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On $|A, \delta|_k$ summability factors

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Summary. - *We obtain sufficient conditions for the series $\sum a_n\lambda_n$ to be absolutely summable of order k by a triangular matrix.*

Recently, in[6] and [5] Özarslan and Ögdük generalized a result of Bor [2] by using a normal matrix. Unfortunately they used an incorrect definition of absolute summability by following Bor's definition. In this paper we consider their results by using correct definition (see, [7]).

Also it should be noted that in [1], an incorrect definition of absolute summability was used. Corollary 2 gives the correct version of Bor's theorem. Let $\{p_n\}$ be a sequence of positive numbers such that
$P_n := \sum_{\nu=0}^{n} p_\nu \to \infty$ as $n \to \infty$, $(P_{-1} = p_{-1} = 0)$.
A series $\sum a_n$ is said to be summable $|\bar{N}, p, \delta|_k, k \geq 1$ and $\delta \geq 0$ if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|t_n - t_{n-1}|^k < \infty,$$

where

$$t_n = (\frac{1}{P_n})\sum_{\nu=0}^{\infty} p_\nu s_\nu.$$

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Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$A_n := \sum_{\nu=0}^{n} a_{n\nu} s_\nu.$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty. \tag{1}$$

and it is said to be summable $|A, \delta|_k, k \geq 1$ and $\delta \geq 0$ if (see,[3])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |A_n - A_{n-1}|^k < \infty. \tag{2}$$

We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ defined as follows:

$$\bar{a}_{n\nu} = \sum_{r=\nu}^{n} a_{nr}, \qquad n, \nu = 0, 1, 2, \dots,$$

and

$$\hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu}, \qquad n = 1, 2, 3, \dots.$$

A triangle is a lower triangular matrix with all nonzero main diagonal entries. For any matrix entry $a_{n\nu}$, $\Delta_\nu a_{n\nu} = a_{n\nu} - a_{n,\nu+1}$ and also $\Delta \lambda_i = \lambda_i - \lambda_{i+1}$

Theorem 0.1. *Let A be a lower triangular matrix with non-negative entries satisfying*

(i) $\bar{a}_{n0} = 1, n = 0, 1, \dots,$

(ii) $a_{n-1,\nu} \geq a_{n\nu} \quad \text{for} \quad n \geq \nu + 1,$

(iii) $na_{nn} = O(1), (n \to \infty),$

(iv) $\sum_{n=\nu+1}^{m+1} n^{\delta k} |\Delta_\nu \hat{a}_{n\nu}| = O\Big(\nu^{\delta k} a_{\nu\nu}\Big), (m \to \infty),$ *and*

(v) $\sum_{n=\nu+1}^{m+1} n^{\delta k} \hat{a}_{n,\nu+1} = O\Big(\nu^{\delta k}\Big), (m \to \infty).$

If the sequence $\{s_n\}$ *is bounded and* (λ_n) *is a sequence such that*

(vi) $\sum_{n=1}^{m} |\Delta\lambda_n| = O(1), (m \to \infty),$

(vii) $\sum_{n=1}^{m} n^{\delta k}|\Delta\lambda_n| = O(1), (m \to \infty),$ *and*

(viii) $\sum_{n=1}^{m} n^{\delta k-1}|\lambda_n|^k = O(1), (m \to \infty),$

then the series $\sum a_n\lambda_n$ *is summable* $|A, \delta|_k, k \geq 1, 0 \leq \delta < 1/k.$

Proof. Let (T_n) be the nth term of the A-transform of $\sum_{i=0}^{n} \lambda_i a_i$. Then

$$T_n = \sum_{\nu=0}^{n} a_{n\nu} \sum_{i=0}^{\nu} a_i\lambda_i = \sum_{i=0}^{n} a_i\lambda_i \sum_{\nu=i}^{n} a_{n\nu} = \sum_{i=0}^{n} \bar{a}_{ni} a_i \lambda_i.$$

It is from (i) , $\hat{a}_{n,0} = 0$. Thus

$$\begin{aligned} T_n - T_{n-1} &= \sum_{i=0}^{n} \bar{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{a}_{n-1,i} a_i \lambda_i \\ &= \sum_{i=0}^{n} \bar{a}_{ni} a_i \lambda_i - \sum_{i=0}^{n} \bar{a}_{n-1,i} a_i \lambda_i \\ &= \sum_{i=0}^{n} (\bar{a}_{ni} - \bar{a}_{n-1,i}) a_i \lambda_i \\ &= \sum_{i=0}^{n} \hat{a}_{ni} a_i \lambda_i \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i(s_i - s_{i-1}) \\
&= \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i s_i - \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i s_{i-1} \\
&= \sum_{i=1}^{n-1} \hat{a}_{ni}\lambda_i s_i + a_{nn}\lambda_n s_n - \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i s_{i-1} \\
&= \sum_{i=1}^{n-1} \hat{a}_{ni}\lambda_i s_i + a_{nn}\lambda_n s_n - \sum_{i=0}^{n-1} \hat{a}_{n,i+1}\lambda_{i+1} s_i \\
&= \sum_{i=1}^{n-1} (\hat{a}_{ni}\lambda_i - \hat{a}_{n,i+1}\lambda_{i+1}) s_i + a_{nn}\lambda_n s_n.
\end{aligned}
$$

We may write

$$
\begin{aligned}
\hat{a}_{ni}\lambda_i - \hat{a}_{n,i+1}\lambda_{i+1} &= \hat{a}_{ni}\lambda_i - \hat{a}_{n,i+1}\lambda_{i+1} - \hat{a}_{n,i+1}\lambda_i + \hat{a}_{n,i+1}\lambda_i \\
&= (\hat{a}_{ni} - \hat{a}_{n,i+1})\lambda_i + \hat{a}_{n,i+1}(\lambda_i - \lambda_{i+1}) \\
&= \lambda_i \Delta_i \hat{a}_{ni} + \hat{a}_{n,i+1}\Delta\lambda_i.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^{n-1} \Delta_i \hat{a}_{ni}\lambda_i s_i + \sum_{i=1}^{n-1} \hat{a}_{n,i+1}\Delta\lambda_i s_i + a_{nn}\lambda_n s_n \\
&= T_{n1} + T_{n2} + T_{n3}, \quad \text{say.}
\end{aligned}
$$

To complete the proof it is sufficient,by Minkowski's inequality, to show that

$$
\sum_{n=1}^{\infty} n^{\delta k + k - 1} |T_{nr}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3.
$$

From the definition of $\hat{A}$ and $\bar{A}$, and using (i) and (ii);

$$
\begin{aligned}
\hat{a}_{n,i+1} = \bar{a}_{n,i+1} - \bar{a}_{n-1,i+1} = \sum_{\nu=i+1}^{n} a_{n\nu} - \sum_{\nu=i+1}^{n-1} a_{n-1,\nu} = \\
= 1 - \sum_{\nu=0}^{i} a_{n\nu} - 1 + \sum_{\nu=0}^{i} a_{n-1,\nu} = \sum_{\nu=0}^{i} \Big(a_{n-1,\nu} - a_{n,\nu} \Big) \geq 0.
\end{aligned} \tag{3}
$$

Using Hölder's inequality,

$$\sum_{n=1}^{m+1} n^{\delta k+k-1}|T_{n1}|^k \le \sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i||s_i|\Big)^k$$
$$\le \sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i|^k|s_i|^k\Big)\Big(\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}|\Big)^{k-1}$$

Also

$$\Delta_i\hat{a}_{ni} = \hat{a}_{ni} - \hat{a}_{n,i+1} = \bar{a}_{ni} - \bar{a}_{n-1,i} - \bar{a}_{n,i+1} + \bar{a}_{n-1,i+1}$$
$$= a_{ni} - a_{n-1,i} \le 0.$$

Thus, using (i),

$$\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}| = \sum_{i=0}^{n-1}(a_{n-1,i} - a_{ni}) = 1 - 1 + a_{nn} = a_{nn}.$$

Using the fact that $\{s_n\}$ is bounded,(iii), and (viii) we have

$$\sum_{n=1}^{m+1} n^{\delta k+k-1}|T_{n1}|^k = O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i|^k|s_i|^k$$
$$= O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{i=0}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i|^k$$
$$= O(1)\sum_{i=0}^{m}|\lambda_i|^k\sum_{n=i+1}^{m+1} n^{\delta k}(na_{nn})^{k-1}|\Delta_i\hat{a}_{ni}|$$
$$= O(1)\sum_{i=0}^{m} i^{\delta k-1}|\lambda_i|^k$$
$$= O(1), (m\to\infty).$$

Again using Hölder's inequality, (iii) and (vi) we get

$$\sum_{n=1}^{m+1} n^{\delta k+k-1}|T_{n2}|^k \le \sum_{n=1}^{m+1} n^{\delta k+k-1}\Big|\sum_{i=0}^{n-1}\hat{a}_{n,i+1}s_i\Delta\lambda_i\Big|^k$$
$$\le \sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{i=0}^{n-1}\hat{a}_{n,i+1}|\Delta\lambda_i||s_i|\Big)^k$$
$$\le \sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{i=0}^{n-1}\hat{a}_{n,i+1}|\Delta\lambda_i||s_i|^k\Big)\Big(\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}||\Delta\lambda_i|\Big)^{k-1}.$$

Using (i)

$$\hat{a}_{n,i+1} = \sum_{\nu=0}^{i} \Big(a_{n-1,\nu} - a_{n,\nu}\Big) \leq \sum_{\nu=0}^{n-1} \Big(a_{n-1,\nu} - a_{n,\nu}\Big) = 1 - 1 + a_{nn}.$$

Therefore from (iii),(v) and (vii)

$$\begin{aligned}\sum_{n=1}^{m+1} n^{\delta k+k-1}|T_{n2}|^k &= O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{i=0}^{n-1}|\hat{a}_{n,i+1}||\Delta\lambda_i||s_i|^k \\ &= O(1)\sum_{i=1}^{m}|\Delta\lambda_i|\sum_{n=i+1}^{m+1} n^{\delta k}(na_{nn})^{k-1}|\hat{a}_{n,i+1}| \\ &= O(1)\sum_{i=1}^{m} i^{\delta k}|\Delta\lambda_i|, \\ &= O(1), \quad (m\to\infty).\end{aligned}$$

Finally we have, using (iii), (viii) and the boundedness of (s_n)

$$\begin{aligned}\sum_{n=1}^{m+1} n^{\delta k+k-1}|T_{n3}|^k &= \sum_{n=1}^{m+1} n^{\delta k+k-1}|a_{nn}\lambda_n s_n|^k \\ &= O(1)\sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1}a_{nn}|\lambda_n|^k|s_n|^k \\ &= O(1)\sum_{n=1}^{m} n^{\delta k-1}|\lambda_n|^k \\ &= O(1), (m\to\infty).\end{aligned}$$

□

Setting $\delta = 0$ in the theorem yields the following corollary:

COROLLARY 0.2. *Let A be a lower triangular matrix with non-negative entries satisfying*

(i) $\bar{a}_{n0} = 1, n = 0, 1, \ldots,$

(ii) $a_{n-1,\nu} \geq a_{n\nu}$ for $n \geq \nu + 1,$ *and*

(iii) $na_{nn} = O(1), (n \to \infty)$.

If the sequence $\{s_n\}$ *is bounded and* (λ_n) *is a sequence such that*

(iv) $\sum_{n=1}^{m} |\Delta\lambda_n| = O(1), (m \to \infty)$, *and*

(v) $\sum_{n=1}^{m} \frac{1}{n}|\lambda_n|^k = O(1), (m \to \infty)$

then the series $\sum a_n\lambda_n$ *is summable* $|A|_k, k \geq 1$.

COROLLARY 0.3. *Let* $\{p_n\}$ *be a positive sequence such that* $P_n := \sum_{k=0}^{n} p_k \to \infty$, *and satisfies*

(i) $np_n = O(P_n), (n \to \infty)$,

(ii) $\sum_{n=\nu+1}^{m+1} n^{\delta k}|\frac{p_n}{P_n P_{n-1}}| = O\Big(\frac{\nu^{\delta k}}{P\ _\nu}\Big), (m \to \infty)$.

If the sequence $\{s_n\}$ *is bounded and* (λ_n) *is a sequence such that*

(iii) $\sum_{n=1}^{m} |\Delta\lambda_n| = O(1), (m \to \infty)$,

(iv) $\sum_{n=1}^{m} n^{\delta k}|\Delta\lambda_n| = O(1), (m \to \infty)$,

(v) $\sum_{n=1}^{m} n^{\delta k-1}|\lambda_n|^k = O(1), (m \to \infty)$,

then the series $\sum a_n\lambda_n$ *is summable* $|\bar{N}, p, \delta|_k, k \geq 1$ *for* $0 \leq \delta < 1/k$.

Proof. Conditions (iii)- (v) of Corollary 2 are, respectively, conditions (vi)-(viii) of Theorem 1. Conditions (i) and (ii) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (iii) of Theorem 1 becomes condition (i) of Corollary 2, and conditions (iv) and (v) of Theorem 1 become condition (ii) of Corollary 2. □

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