of γ' one can find points of $RU \cap U$, which can be joined to P through paths inside $RU \cap U$, it follows that $\gamma' \subset \partial A$. Therefore the set $R(\partial\Omega) \cap U \cap \partial A$ is non-empty and on such a set $v = 0 < u$, contradicting (13).

We have therefore proved that for any direction ξ there exists a plane Π orthogonal to ξ which is a plane of symmetry for Ω. Hence Ω is a ball $B_R(x_0)$. It remains to prove that $x_0 = O$ and that u is radially symmetric.

Let $b = b(s)$ be the inverse function to $ta(t)$, let $K = R^{n-1}c\, a(c)$ and set

$$v(r) = \int_r^R b(K\rho^{1-n})d\rho, \quad 0 < r \leq R.$$

One can easily verify that $v(|x - x_0|)$ solves

$$\begin{cases} \mathrm{div}(a(|\nabla v|)\nabla v) = 0, & \text{in } B_R(x_0) \setminus \{x_0\}, \\ v = 0, & \text{on } \partial B_R(x_0), \\ \frac{\partial v}{\partial \nu} = -c, & \text{on } \partial B_R(x_0). \end{cases}$$

Now, in $B_R(x_0)\setminus\{O, x_0\}$, $u(x)$ and $v(|x-x_0|)$ satisfy the same elliptic equation and have the same Cauchy data on $\partial B_R(x_0)$.

We recall that for such an equation the unique continuation property holds as long as one of the two solutions has non-vanishing gradient. It follows from a standard continuity argument that $u(x) = v(|x - x_0|)$ for every $x \neq O, x_0$.

Consequently u and v must have the same singular point, that is $x_0 = O$ and finally $u(x) = v(|x|)$. We conclude the proof observing that the singular value M can be computed as follows

$$M = \int_0^R b\left(\left(\frac{R}{\rho}\right)^{n-1} c\, a(c)\right) d\rho.$$

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