Rend. Istit. Mat. Univ. Trieste
Vol. XXXVIII, 73–93 (2006)

The Transmission Acoustic Scattering Problem for Bi-Spheres in Low-Frequency Regime

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Summary. - *An acoustically soft sphere covered by a penetrable eccentric spherical shell disturbs the propagation of an incident plane wave field. It is shown that there exists exactly one bispherical coordinate system that describes the given geometry. The incident wave is assumed to have a wavelength which is much larger than the characteristic dimension of the scatterer and thus the low-frequency approximation method is applicable to the scattering problem. The incomplete R-separation of variables in bispherical coordinates and the normal differentiation involved in the transmission boundary conditions lead to a three-term recurrence relation for the series coefficients corresponding to the scattered fields. Thus, the potential boundary-value problem for the leading low-frequency approximations is reduced to infinite tridiagonal linear systems, which are solved analytically.*

1. Introduction

The direct and inverse scattering problems for penetrable obstacles with concentric cores has been extensively investigated in acoustics,

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Keywords: Helmholtz Equation, Low-Frequency Scattering, Bispherical Coordinates, Infinite Tridiagonal Systems, Continuous Fractions.

as well as in the electromagnetic propagation theory [1, 5, 8, 27]. If the scattering region is composed by two obstacles, than we refer to multiple scattering [26], which is by far less studied in the field of applied mathematics. There is a great interest for such applied problems, which comes mainly from the theory of acoustic emission, the blood analysis and biological studies at the cell level, from the modelling in medicine and health sciences. For example, biological cells involve a nucleus coated by an eccentric cytoplasmic shell, human organs also form scattering obstacles buried within the body and having the same geometrical characteristic. Nevertheless, the most important argument is due to the fact that the far field scattering data involve the parameters needed for identifying the location and the geometry of the core in solving the inverse scattering problem.

The present work treats in low-frequency regime the effects that a small acoustically soft sphere coated by an eccentric spherical shell has on the propagation of a plane wave field. The shell is not allowed to touch the core, so they describe a genuine two-body scattering process. For the geometrical investigation of the two eccentric spheres we refer to Morse and Feshbach [22] and Twersky [26].

The low-frequency series expansion method was first proposed by Kleinman and Vainberg [15] and was later developed and numerically tested by Dassios and Kleinman [8]. We also refer to the works of Arnaoudov, Dassios and Kostopoulos [1], Dassios [5, 6], Dassios and Kamvyssas [7], MacCamy [19] and Venkov [27, 28], where the method is applied to different boundary value problems for acoustic and electromagnetic wave propagation. Using the low-frequency technique the scattering problem is transformed to a sequence of potential boundary value problems for the approximation coefficients of both the scattered exterior field and the induced interior field.

It is shown that given the radii and the position of the external and the internal spheres, there is exactly one bispherical coordinate system that fit the geometry of the composite scatterer. The bispherical coordinates belong to a class of six curvilinear orthogonal coordinate systems, known as cyclides [20], which do not admit complete separation of variables in the harmonic equation, but still allow for solving scalar and vector boundary-value problems of mathematical physics [2, 3, 16, 18, 22]. The incomplete R-separation of variables for Laplace equation in cyclidal coordinates resulted in a relatively rare application of these coordinate systems for solving applied and engineering problems.

Section 2 contains the specification of the particular bispherical system and the statement of the scattering problem. The low-frequency treatment of the problem is presented in Section 3. The impact of normal differentiation in the transmission boundary conditions on the surface of the penetrable shell leads to infinite tridiagonal systems of linear algebraic equations, which we overcome by an appropriate use of continuous fractions. Finally, after the straightforward calculation of several surface integrals we approximate the scattering amplitude and the scattering cross-section up to the order of k^3. The obtained approximations both for the near and the far field are given in terms of rapidly converging series, which is satisfactory for any practical purpose.

2. Statement of the scattering problem

Let us consider two eccentric spheres S_a and S_b of radii a and b respectively ($a > b$), with centers that are located a distance $d < a - b$ apart. Our first step is to introduce a bispherical coordinate system which describes the given spheres with two specified values of one of the space variables. The bispherical system is an orthogonal coordinate system [22], which is connected to the Cartesian system through the equations

$$\begin{aligned} x &= c\frac{\sin\theta\cos\varphi}{\cosh\rho - \cos\theta}, \\ y &= c\frac{\sin\theta\sin\varphi}{\cosh\rho - \cos\theta}, \\ z &= c\frac{\sinh\rho}{\cosh\rho - \cos\theta} \end{aligned} \tag{1}$$

where $2c$ denotes the interfocal distance, $\rho \in (-\infty, +\infty)$ specifies the nonintersecting spheres, $\theta \in [0, \pi]$ specifies the intersecting spheres and $\varphi \in [0, 2\pi]$ is the azimuthal angle, which represents the axial symmetry of the system. The ρ-coordinate surface is a sphere centered at the point $(0, 0, c/\tanh\rho)$ with radius $c/|\sinh\rho|$. As ρ

runs from $-\infty$ to $+\infty$ the corresponding sphere springs at the focus $(0,0,-a)$ for $\rho \to -\infty$, sweeps the $z<0$ half-space for $\rho<0$, passes through the $z=0$ plane for $\rho=0$ and then sweeps the $z>0$ half-space for $\rho>0$ to end up at the focus $(0,0,a)$ for $\rho \to +\infty$.

Let us demand that the sphere S_a that describes the boundary of the shell corresponds to the value $\rho = r_1$ and the sphere S_b that describes the surface of the core corresponds to the value $\rho = r_2$. In order to adapt a bispherical coordinate system to the spheres S_a and S_b we need to find positive numbers r_1, r_2, $r_1 < r_2$ which satisfy the conditions

$$\begin{aligned} \frac{c}{\sinh r_1} &= a, \\ \frac{c}{\sinh r_2} &= b, \\ \frac{c}{\tanh r_1} - \frac{c}{\tanh r_2} &= d. \end{aligned} \tag{2}$$

Solving the system (2) with respect to the unknown r_1, r_2 and c, we arrive at

$$\begin{aligned} c &= \frac{\sqrt{(a+b)^2-d^2}\sqrt{(a-b)^2-d^2}}{2d}, \\ r_1 &= \ln\frac{c+\sqrt{c^2+a^2}}{a}, \\ r_2 &= \ln\frac{c+\sqrt{c^2+b^2}}{b}, \end{aligned} \tag{3}$$

which determine exactly one bispherical system that fits the given two-sphere obstacle.

The exterior region V^+ where the scattered acoustic wave propagates is the exterior of the sphere S_a and corresponds to the domain

$$V^+ = \{(\rho,\theta,\varphi) | \rho \in (-\infty, r_1), \theta \in [0,\pi], \varphi \in [0,2\pi)\},$$

while the shell V^- between S_a and S_b is defined as

$$V^- = \{(\rho,\theta,\varphi) | \rho \in (r_1, r_2), \theta \in [0,\pi], \varphi \in [0,2\pi)\}.$$

In the new coordinate system the radial distance is given by

$$r = c\sqrt{\frac{\cosh\rho + \cos\theta}{\cosh\rho - \cos\theta}}, \tag{4}$$

which implies that the far-field region corresponds to a small neighborhood of $(\rho, \theta) = (0, 0)$.

Suppressing the harmonic time dependence $\exp\{-i\omega t\}$, where ω denotes the angular frequency, we assume that an incident plane acoustic field of the form

$$u^i(\mathbf{r}) = e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}}, \tag{5}$$

illuminates the target. Here k stands for the wave number in V^+ and $\hat{\mathbf{k}}$ is the direction of propagation. The scattering problem we consider here is the following: Find the total field

$$u^+(\mathbf{r}) = u^i(\mathbf{r}) + u^s(\mathbf{r}), \qquad \mathbf{r} \in V^+, \tag{6}$$

which solve the Helmholtz equation

$$\Delta u^+(\mathbf{r}) + k^2 u^+(\mathbf{r}) = 0, \qquad \mathbf{r} \in V^+ \tag{7}$$

and the interior excess pressure field u^-, which solve the equation

$$\Delta u^-(\mathbf{r}) + (k^-)^2 u^-(\mathbf{r}) = 0, \qquad \mathbf{r} \in V^-, \tag{8}$$

with k^- be the corresponding wave number in V^-.

We assume that the exterior and interior regions V^+ and V^- are occupied by linear, homogeneous, isotropic and lossless acoustic mediums and the interior wave number k^- is related to k through the expression $(k^-)^2 = \eta^2 k^2$, where η is the real index of refraction connecting the two mediums.

The scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\Big(\frac{\partial u^s}{\partial r} - iku^s\Big) = 0 \tag{9}$$

uniformly over the unit sphere S^2 of the three dimensional Euclidean space [4].

On the sphere S_a we demand the transmission conditions

$$u^+(\mathbf{r}) = u^-(\mathbf{r}), \qquad \frac{\partial u^+(\mathbf{r})}{\partial n} = \beta\frac{\partial u^-(\mathbf{r})}{\partial n}, \tag{10}$$

where $\partial/\partial n$ denotes outward normal differentiation and β is the average density. The transmission boundary conditions (10) describe the continuity of the excess pressure field, as well as that of the normal component of the velocity field, as we cross the penetrable surface S_a.

On the surface S_b of the core we assume the Dirichlet boundary condition

$$u^-(\mathbf{r}) = 0, \tag{11}$$

which describes the fact that S_b cannot sustain any pressure and identifies S_b as an acoustically soft boundary.

If we denote by

$$h(k|\mathbf{r}-\mathbf{r}'|) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{ik|\mathbf{r}-\mathbf{r}'|} \tag{12}$$

the fundamental solution of the Helmholtz operator with wave number k, then the exterior field u^+ assumes the well known integral representation [8]

$$\begin{aligned} u^+(\mathbf{r}) &= u^i(\mathbf{r}) + \\ &+\frac{ik}{4\pi}\int_{S_a}\left[u^+(\mathbf{r}')\frac{\partial}{\partial n'}h(k|\mathbf{r}-\mathbf{r}'|) - h(k|\mathbf{r}-\mathbf{r}'|)\frac{\partial}{\partial n'}u^+(\mathbf{r}')\right]ds(\mathbf{r}'). \end{aligned} \tag{13}$$

In the far field we have the asymptotic form

$$u^s(\mathbf{r}) = g(\hat{\mathbf{r}})h(kr) + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad r\to\infty, \tag{14}$$

where the scattering amplitude g is normalized to the same dimensions as u^s and is given by

$$g(\hat{\mathbf{r}}) = -\frac{ik}{4\pi}\int_{S_a}\left[\frac{\partial}{\partial n'}u^+(\mathbf{r}') + ik(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u^+(\mathbf{r}')\right]e^{ik\hat{\mathbf{r}}\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{15}$$

for $\hat{\mathbf{r}} \in S^2$. Once the scattering amplitude is obtained, the scattering cross-section

$$\sigma_s = \frac{1}{k^2}\int_{S^2}|g(\hat{\mathbf{r}})|^2 ds(\hat{\mathbf{r}}) \tag{16}$$

is defined as the L^2-norm of g on the unit sphere [8].

3. Low-frequency approximation method for a small penetrable sphere with an impenetrable core

Motivation for applying low-frequency expansions, especially for curvilinear coordinate systems that does not allow separation of variables for the Helmholtz equation, is the replacement of the initial scattering problem with a sequence of potential boundary-value problems that can be solved iteratively [1, 2, 3, 5, 6, 7, 27, 28]. When the wavelength of the incident field is much larger than the radius of the exterior sphere, namely $ka << 1$, all the fields involved are analytic functions of the wavenumber [8, 14, 15].

The total acoustic field u^+ allows power series expansion of the form

$$u^+(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^+(\mathbf{r}), \qquad \mathbf{r} \in V^+ \cup S_a. \tag{17}$$

To facilitate our further investigations the interior field u^- is expanded in powers of ik rather than ik^-, that is

$$u^-(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^-(\mathbf{r}), \qquad \mathbf{r} \in V^- \cup S_a \cup S_b, \tag{18}$$

where we use the relation $k^- = \eta k$ and the powers of η are absorbed in the coefficients u_n^-. The validity of the expansions (17), (18) is established by Kleinman [13].

The low-frequency approximations u_n^+, u_n^- solve the equations

$$\begin{aligned} \Delta u_n^+(\mathbf{r}) &= n(n-1)u_{n-2}^+(\mathbf{r}) \ , && \mathbf{r} \in V^+ \ , && (19) \\ \Delta u_n^-(\mathbf{r}) &= n(n-1)\eta^2 u_{n-2}^-(\mathbf{r}) \ , && \mathbf{r} \in V^- \ , && (20) \end{aligned}$$

the transmission conditions on the shell

$$u_n^+(\mathbf{r}) = u_n^-(\mathbf{r}), \quad \frac{\partial u_n^+(\mathbf{r})}{\partial n} = \beta \frac{\partial u_n^-(\mathbf{r})}{\partial n}, \qquad \mathbf{r} \in S_a \tag{21}$$

and the Dirichlet boundary condition on the core

$$u_n^-(\mathbf{r}) = 0, \qquad \mathbf{r} \in S_b \tag{22}$$

for every $n = 0, 1, \ldots$.

For the determination of the form of u_n^+ we employ the low-frequency expansions both of the plane incident field (5) and the fundamental solution (12)

$$\begin{aligned} u^i(\mathbf{r}) &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}(\hat{\mathbf{k}}\cdot\mathbf{r})^n, \\ h(k|\mathbf{r}-\mathbf{r}'|) &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}|\mathbf{r}-\mathbf{r}'|^{n-1}, \end{aligned} \tag{23}$$

and substituting into (13) we obtain

$$\begin{aligned} u_n^+(\mathbf{r}) = (\hat{\mathbf{k}}\cdot\mathbf{r})^n &+ \\ &+\frac{1}{4\pi}\int_{S_a} u_n^+(\mathbf{r}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial u_n^+(\mathbf{r}')}{\partial n'}ds(\mathbf{r}') + \\ &+\frac{1}{4\pi}\sum_{m=1}^{n}\binom{n}{m}\int_{S_a} u_{n-m}^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{m-1} + \\ &\qquad -|\mathbf{r}-\mathbf{r}'|^{m-1}\frac{\partial u_{n-m}^+(\mathbf{r}')}{\partial n'}ds(\mathbf{r}'). \end{aligned} \tag{24}$$

In order to establish the so-called asymptotic integral representation we use the fact that

$$\int_{S_a} u_n^+(\mathbf{r}')\frac{\partial}{\partial n'}\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial u_n^+(\mathbf{r}')}{\partial n'}ds(\mathbf{r}') = \mathcal{O}\Big(\frac{1}{r}\Big), \quad r\to\infty \tag{25}$$

and thus (24) becomes

$$\begin{aligned} u_n^+(\mathbf{r}) = (\hat{\mathbf{k}}\cdot\mathbf{r})^n &+ \\ &+\frac{1}{4\pi}\sum_{m=1}^{n}\binom{n}{m}\int_{S_a} u_{n-m}^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{m-1} + \\ &\qquad -|\mathbf{r}-\mathbf{r}'|^{m-1}\frac{\partial u_{n-m}^+(\mathbf{r}')}{\partial n'}ds(\mathbf{r}') + \mathcal{O}\Big(\frac{1}{r}\Big). \end{aligned} \tag{26}$$

One can observe that the nonvanishing part of the asymptotic representation (26) provides a particular solution of the nonhomogeneous Laplace equation (19). Thus, any solution of (19) can be written as a sum of the corresponding particular solution plus a solution of the homogeneous Laplace equation, which must also be of order of $1/r$, as $r \to \infty$.

The Rayleigh approximations: The leading low-frequency coefficients u_0^+ and u_0^-, known as the Rayleigh approximations [14], solve the following boundary-value problem

$$\Delta u_0^+(\mathbf{r}) = 0, \qquad \mathbf{r} \in V^+, \tag{27}$$

$$\Delta u_0^-(\mathbf{r}) = 0, \qquad \mathbf{r} \in V^- \tag{28}$$

with boundary conditions

$$u_0^+(\mathbf{r}) = u_0^-(\mathbf{r}), \quad \frac{\partial u_0^+(\mathbf{r})}{\partial n} = \beta \frac{\partial u_0^-(\mathbf{r})}{\partial n}, \qquad \mathbf{r} \in S_a, \tag{29}$$

and

$$u_0^-(\mathbf{r}) = 0, \qquad \mathbf{r} \in S_b. \tag{30}$$

The low-frequency expansion of the incident field and integral representation formula (26) determine the asymptotic behavior

$$u_0^+(\mathbf{r}) = 1 + \mathcal{O}\left(\frac{1}{r}\right), \qquad r \to \infty. \tag{31}$$

In bispherical coordinates the boundary value problem (27)–(30) leads to the axially symmetric solutions [22] and a typical eigenfunction expansion, regular on the axis, assumes the form

$$f(\rho, \theta) = \sqrt{\cosh \rho - \cos \theta} \sum_{n=0}^{\infty} \left[A_n e^{(n+\frac{1}{2})\rho} + B_n e^{-(n+\frac{1}{2})\rho} \right] P_n(\cos \theta), \tag{32}$$

where P_n are the Legendre polynomials.

Taking the expansion of $(1 - 2h\cos\theta + h^2)^{-1/2}$ for $h = e^{-\rho}$ we obtain the uniformly convergent expansion of $(\cosh\rho - \cos\theta)^{-1/2}$ in terms of zonal harmonics of $\cos\theta$, given by

$$\frac{1}{\sqrt{\cosh \rho - \cos \theta}} = \sqrt{2} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})|\rho|} P_n(\cos \theta). \tag{33}$$

In view of (32), (33) and the asymptotic behavior (31) of the zeroth order coefficient u_0^+ we seek solutions in the form

$$u_0^+(\rho,\theta)=\sqrt{\cosh\rho-\cos\theta}\sum_{n=0}^{\infty}\left[\sqrt{2}e^{-(n+\frac{1}{2})|\rho|}+A_n^+e^{(n+\frac{1}{2})\rho}\right]P_n(\cos\theta)\ , \tag{34}$$

for $\rho\in(-\infty,r_1)$ and

$$u_0^-(\rho,\theta)=\sqrt{\cosh\rho-\cos\theta}\sum_{n=0}^{\infty}\left[A_n^-e^{(n+\frac{1}{2})\rho}+B_n^-e^{-(n+\frac{1}{2})\rho}\right]P_n(\cos\theta)\ , \tag{35}$$

for $\rho\in(r_1,r_2)$. Note, that the approximation u_0^+ of the exterior field, has coefficients $B_n^+=0$ due to the convergence of the series (34) in $(-\infty,r_1)$.

The coefficients A_n^- and B_n^- are related through Dirichlet condition (30), implying

$$B_n^-=-A_n^-e^{(2n+1)r_2}. \tag{36}$$

In view of the normal derivative in bispherical coordinates

$$\begin{aligned}\frac{\partial}{\partial n}&=\hat{\mathbf{n}}\cdot\nabla\\&=\frac{\cosh\rho-\cos\theta}{c}\frac{\partial}{\partial\rho},\end{aligned} \tag{37}$$

the transmission conditions (29) produce the relations

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[A_n^+e^{(n+\frac{1}{2})r_1}+\sqrt{2}e^{-(n+\frac{1}{2})r_1}\right]P_n(\cos\theta)=\\&\qquad=\sum_{n=0}^{\infty}A_n^-\left[e^{(n+\frac{1}{2})r_1}-e^{(2n+1)r_2}e^{-(n+\frac{1}{2})r_1}\right]P_n(\cos\theta)\end{aligned} \tag{38}$$

and

$$\frac{\sinh r_1}{2\sqrt{\cosh r_1 - \cos\theta}} \sum_{n=0}^{\infty} \left[A_n^+ e^{(n+\frac{1}{2})r_1} + \sqrt{2} e^{-(n+\frac{1}{2})r_1} \right] P_n(\cos\theta)$$
$$+ \sqrt{\cosh r_1 - \cos\theta} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \left[A_n^+ e^{(n+\frac{1}{2})r_1} + \right.$$
$$\left. - \sqrt{2} e^{-(n+\frac{1}{2})r_1} \right] P_n(\cos\theta)$$
$$= \beta \frac{\sinh r_1}{2\sqrt{\cosh r_1 - \cos\theta}} \sum_{n=0}^{\infty} A_n^- \left[e^{(n+\frac{1}{2})r_1} \right.$$
$$\left. - e^{(2n+1)r_2} e^{-(n+\frac{1}{2})r_1} \right] P_n(\cos\theta)$$
$$+ \beta \sqrt{\cosh r_1 - \cos\theta} \sum_{n=0}^{\infty} A_n^- \left(n + \frac{1}{2} \right) \left[e^{(n+\frac{1}{2})r_1} + \right.$$
$$\left. + e^{(2n+1)r_2} e^{-(n+\frac{1}{2})r_1} \right] P_n(\cos\theta). \tag{39}$$

From (38) we obtain

$$A_n^- = \frac{\sqrt{2} + A_n^+ e^{(2n+1)r_1}}{e^{(2n+1)r_1} - e^{(2n+1)r_2}}, \tag{40}$$

and then (39) becomes

$$\sum_{n=0}^{\infty} (1-\beta) \sinh r_1 \left[A_n^+ e^{(n+\frac{1}{2})r_1} + \sqrt{2} e^{-(n+\frac{1}{2})r_1} \right] P_n(\cos\theta)$$
$$= \sum_{n=0}^{\infty} (2n+1) \cosh r_1 \left[\beta \Lambda_n \left(A_n^+ e^{(n+\frac{1}{2})r_1} + \sqrt{2} e^{-(n+\frac{1}{2})r_1} \right) \right.$$
$$\left. - \left(A_n^+ e^{(n+\frac{1}{2})r_1} - \sqrt{2} e^{-(n+\frac{1}{2})r_1} \right) \right] P_n(\cos\theta)$$
$$- \sum_{n=0}^{\infty} \left[\beta \Lambda_n \left(A_n^+ e^{(n+\frac{1}{2})r_1} + \sqrt{2} e^{-(n+\frac{1}{2})r_1} \right) \right.$$
$$\left. - \left(A_n^+ e^{(n+\frac{1}{2})r_1} - \sqrt{2} e^{-(n+\frac{1}{2})r_1} \right) \right] (2n+1) \cos\theta P_n(\cos\theta) \tag{41}$$

where we denote

$$\Lambda_n = \frac{e^{(2n+1)r_1} + e^{(2n+1)r_2}}{e^{(2n+1)r_1} - e^{(2n+1)r_2}}. \tag{42}$$

Substituting the recursion formula for Legendre polynomials [11, 22]

$$(2n+1)\cos\theta P_n(\cos\theta) = (n+1)P_{n+1}(\cos\theta) + nP_{n-1}(\cos\theta) \tag{43}$$

into (41) and equating degrees of P_n, we arrive at the three-term recurrence relation

$$\begin{aligned}
&(n+1)e^{(2n+3)r_1}(\Lambda_{n+1}\beta - 1)A^+_{n+1}\\
&-e^{(2n+2)r_1}[(2n+1)(\Lambda_n\beta - 1)\cosh r_1 + (\beta - 1)\sinh r_1]A^+_n\\
&+ne^{(2n+1)r_1}(\Lambda_{n-1}\beta - 1)A^+_{n-1} =\\
&\quad = \sqrt{2}[(n+1)(\Lambda_{n+1}\beta + 1) + ne^{2r_1}(\Lambda_{n-1}\beta + 1)\\
&\qquad +e^{r_1}((1-\beta)\sinh r_1 - (2n+1)(\Lambda_n\beta + 1)\cosh r_1)],
\end{aligned} \tag{44}$$

due to the factor $\sqrt{\cosh\rho - \cos\theta}$ and the non-orthogonality of the bispherical [10, 18, 24, 25] (and toroidal [2, 17]) harmonics in θ. Thus, the transmission acoustic scattering problem for bi-spheres reduces to infinite tridiagonal systems of linear algebraic equations.

Denoting by

$$\begin{aligned}
x_n &= A^+_n, \quad a_n = (n+1)e^{(2n+3)r_1}(\Lambda_{n+1}\beta - 1),\\
b_n &= e^{(2n+2)r_1}[(2n+1)(\Lambda_n\beta - 1)\cosh r_1 + (\beta - 1)\sinh r_1],\\
c_n &= ne^{(2n+1)r_1}(\Lambda_{n-1}\beta - 1),\\
d_n &= \sqrt{2}[(n+1)(\Lambda_{n+1}\beta + 1) + ne^{2r_1}(\Lambda_{n-1}\beta + 1)\\
&\qquad +e^{r_1}((1-\beta)\sinh r_1 - (2n+1)(\Lambda_n\beta + 1)\cosh r_1)],
\end{aligned} \tag{45}$$

then the recursive equation (44) reduces to an infinite system of algebraic equations with respect to x_n of the form

$$a_n x_{n+1} - b_n x_n + c_n x_{n-1} = d_n, \quad a_n \neq 0, \quad c_0 = 0, \tag{46}$$

for $n = 0, 1, 2, ...$

It is clear that the transmission boundary-value problem for bi-spheres reduces to equations of type (46), implying that harmonic function in bispherical coordinates cannot be determined simpler than by a set of tridiagonal infinite systems of algebraic equations [2, 16]. One of the usual techniques for solving recursive equations of the form (46) is based on the application of Greens function method for difference equations as suggested by Milne-Thomson [21] and used by Love [18]. In the present work we introduce more relevant and effective analytical method for solving tridiagonal algebraic systems using the theories of continued fractions and sequence transformations (see Good [9], Householder [12], Stewart and Sun [23]).

The tridiagonal infinite system of algebraic equations (46) reduces to successive solving of two bi-diagonal systems

$$\begin{aligned} \tilde{a}_n x_{n+1} - (1-\kappa_n)x_n &= (1-\kappa_n)y_n, \quad n = 0,1,2,\dots \\ (1-\kappa_n)y_n - \tilde{c}_n y_{n-1} &= \tilde{d}_{n,s} \end{aligned} \tag{47}$$

where $\tilde{a}_n = a_n/b_n$, $\tilde{c}_n = c_n/b_n$, $\tilde{d}_n = d_n/b_n$ and κ_n is a finite continuous fraction, defined by

$$\kappa_n = \cfrac{\tilde{c}_n \tilde{a}_{n-1}}{1 - \cfrac{\tilde{c}_{n-1}\tilde{a}_{n-2}}{1 - \cfrac{\tilde{c}_{n-2}\tilde{a}_{n-3}}{1 - \dots}}}, \quad \kappa_0 = 0. \tag{48}$$

The above bi-diagonal systems have analytical solutions

$$x_n = -y_n - \sum_{m=n+1}^{\infty} y_m \prod_{s=n}^{m-1} \frac{\tilde{a}_s}{1-\kappa_s}, \tag{49}$$

$$y_n = \frac{\tilde{d}_n}{1-\kappa_n} + \sum_{m=0}^{n-1} \frac{\tilde{d}_m}{1-\kappa_m} \prod_{s=m+1}^{n} \frac{\tilde{c}_s}{1-\kappa_s}. \tag{50}$$

It should be mentioned that y_n, $n = 0, 1, 2, ...$ are uniquely determined by the second recurrence relation in (47) (since $\tilde{c}_0 = 0$). Then the values of x_n are generated by the infinite series (49).

The convergence in formulas (49) can be verified by the analysis of the coefficients $\tilde{a}_n, \tilde{c}_n$ and $\tilde{d}_n$ of the system (47), which in our case converge exponentially to zero, making their solutions converge as well. Expressions (36), (40) and (44) reduce to the corresponding formulae for the soft sphere when $\beta = 1$, while the constants Λ_n are to be interpreted as the influence of the coating.

The first-order approximation: The first-order low-frequency coefficients u_1^+, u_1^- are obtained by solving the following boundary value problems:

$$\begin{aligned} \Delta u_1^+(\mathbf{r}) &= 0, & \mathbf{r} \in V^+, \quad (51)\\ \Delta u_1^-(\mathbf{r}) &= 0, & \mathbf{r} \in V^-, \quad (52)\\ u_1^+(\mathbf{r}) = u_1^-(\mathbf{r}), \qquad \frac{\partial u_1^+(\mathbf{r})}{\partial n} &= \beta\frac{\partial u_1^-(\mathbf{r})}{\partial n}, & \mathbf{r} \in S_a, \quad (53)\\ u_1^-(\mathbf{r}) &= 0, & \mathbf{r} \in S_b. \quad (54) \end{aligned}$$

Substituting $n = 1$ into (26) we establish the asymptotic behavior of the exterior field as

$$u_1^+(\mathbf{r}) = \hat{\mathbf{k}} \cdot \mathbf{r} - \frac{1}{4\pi}\int_{S_a} \frac{\partial u_0^+(\mathbf{r})}{\partial n'} ds(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right), \qquad r \to \infty. \quad (55)$$

Because of the $\hat{\mathbf{k}} \cdot \mathbf{r}$–term in the asymptotic form (55), the first-order approximations are not azimuthal independent anymore. In bispherical coordinates the general form of the Laplace equation [8] is

$$(\cosh\rho - \cos\theta)\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\cosh\rho - \cos\theta}\frac{\partial}{\partial\rho}\right) + \right.$$
$$\left. + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\rho - \cos\theta}\frac{\partial}{\partial\theta}\right)\right] f(\rho,\theta,\varphi) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} f(\rho,\theta,\varphi) = 0.$$

From (1) and (33) we derive the expansions

$$\begin{aligned} x &= 2\sqrt{2}c\sqrt{\cosh\rho - \cos\theta}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})|\rho|} P_n^1(\cos\theta)\cos\varphi,\\ y &= 2\sqrt{2}c\sqrt{\cosh\rho - \cos\theta}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})|\rho|} P_n^1(\cos\theta)\sin\varphi,\\ z &= 2\sqrt{2}c\sqrt{\cosh\rho - \cos\theta}\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)(\mathrm{sign}\rho) e^{-(n+\frac{1}{2})|\rho|} P_n(\cos\theta). \end{aligned} \quad (56)$$

Therefore, the $\hat{\mathbf{k}} \cdot \mathbf{r}$–term dictates that u_1^+, u_1^- live not only in the axially symmetric subspace generated by $P_n(\cos\theta)$, but also in the subspace of the first-order azimuthal dependence generated by $P_n^1(\cos\theta)\cos\varphi$ and $P_n^1(\cos\theta)\sin\varphi$. Thus, we conclude that the solutions of (51)–(55) are expressed in the form

$$\begin{aligned} u_1^+(\rho,\theta,\varphi) = \sqrt{\cosh\rho-\cos\theta}\Bigg\{ \sum_{n=0}^{\infty}\Bigg[C_n^+ e^{(n+\frac{1}{2})\rho} + & \\ +2\sqrt{2}ck_1 e^{-(n+\frac{1}{2})|\rho|}\Bigg] P_n^1(\cos\theta)\cos\varphi & \\ +\sum_{n=0}^{\infty}\Bigg[E_n^+ e^{(n+\frac{1}{2})\rho} + 2\sqrt{2}ck_2 e^{-(n+\frac{1}{2})|\rho|}\Bigg] P_n^1(\cos\theta)\sin\varphi & \\ +\sum_{n=0}^{\infty}\Bigg[G_n^+ e^{(n+\frac{1}{2})\rho} + \sqrt{2}((2n+1)(\mathrm{sign}\rho)ck_3 - I)e^{-(n+\frac{1}{2})|\rho|}\Bigg]\cdot & \\ \cdot P_n(\cos\theta)\Bigg\}, & \end{aligned} \tag{57}$$

for $\rho\in(-\infty,r_1)$ and

$$\begin{aligned} u_1^-(\rho,\theta,\varphi) = \sqrt{\cosh\rho-\cos\theta}\Bigg\{ \sum_{n=0}^{\infty}\Bigg[C_n^- e^{(n+\frac{1}{2})\rho} + & \\ +D_n^- e^{-(n+\frac{1}{2})\rho}\Bigg] P_n^1(\cos\theta)\cos\varphi & \\ +\sum_{n=0}^{\infty}\Bigg[E_n^- e^{(n+\frac{1}{2})\rho} + F_n^- e^{-(n+\frac{1}{2})\rho}\Bigg] P_n^1(\cos\theta)\sin\varphi & \\ +\sum_{n=0}^{\infty}\Bigg[G_n^- e^{(n+\frac{1}{2})\rho} + H_n^- e^{-(n+\frac{1}{2})\rho}\Bigg] P_n(\cos\theta)\Bigg\}, & \end{aligned} \tag{58}$$

for $\rho\in(r_1,r_2)$, where

$$I = \frac{1}{4\pi}\int_{S_a}\frac{\partial u_0^+(\mathbf{r})}{\partial n}ds(\mathbf{r}) \tag{59}$$

and $\hat{\mathbf{k}} = (k_1,k_2,k_3)$ is the unit propagation vector.

The coefficients C_n^+, E_n^+, G_n^+, C_n^-, D_n^-, E_n^-, F_n^-, G_n^- and H_n^- have to be chosen in such a way as to satisfy the boundary conditions (53) and (54). Using the same techniques for obtaining the coefficients of the zeroth-order approximations and the orthogonality properties of the Legendre polynomials and after a long series of tedious calculations, we find the following relations

$$\begin{aligned} D_n^- &= -C_n^- e^{(2n+1)r_2}, \\ F_n^- &= -E_n^- e^{(2n+1)r_2}, \\ H_n^- &= -G_n^- e^{(2n+1)r_2}, \end{aligned} \tag{60}$$

$$\begin{aligned} C_n^- &= \frac{2\sqrt{2}ck_1 + C_n^+ e^{(2n+1)r_1}}{e^{(2n+1)r_1} - e^{(2n+1)r_2}}, \\ E_n^- &= \frac{2\sqrt{2}ck_2 + E_n^+ e^{(2n+1)r_1}}{e^{(2n+1)r_1} - e^{(2n+1)r_2}}, \\ G_n^- &= \frac{\sqrt{2}((2n+1)ck_3 - I) + G_n^+ e^{(2n+1)r_1}}{e^{(2n+1)r_1} - e^{(2n+1)r_2}}, \end{aligned} \tag{61}$$

and the recursive equations with respect to the coefficients C_n^+, E_n^+ and G_n^+

$$\begin{aligned} (n+2)(1-\beta\Lambda_{n+1})C_{n+1}^+ &= \{C_n^+[(1-\beta)\sinh r_1 + \\ &\quad (2n+1)(1-\beta\Lambda_n)\cosh r_1]e^{-r_1} - (n-1)C_{n-1}^+(1-\beta\Lambda_{n-1})e^{-2r_1}\} \\ &\quad +2\sqrt{2}ck_1\{[(1-\beta)\sinh r_1 - (2n+1)(1+\beta\Lambda_n)\cosh r_1]e^{-(2n+2)r_1} \\ &\quad +(n-1)(1+\beta\Lambda_{n-1})e^{-(2n+1)r_1} + (n+2)(1+\beta\Lambda_{n+1})e^{-(2n+3)r_1}\}, \\ &\qquad n = 0, 1, \dots, \end{aligned} \tag{62}$$

$$\begin{aligned} (n+2)(1-\beta\Lambda_{n+1})E_{n+1}^+ &= \{E_n^+[(1-\beta)\sinh r_1 + \\ &\quad (2n+1)(1-\beta\Lambda_n)\cosh r_1]e^{-r_1} - (n-1)E_{n-1}^+(1-\beta\Lambda_{n-1})e^{-2r_1}\} \\ &\quad +2\sqrt{2}ck_2\{[(1-\beta)\sinh r_1 - (2n+1)(1+\beta\Lambda_n)\cosh r_1]e^{-(2n+2)r_1} \\ &\quad +(n-1)(1+\beta\Lambda_{n-1})e^{-(2n+1)r_1} + (n+2)(1+\beta\Lambda_{n+1})e^{-(2n+3)r_1}\}, \\ &\qquad n = 0, 1, \dots, \end{aligned} \tag{63}$$

$$
\begin{aligned}
(n+1)(1-\beta\Lambda_{n+1})G^{+}_{n+1} \;=\; & \{G^{+}_{n}[(1-\beta)\sinh r_1 + \\
& (2n+1)(1-\beta\Lambda_n)\cosh r_1]e^{-r_1} - nG^{+}_{n-1}(1-\beta\Lambda_{n-1})e^{-2r_1}\} \\
& +\sqrt{2}\{((2n+1)ck_3 - I)[(1-\beta)\sinh r_1 + \\
& \qquad -(2n+1)(1+\beta\Lambda_n)\cosh r_1]e^{-(2n+2)r_1} \\
& +n((2n-1)ck_3 - I)(1+\beta\Lambda_{n-1})e^{-(2n+1)r_1} \\
& +(n+1)((2n+3)ck_3 - I)(1+\beta\Lambda_{n+1})e^{-(2n+3)r_1}\}, \\
& \qquad\qquad n = 0,1,\ldots, \qquad (64)
\end{aligned}
$$

The three-term recurrence relations (62)–(64) are reduced to infinite tridiagonal systems of the form (46) and accept solution given by (49).

4. The far field approximations

The scattering amplitude $g(\hat{\mathbf{r}})$ provides a directional analysis of the interaction between the incident field and the obstacle as it is established far away from the region of interaction. It has been proved that the scattering amplitude, also known as far-field pattern, registers all the geometrical and physical information about the scatterer [4].

Low-frequency analysis of the scattering amplitude [8] leads to the following expansion

$$
\begin{aligned}
g(\hat{\mathbf{r}}) \;=\; & \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^{n+1}}{n!}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l+1}\cdot \qquad (65) \\
& \qquad \cdot\int_{S_a}(\hat{\mathbf{r}}\cdot\mathbf{r}')^{l}\frac{\partial}{\partial n'}u^{+}_{n-l}(\mathbf{r}')ds(\mathbf{r}') \\
& +\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^{n+2}}{n!}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l+1}\cdot \\
& \qquad \cdot\int_{S_a}(\hat{\mathbf{r}}\cdot\mathbf{r}')^{l}(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u^{+}_{n-l}(\mathbf{r}')ds(\mathbf{r}')
\end{aligned}
$$

for $\hat{\mathbf{r}} \in S^2$. The approximation of the far field, based on the knowledge of $u_0^{\pm}$ and $u_1^{\pm}$, is derived from (65) as

$$\begin{aligned} g(\hat{\mathbf{r}}) &= -\frac{ik}{4\pi}\int_{S_a}\frac{\partial}{\partial n'}u_0^+(\mathbf{r}')ds(\mathbf{r}') \\ &+\frac{k^2}{4\pi}\int_{S_a}\left[\frac{\partial}{\partial n'}u_1^+(\mathbf{r}') - (\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial}{\partial n'}u_0^+(\mathbf{r}') + \right. \\ &\left. +(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u_0^+(\mathbf{r}')\right]ds(\mathbf{r}') + \mathcal{O}(k^3). \qquad (66) \end{aligned}$$

In order to obtain an analytic expression for the low-frequency approximation of the scattering amplitude, the four integrals in (66) must be evaluated. In view of the unit normal vector on the $\rho =$ *constant* surface and the expressions for the metric coefficients

$$\hat{\mathbf{n}} = \frac{1}{\cosh\rho - \cos\theta}\begin{pmatrix}\sinh\rho\sin\theta\cos\varphi \\ \sinh\rho\sin\theta\sin\varphi \\ 1-\cosh\rho\cos\theta\end{pmatrix} \qquad (67)$$

$$g_{\rho\rho} = g_{\theta\theta} = \frac{g_{\varphi\varphi}}{\sin^2\theta} = \frac{c^2}{(\cosh\rho - \cos\theta)^2} \qquad (68)$$

we confirm the expression

$$ds(\mathbf{r}) = \frac{c^2\sin\theta}{(\cosh\rho-\cos\theta)^2}d\theta d\varphi \qquad (69)$$

for the surface element.

Substituting (34), (67) and (69) in the first integral on the right-hand side of (66) and using orthogonality arguments for the Legendre polynomials we obtain

$$\int_{S_a}\frac{\partial}{\partial n'}u_0^+(\mathbf{r}')ds(\mathbf{r}') = 4\sqrt{2}\pi c\sum_{n=0}^{\infty}A_n^+, \qquad (70)$$

which also implies that

$$I = \sqrt{2}c\sum_{n=0}^{\infty}A_n^+. \qquad (71)$$

In a similar way it is shown that

$$\int_{S_a}\frac{\partial}{\partial n'}u_1^+(\mathbf{r}')ds(\mathbf{r}') = 4\sqrt{2}\pi c\sum_{n=0}^{\infty}G_n^+, \qquad (72)$$

and finally

$$\int_{S_a} (\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}') u_0^+(\mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} u_0^+(\mathbf{r}') ds(\mathbf{r}') = = -8\sqrt{2}\pi c^2 \Omega_3 \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) A_n^+, \quad (73)$$

where we denote with $(\Omega_1, \Omega_2, \Omega_3)$ the Cartesian coordinates of the unit position vector $\hat{\mathbf{r}}$.

Then, the low-frequency approximation of the scattering amplitude is expressed by

$$g(\hat{\mathbf{r}}) = -\sqrt{2} ikcA + \sqrt{2}(kc)^2 \left[\, G - 2\Omega_3 \tilde{A}\, \right] + \mathcal{O}\big((kc)^3\big), \quad kc \to 0\,, \quad (74)$$

where

$$A = \sum_{n=0}^{\infty} A_n^+\,, \quad G = \frac{1}{c} \sum_{n=0}^{\infty} G_n^+\,, \quad \tilde{A} = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) A_n^+. \quad (75)$$

Substituting (74) with $\Omega_3 = \cos\Theta$, $\Theta \in [0, \pi]$ into the formula (16) which defines the scattering cross-section, we obtain

$$\begin{aligned} \sigma_s &= \frac{1}{k^2} \int_{S^2} |g(\hat{\mathbf{r}})|^2 ds(\hat{\mathbf{r}}) \\ &= 2c^2 \int_{S^2} \left[A^2 + (kc)^2 \big(G - 2\tilde{A}\cos\Theta\big)^2 \right] ds(\hat{\mathbf{r}}) \qquad (76) \\ &= 8\pi c^2 \left[A^2 + (kc)^2 \left(G^2 + \frac{4}{3}\tilde{A}^2 \right) \right] + \mathcal{O}\big((kc)^3\big), \quad kc \to 0. \end{aligned}$$

In accord with the theory of multiple scattering, all the results obtained for the far fields are expressed in a series from which confirms the multiple interactions between the internal and the external eccentric spheres. Nevertheless, because of the exponential behavior of the terms defining the series, the convergence is so fast that the results are satisfactory for any practical purpose. The bispherical coordinate system provides the appropriate environment for solving multiple scattering problems by two spheres [3, 10, 11, 18, 20]. This is true only in the low-frequency realm since the Laplace equation accepts separation in bispherical coordinates while the Helmholtz equation does not [16, 20, 22]. We remark that the effectiveness of this system in producing analytic results for multiple scattering problems could be utilized to obtain more analytic results which can be used as reference tools for numerical computations.

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Received December 1, 2006.