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A General View to Classification of Almost Hermitian Manifolds

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Summary. - *In this paper, firstly, we obtain the higher order vertical and complete lifts of type (1,1) and type (0,2) by means of lifted geometric structures on extended complex manifolds. Then using higher order lifting theory, we generalize the classification of almost Hermitian manifolds M to the extended complex manifolds ${}^kM$.*

1. Introduction and notations

In studies before, the classes of almost Hermitian manifolds M had been given as follows:

Kähler manifolds	:	$[\nabla_X J_0]\,Y = 0$
Nearly Kähler manifolds	:	$[\nabla_X\,, J_0]\,X = 0$
Almost Kähler manifolds	:	$d\Phi = 0$
Quasi Kähler manifolds	:	$[\nabla_{J_0 X}\,, J_0] = -J_0\,[\nabla_X\,, J_0]$
Hermitian manifolds	:	$[\nabla_{J_0 X}\,, J_0] = J_0\,[\nabla_X\,, J_0]$
Semi Kähler manifolds	:	$\sum_{i=1}^{m}\{\nabla_{E_i}(J_0)E_i + \nabla_{J_0 E_i}(J_0)J_0 E_i\} = 0$

for all $X, Y \in \chi(M)$, where $\chi(M)$ denotes the lie algebra of C^∞ vector fields on M, ∇ the Riemannian connection, J_0 the almost complex structure, Φ the fundamental 2-form, and $\{E_i, J_0 E_i\}$ a local orthonormal frame field [1, 2]. Lifts of complex structures on Hermitian and Kähler manifolds and the first order lift classes of the classes given in [4] were studied.

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The paper is structured as follows. In section 2, firstly we recall extended complex manifold and give higher order vertical and complete lifts of differential objects a complex manifold to extended complex manifold [3, 5]. Then higher order lifts of an almost complex structure will be obtained. In sections 3 and 4, we obtain the vertical and complete lifts of complex tensor fields of type (1,1) and type (0,2) defined on any complex manifold M to extended complex manifolds. In section 5, we give definitions about higher order vertical and complete lifts of a Hermitian metric and Kähler form defined on a complex manifold M to its extensions.

In section 6 it is generalized the classification of almost Hermitian manifolds M to the extended complex manifolds ${}^{k}M$.

Along this study, all mappings and manifolds will be assumed to be of class C^{∞} and the sum is taken over repeated indices. Also, v (resp. c) denotes the vertical (resp. complete) lift of any differentiable geometric complex structure defined on extended complex manifolds ${}^{k-1}M$ to ${}^{k}M$.

2. Preliminaries

2.1. Extended complex manifolds

Let ${}^{k}\ M$ be extension of order k of manifold M. A tensor field J_k on ${}^{k}\ M$ is called an *extended almost complex structure* on ${}^{k}\ M$ if at every point p of ${}^{k}\ M$, J_k is endomorphism of the tangent space $T_p({}^{k}\ M)$ such that $(J_k)^2 = -I$. An extended manifold ${}^{k}\ M$ with fixed extended almost complex structure J_k is called *extended almost complex manifold*. If $k = 0$, J_0 is called *almost complex structure* and the manifold M with fixed almost complex structure J_0 is called *almost complex manifold.*

Let (x^{ri}, y^{ri}) be a real coordinate system on a neighborhood ${}^{k}U$ of any point p of ${}^{k}\ M$. In this case, it is respectively defined by $\{\frac{\partial}{\partial x^{ri}}_p, \frac{\partial}{\partial y^{ri}}_p\}$ and $\{dx^{ri}|_p, dy^{ri}|_p\}$ natural bases over $\mathbf{R}$ of tangent space $T_p({}^{k}\ M)$ and cotangent space $T_p^*({}^{k}\ M)$ of ${}^{k}\ M$.

Let ${}^{k}\ M$ be extended almost complex manifold with fixed extended almost complex structure J_k. Then ${}^{k}\ M$ is called *extended complex manifold* if there exists an open covering $\left\{{}^{k}U\right\}$ of ${}^{k}\ M$ satisfying the following condition: There is a local coordinate system (x^{ri}, y^{ri}) on each ${}^kU$, such that

$$J_k(\frac{\partial}{\partial x^{ri}}) = \frac{\partial}{\partial y^{ri}},\ J_k(\frac{\partial}{\partial y^{ri}}) = -\frac{\partial}{\partial x^{ri}}, \tag{1}$$

for each point of ${}^kU$.

If $k = 0$, then a manifold M with fixed canonical almost complex structure J_0 is called *complex manifold*.

Let $(z^{ri}, \overline{z}^{ri})$ be an extended complex local coordinate system on a neighborhood ${}^kU$ of any point p of ${}^k\ M$ where $z^{ri} = x^{ri} + \mathrm{i}\, y^{ri}, \mathrm{i} = \sqrt{-1}$. The real dimension of ${}^k\ M$ is equal to 2(k+1)m. Therefore, these coordinates are locally defined by $x^{ri},\ y^{ri}\ :{}^k\ M \to \mathbf{R}^{(k+1)m}, \forall A, \overline{A} \in{}^k\ M,\ A = a^{ri} + \mathrm{i}\, b^{ri},\ x^{ri}(A) = a^{ri},\ y^{ri}(A) = b^{ri}$ and $z^{ri}, \overline{z}^{ri} :{}^k\ M \to \mathbf{C}^{(k+1)m},\ z^{ri}(A) = A,\ \overline{z}^{ri}(A) = \overline{A}$.

We define the vector fields

$$\frac{\partial}{\partial z^{ri}}_p = \frac{1}{2}\{\frac{\partial}{\partial x^{ri}}_p - \mathbf{i}\frac{\partial}{\partial y^{ri}}_p\}, \frac{\partial}{\partial \overline{z}^{ri}}_p = \frac{1}{2}\{\frac{\partial}{\partial x^{ri}}_p + \mathbf{i}\frac{\partial}{\partial y^{ri}}_p\} \tag{2}$$

and the dual covector fields

$$dz^{ri}\Big|_p = dx^{ri}\Big|_p + \mathbf{i}\ dy^{ri}\Big|_p,\ d\overline{z}^{ri}\Big|_p = dx^{ri}\Big|_p - \mathbf{i}\ dy^{ri}\Big|_p \tag{3}$$

which represent bases of the tangent space $T_p({}^k\ M)$ and cotangent space $T_p^*({}^k\ M)$ of ${}^k\ M$ respectively. Then the endomorphism J_k is given as:

$$J_k(\frac{\partial}{\partial z^{ri}}) = \mathbf{i}\frac{\partial}{\partial z^{ri}}, J_k(\frac{\partial}{\partial \overline{z}^{ri}}) = -\mathbf{i}\frac{\partial}{\partial \overline{z}^{ri}} \tag{4}$$

In this study, we recall the higher order vertical and complete lifts of functions, vector fields, 1-forms on M to ${}^kM$ given in [3]. Then we obtain higher order vertical and complete lifts of complex tensor fields of type (1,1) and of type (0,2) defined on M to ${}^kM$.

Throughout the paper, all mappings and manifolds are assumed to be differentiable of class C^∞ and the sum is taken over repeated indices.

2.2. Higher order lifts of complex structures

In this section, we recall extensions of definitions and properties about vertical and complete lifts of complex geometrical structures defined on a complex manifold M to extended manifolds ${}^k M$.

The *vertical lift* of function $f \in \mathcal{F}(M)$ to ${}^k M$ is the function f^{v^k} on ${}^k M$ such that

$$f^{v^k} = f \circ \tau_{0_M} \circ \tau_{1_M} \circ ... \circ \tau_{k-1_M}. \tag{5}$$

where, for any $h \in \{1, 2, ..., k-1\}$, $\tau_h : T({}^h M) \to^h M$ denotes the canonical projection. The function $f^{c^k} \in \mathcal{F}({}^k M)$ such that

$$f^{c^k} = \dot{z}^{ri} \left(\frac{\partial f^{c^{k-1}}}{\partial z^{ri}}\right)^v + \dot{\overline{z}}^{ri} \left(\frac{\partial f^{c^{k-1}}}{\partial \overline{z}^{ri}}\right)^v. \tag{6}$$

is called *complete lift* of function f to ${}^k M$. The *vertical* and *complete lifts* of Z to ${}^k M$ are the complex vector fields Z^{v^k} and Z^{c^k} on ${}^k M$ such that

$$Z^{v^k}(f^{c^k}) = (Zf)^{v^k} \text{ and } Z^{c^k}(f^{c^k}) = (Zf)^{c^k} \tag{7}$$

Proposition 2.1. *Let M be a complex manifold and ${}^k M$ its k order extended complex manifold. If the complex vector field Z defined on M is taken by*

$$Z = Z^{0i}\frac{\partial}{\partial z^{0i}} + \overline{Z}^{0i}\frac{\partial}{\partial \overline{z}^{0i}},$$

the vertical and complete lifts of Z to ${}^k M$ are

$$\begin{aligned} Z^{v^k} &= (Z^{0i})^{v^k}\frac{\partial}{\partial z^{ki}} + (\overline{Z}^{0i})^{v^k}\frac{\partial}{\partial \overline{z}^{ki}} \\ &\text{and} \\ Z^{c^k} &= \binom{k}{r}(Z^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial z^{ri}} + \binom{k}{r}(\overline{Z}^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial \overline{z}^{ri}}. \end{aligned} \tag{8}$$

The *vertical* and *complete lifts* of $\omega \in \chi^*(M)$ to ${}^k M$ are the complex 1-forms ω^{v^k} and ω^{c^k} on ${}^k M$ determined by

$$\omega^{v^k}(Z^{c^k}) = (\omega Z)^{v^k} \text{ and } \omega^{c^k}(Z^{c^k}) = (\omega Z)^{c^k}. \tag{9}$$

PROPOSITION 2.2. *Let $^k M$ be extension of order k of complex manifold M. If the complex 1-form ω defined on M is given*

$$\omega = \omega_{0i} dz^{0i} + \overline{\omega}_{0i} d\overline{z}^{0i},$$

the vertical and complete lifts of ω to $^k M$ are

$$\begin{aligned} \omega^{v^k} &= (\omega_{0i})^{v^k} dz^{0i} + (\overline{\omega}_{0i})^{v^k} d\overline{z}^{0i} \\ &\textit{and} \\ \omega^{c^k} &= (\omega_{0i})^{c^{k-r} v^r} dz^{ri} + (\overline{\omega}_{0i})^{c^{k-r} v^r} d\overline{z}^{ri}. \end{aligned} \tag{10}$$

3. Higher order lifts of complex tensor fields of type (1,1)

In this section, we obtain the definitions and properties about vertical and complete lifts of a complex tensor field of type (1,1) defined on any complex manifold M to extended complex manifolds $^k M$.

Let M be any complex manifold and ^{k-1}M its $(k-1)$ order extended complex manifold. Denote by $\widetilde{F}$ a complex tensor field of type (1,1) and by $\widetilde{Z}$ be a complex vector field defined on ^{k-1}M. Then the *vertical lift* of a complex tensor field of type (1,1) $\widetilde{F} \in \Im^1_1({}^{k-1}M)$ to $^k M$ is the structure $\widetilde{F}^v \in \Im^1_1({}^k M)$ such that

$$\widetilde{F}^v(\widetilde{Z}^c) = (\widetilde{F}\widetilde{Z})^v. \tag{11}$$

Now, let $Z^{c^{k-1}}$ and $F^{v^{k-1}}$be respectively complete and vertical lifts of a complex vector field $Z \in \chi(M)$ and a complex tensor field of type (1,1) $F \in \Im^1_1(M)$ to ^{k-1}M. In (11), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{F} = F^{v^{k-1}}$, then the *vertical lift* of $F \in \Im^1_1(M)$ to $^k M$ is the tensor field F^{v^k} on $^k M$ such that

$$F^{v^k}(Z^{c^k}) = (FZ)^{v^k}. \tag{12}$$

Local components of vertical lift of a complex tensor field

$$F = F^{0i}_{0j} \frac{\partial}{\partial z^{0i}} \otimes dz^{0j} + \overline{F}^{0i}_{0j} \frac{\partial}{\partial \overline{z}^{0i}} \otimes d\overline{z}^{0j} \tag{13}$$

are

$$F^{v^k} = (F^{0i}_{0j})^{v^k} \frac{\partial}{\partial z^{ki}} \otimes dz^{0j} + (\overline{F}^{0i}_{0j})^{v^k} \frac{\partial}{\partial \overline{z}^{ki}} \otimes d\overline{z}^{0j}, \tag{14}$$

i.e.,

$$F^{v^k} : \begin{pmatrix} K & 0 \\ 0 & \overline{K} \end{pmatrix} \qquad (15)$$

where

$$K : \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ i(F_{0j}^{0i})^{v^k} & 0 & \dots & 0 \end{pmatrix}$$

and $\overline{K}$ is the conjugate of K.

If F is taken as complex structure $J_0 = \frac{\partial}{\partial z^{0i}} \otimes dz^{0j} + \frac{\partial}{\partial \overline{z}^{0i}} \otimes d\overline{z}^{0j}$, then it is

$$J_0^{v^k} : \begin{pmatrix} H & 0 \\ 0 & \overline{H} \end{pmatrix}, \qquad (16)$$

where

$$H : \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ iI_m & 0 & \dots & 0 \end{pmatrix}$$

and $\overline{H}$ is the conjugate of H.

Vertical lift of order k of complex structure J_0 is tangent structure on ${}^kM$ such that $(J_0^{v^k})^2 = 0$.

Let M be a complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Denote by $\widetilde{F}$ a complex tensor field of type (1,1) and by $\widetilde{Z}$ a complex vector field defined on ${}^{k-1}M$. Then the *complete lift* of $\widetilde{F} \in \Im_1^1({}^{k-1}M)$ to ${}^kM$ is the complex tensor field $\widetilde{F}^c \in \Im_1^1({}^kM)$ such that

$$\widetilde{F}^c(\widetilde{Z}^c) = (\widetilde{F}\widetilde{Z})^c. \qquad (17)$$

Now, let $Z^{c^{k-1}}$ and $F^{c^{k-1}}$be respectively complete lifts of a complex vector field $Z \in \chi(M)$ and a complex tensor field of type (1,1) $F \in \Im_1^1(M)$ to ${}^{k-1}M$. In (17), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{F} = F^{c^{k-1}}$, then the *complete lift* of $F \in \Im_1^1(M)$ to ${}^kM$ is the tensor field F^{c^k} on ${}^kM$ such that

$$F^{c^k}(Z^{c^k}) = (FZ)^{c^k}. \qquad (18)$$

The local components of the complete lift of a complex tensor field expressed as in (13) are

$$\begin{aligned} F^{c^k} &= (F^{0i}_{0j})^{c^k}\frac{\partial}{\partial z^{ki}}\otimes dz^{0j}+(F^{0i}_{0j})^{v^k}\frac{\partial}{\partial z^{0i}}\otimes dz^{0j}+ \\ &+(F^{0i}_{0j})^{v^k}\frac{\partial}{\partial z^{ki}}\otimes dz^{kj}+(\overline{F}^{0i}_{0j})^{c^k}\frac{\partial}{\partial \overline{z}^{ki}}\otimes d\overline{z}^{0j}+ \\ &+(\overline{F}^{0i}_{0j})^{v^k}\frac{\partial}{\partial \overline{z}^{0i}}\otimes d\overline{z}^{0j}+(\overline{F}^{0i}_{0j})^{v^k}\frac{\partial}{\partial \overline{z}^{ki}}\otimes d\overline{z}^{kj}, \end{aligned} \tag{19}$$

i.e.,

$$F^{c^k}:\begin{pmatrix} L & 0 \\ 0 & \overline{L}\end{pmatrix}, \tag{20}$$

where

$$L:\begin{pmatrix} \mathbf{i}\binom{k}{0}(F^{0i}_{0j})^{v^k c^0} & & & \\ & . & . & 0 \\ & . & . & \\ & . & & . \\ \mathbf{i}\binom{k}{k}(F^{0i}_{0j})^{v^0 c^k} & . & . & . \ \ \mathbf{i}\binom{k}{k}(F^{0i}_{0j})^{v^k c^0} \end{pmatrix}$$

and $\overline{L}$ is the conjugate of L.

If F is taken as complex structure $J_0=\frac{\partial}{\partial z^{0i}}\otimes dz^{0j}+\frac{\partial}{\partial \overline{z}^{0i}}\otimes d\overline{z}^{0j}$, then complete lift of order k of complex structure J is

$$J_0^{c^k}:\begin{pmatrix} T & 0 \\ 0 & \overline{T}\end{pmatrix}, \tag{21}$$

where

$$T:\begin{pmatrix} \mathbf{i}I_m & & & \\ & . & 0 & \\ & & . & \\ & 0 & . & \\ & & & \mathbf{i}I_m \end{pmatrix}$$

and $\overline{T}$ is the conjugate of T.

Complete lift of order k of complex structure J_0 is complex structure on ${}^kM$ such that $(J_0^{c^k})^2=-I$.

The higher order vertical and complete lifts of a complex tensor field of type (1,1) on any complex manifold M obey the following generic properties

$$\begin{array}{lrcl} i) & F^{v^k}(Z^{v^k}) & = & 0 \\ ii) & F^{v^k}(Z^{c^k})=F^{c^k}(Z^{v^k}) & = & (FZ)^{v^k} \\ iii) & F^{c^k}(Z^{c^k}) & = & (FZ)^{c^k}, \end{array}$$

for all $Z\in\chi(M)$ and $F\in\Im_1^1(M)$.

4. Higher order lifts of complex tensor fields of type (0,2)

In this section, we obtain the definitions and properties about vertical and complete lifts of a complex tensor field of type (0,2) defined on any complex manifold M to extended complex manifolds ${}^kM$.

Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Let $\widetilde{G}$ be a tensor field of type (0,2) and $\widetilde{Z},\widetilde{W}$ be complex vector fields defined on ${}^{k-1}M$. Then the *vertical lift* of $\widetilde{G}\in\Im_2^0({}^{k-1}M)$ to ${}^kM$ is the tensor field of type (0,2) $\widetilde{G}^v\in\Im_2^0({}^kM)$ such that

$$\widetilde{G}^v(\widetilde{Z}^c,\widetilde{W}^c)=\widetilde{G}(\widetilde{Z},\widetilde{W})^v. \tag{22}$$

Now, let $Z^{c^{k-1}},W^{c^{k-1}}$ and $G^{v^{k-1}}$be respectively complete and vertical lifts of complex vector fields $Z,W\in\chi(M)$ and a tensor field of type (0,2) $G\in\Im_2^0(M)$ to ${}^{k-1}M$. In (22), if $\widetilde{Z}=Z^{c^{k-1}},\widetilde{W}=W^{c^{k-1}}$ and $\widetilde{G}=G^{v^{k-1}}$, then the *vertical lift* of $G\in\Im_2^0(M)$ to ${}^kM$ is the tensor field of type (0,2) $G^{v^k}\in\Im_2^0({}^kM)$ such that

$$G^{v^k}(Z^{c^k},W^{c^k})=(G(Z,W))^{v^k}. \tag{23}$$

Local components of vertical lift of a complex tensor field

$$g=g_{\overline{0i}0j}d\overline{z}^{0i}\otimes dz^{0j}+g_{0i\overline{0j}}dz^{0i}\otimes d\overline{z}^{0j} \tag{24}$$

are

$$g^{v^k} = (g_{\overline{0i}0j})^{v^k} d\overline{z}^{0i} \otimes dz^{0j} + (g_{0i\overline{0j}})^{v^k} dz^{0i} \otimes d\overline{z}^{0j}, \tag{25}$$

i.e.,

$$g^{v^k} : \begin{pmatrix} 0 & M \\ \overline{M} & 0 \end{pmatrix}, \tag{26}$$

where

$$M : \begin{pmatrix} (g_{0i\overline{0j}})^{v^k} & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $\overline{M}$ is the conjugate of M.

We say the *complete lift* of $\widetilde{G} \in \Im_2^0({}^{k-1}M)$ to ${}^kM$ is the tensor field of type (0,2) $\widetilde{G}^c \in \Im_2^0({}^kM)$ such that

$$\widetilde{G}^c(\widetilde{Z}^c, \widetilde{W}^c) = \widetilde{G}(\widetilde{Z}, \widetilde{W})^c. \tag{27}$$

Let $Z^{c^{k-1}}, W^{c^{k-1}}$ and $G^{c^{k-1}}$be respectively complete lifts of complex vector fields Z, W and a tensor field of type (0,2) $G \in \Im_2^0(M)$ defined on M to ${}^{k-1}M$. In (27), if $\widetilde{Z} = Z^{c^{k-1}}, \widetilde{W} = W^{c^{k-1}}$ and $\widetilde{G} = G^{c^{k-1}}$, then the *complete lift* of $G \in \Im_2^0(M)$ to ${}^kM$ is the tensor field of type (0,2) G^{c^k} on ${}^kM$ such that

$$G^{c^k}(Z^{c^k}, W^{c^k}) = (G(Z, W))^{c^k}. \tag{28}$$

Local components of vertical lift of a complex tensor field g given in (24) are

$$\begin{aligned} g^{c^k} \quad = \quad & (g_{\overline{0i}0j})^{c^k} d\overline{z}^{0i} \otimes dz^{0j} + (g_{\overline{0i}0j})^{v^k} d\overline{z}^{ki} \otimes dz^{0j} \\ & + (g_{\overline{0i}0j})^{v^k} d\overline{z}^{0i} \otimes dz^{kj} + (g_{0i\overline{0j}})^{c^k} dz^{0i} \otimes d\overline{z}^{0j} \\ & + (g_{0i\overline{0j}})^{v^k} dz^{ki} \otimes d\overline{z}^{0j} + (g_{0i\overline{0j}})^{v^k} dz^{0i} \otimes d\overline{z}^{kj} \end{aligned} \tag{29}$$

i.e.,

$$g^{c^k} : \begin{pmatrix} 0 & N \\ \overline{N} & 0 \end{pmatrix}, \tag{30}$$

where

$$N:\begin{pmatrix} \binom{k}{0}(g_{0i\overline{0j}})^{c^k} & \cdots & \binom{k}{r}(g_{0i\overline{0j}})^{c^{k-r}v^r} & \cdots & \binom{k}{k}(g_{0i\overline{0j}})^{v^k} \\ \vdots & & & & \\ \binom{k}{r}(g_{0i\overline{0j}})^{c^{k-r}v^r} & & 0 & & \\ \vdots & & & & \\ \binom{k}{k}(g_{0i\overline{0j}})^{v^k} & & & & \end{pmatrix}$$

and $\overline{N}$ is the conjugate of N.

The general properties of higher order vertical and complete lifts of complex tensor fields of type (0,2) on complex manifold M are

$$\begin{array}{llll} i) & G^{v^k}(Z^{v^k},W^{v^k}) & = & 0 \\ ii) & G^{c^k}(Z^{c^k},W^{v^k}) & = & G^{c^k}(Z^{v^k},W^{c^k}) = G^{v^k}(Z^{c^k},W^{c^k}) = (G(Z,W))^{v^k} \\ iii) & G^{c^k}(Z^{c^k},W^{c^k}) & = & (G(Z,W))^{c^k}, \end{array}$$

for all $Z,W\in\chi(M)$ and $G\in\Im_2^0(M)$.

5. Higher order lifts of hermitian metric and Kähler form

In this section, we give definitions about higher order vertical and complete lifts of an Hermitian metric and Kähler form defined on any complex manifold M to extended complex manifolds ${}^kM$.

In order to define higher order vertical and complete lifts of an Hermitian metric defined on M, we first define higher order vertical and complete lifts of complex tensor fields of type (0,2).

Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Let $\widetilde{G}$ be a tensor field of type (0,2) and $\widetilde{Z},\widetilde{W}$ be complex vector fields defined on ${}^{k-1}M$. Then the *vertical lift* of $\widetilde{G}$ to ${}^kM$ is the tensor field of type (0,2) $\widetilde{G}^v$ on ${}^kM$ such that

$$\widetilde{G}^v(\widetilde{Z}^c,\widetilde{W}^c)=\widetilde{G}(\widetilde{Z},\widetilde{W})^v. \tag{31}$$

Now, let $Z^{c^{k-1}},W^{c^{k-1}}$ and $G^{v^{k-1}}$be respectively complete and vertical lifts of complex vector fields Z,W and a tensor field of type

(0,2) G defined on M to ${}^{k-1}M$. In (31), if $\widetilde{Z} = Z^{c^{k-1}}, \widetilde{W} = W^{c^{k-1}}$ and $\widetilde{G} = G^{v^{k-1}}$, then the *vertical lift* of G to ${}^{k}M$ is the tensor field of type (0,2) G^{v^k}on ${}^{k}M$ such that

$$G^{v^k}(Z^{c^k}, W^{c^k}) = (G(Z,W))^{v^k}. \tag{32}$$

Given a Hermitian metric g and an almost complex structure J_0 defined on any complex manifold M. Since g is a tensor field of type (0,2), we have equality

$$g^{v^k}(Z^{c^k}, W^{c^k}) = g^{v^k}(J_0^{c^k} Z^{c^k}, J_0^{c^k} W^{c^k}). \tag{33}$$

for any complex vector fields Z, W on M. Hence we call the *vertical lift* of g to extended complex manifold ${}^{k}M$ as Hermitian metric g^{v^k}on ${}^{k}M$. extended complex manifold ${}^{k}M$ with fixed Hermitian metric g^{v^k} is called *vertical lift* of order k of Hermitian manifold which is complex manifold M with fixed Hermitian metric g.

We say that the *complete lift* of $\widetilde{G}$ to ${}^{k}M$ is the tensor field of type (0,2) $\widetilde{G}^c$ on ${}^{k}M$ such that

$$\widetilde{G}^c(\widetilde{Z}^c, \widetilde{W}^c) = \widetilde{G}(\widetilde{Z}, \widetilde{W})^c. \tag{34}$$

Let $Z^{c^{k-1}}, W^{c^{k-1}}$ and $G^{c^{k-1}}$be respectively complete lifts of complex vector fields Z, W and a tensor field of type (0,2) G defined on M to ${}^{k-1}M$. In (34), if $\widetilde{Z} = Z^{c^{k-1}}, \widetilde{W} = W^{c^{k-1}}$ and $\widetilde{G} = G^{c^{k-1}}$, then the *complete lift* of G to ${}^{k}M$ is the tensor field of type (0,2) G^{c^k} on ${}^{k}M$ given by equality

$$G^{c^k}(Z^{c^k}, W^{c^k}) = (G(Z,W))^{c^k}. \tag{35}$$

Given a Hermitian metric g and an almost complex structure J_0 defined on any complex manifold M. Since g is a tensor field of type (0,2), we have equality

$$g^{c^k}(Z^{c^k}, W^{c^k}) = g^{c^k}(J_0^{c^k} Z^{c^k}, J_0^{c^k} W^{c^k}). \tag{36}$$

for any complex vector fields Z, W on M. Hence we call the *complete lift* of g to extended complex manifold ${}^{k}M$ as g^{c^k}on ${}^{k}M$. extended complex manifold ${}^{k}M$ with fixed Hermitian metric g^{c^k} is called *complete lift* of order k of Hermitian manifold which is complex manifold M with fixed Hermitian metric g.

Given a Kähler form Φ, Hermitian metric g and an almost complex structure J_0 defined on any Hermitian manifold M. Since Φ is a complex tensor field of type (0,2), we have equality

$$\Phi^{v^k}(Z^{c^k}, W^{c^k}) = g^{v^k}(Z^{c^k}, J_0^{c^k} W^{c^k}) \tag{37}$$

for any complex vector fields Z, W on M. Hence we call the *vertical lift* of Φ to extended complex manifold ${}^kM$ as Kähler form Φ^{v^k} on ${}^kM$.

In this way we shall definition the following.

Given a Kähler form Φ,Hermitian metric g and an almost complex structure J_0 defined on any Hermitian manifold M. Since Φ is a complex tensor field of type (0,2), we have equality

$$\Phi^{c^k}(Z^{c^k}, W^{c^k}) = g^{c^k}(Z^{c^k}, J_0^{c^k} W^{c^k}) \tag{38}$$

for any complex vector fields Z, W on M. Hence we call the *complete lift* of Φ to extended complex manifold ${}^kM$ as Kähler form Φ^{c^k} on ${}^kM$.

Now, let define the complete lift to ${}^kM$ of an Affine connection $\bigtriangledown$ on M for using in Section 6. So, we extend using induction method the properties about vertical and complete lifts of complex tensor fields given the following such that all $X, Y, Z \in \Im_0^1(M)$, $f \in \Im_0^0(M)$, $\omega \in \Im_1^0(M)$, $K \in \Im_s^r(M)$:

$$\begin{aligned}
i) \quad \bigtriangledown^{c^k}_{X^{v^k}} f^{v^k} &= 0, \\
\bigtriangledown^{c^k}_{X^{v^k}} f^{c^k} &= \bigtriangledown^{c^k}_{X^{c^k}} f^{v^k} = (\bigtriangledown_X f)^{v^k}, \\
\bigtriangledown^{c^k}_{X^{c^k}} f^{c^k} &= (\bigtriangledown_X f)^{c^k} \\
ii) \quad \bigtriangledown^{c^k}_{X^{v^k}} Y^{v^k} &= 0, \\
\bigtriangledown^{c^k}_{X^{v^k}} Y^{c^k} &= \bigtriangledown^{c^k}_{X^{c^k}} Y^{v^k} = (\bigtriangledown_X Y)^{v^k}, \\
\bigtriangledown^{c^k}_{X^{c^k}} Y^{c^k} &= (\bigtriangledown_X Y)^{c^k}
\end{aligned}$$

$$\begin{aligned}
iii)\quad \nabla^{c^k}_{X^{v^k}}\omega^{v^k} &= 0,\\
\nabla^{c^k}_{X^{v^k}}\omega^{c^k} &= \nabla^{c^k}_{X^{c^k}}\omega^{v^k} = (\nabla_X\omega)^{v^k},\\
\nabla^{c^k}_{X^{c^k}}\omega^{c^k} &= (\nabla_X\omega)^{c^k}\\
iv)\quad \nabla^{c^k}_{X^{v^k}}K^{v^k} &= 0,\\
\nabla^{c^k}_{X^{v^k}}K^{c^k} &= \nabla^{c^k}_{X^{c^k}}K^{v^k} = (\nabla_X K)^{v^k},\\
\nabla^{c^k}_{X^{c^k}}K^{c^k} &= (\nabla_X K)^{c^k}
\end{aligned}$$

6. Classification of extended almost Hermitean manifold

In this section we may generalize the classification made for almost Hermitian manifold to extended almost Hermitian manifolds using lift theory as follows.

PROPOSITION 6.1. *Let M be a Kähler manifold with almost complex structure J_0. Then the extended manifold $({}^kM, J_0^{c^k})$ is a Kähler manifold.*

Proof. Given by complex vector fields Z, W and by J_0 almost complex structure on any Kähler manifold M. Also, let Z^{c^k}, W^{c^k}, and $J_0^{c^k}$ be respectively complete lifts of this tensor fields to extended complex manifold ${}^kM$. Then we have

$$\begin{aligned}
\left[\nabla^{c^k}_{Z^{c^k}}, J_0^{c^k}\right] W^{c^k} &= \nabla^{c^k}_{Z^{c^k}}(J_0^{c^k} W^{c^k}) - J_0^{c^k}\,\nabla^{c^k}_{Z^{c^k}} W^{c^k}\\
&= \left([\nabla_Z, J_0]\,W\right)^{c^k}\\
&= 0.
\end{aligned}$$

Finally, the proof is completed. □

PROPOSITION 6.2. *Let M be a nearly Kähler manifold with almost complex structure J_0. Then the extended manifold $({}^kM, J_0^{c^k})$ is a nearly Kähler manifold.*

Proof. We easily make as similar the proof of proposition 6.1. □

PROPOSITION 6.3. *Let M be an almost Kähler manifold with almost complex structure J_0. Then the extended manifold $({}^kM, J_0^{c^k})$ is an almost Kähler manifold.*

Proof. For any 2-form ω, one has:

$$3d\omega(X,Y,Z) = (\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla_Z\omega)(X,Y). \quad (39)$$

So, we consider an almost Kähler manifold (M, g, J_0) with fundamental form Φ. Since $(\nabla^{c^k}_{X^{c^k}}\Phi^{c^k})(Y^{c^k}, Z^{c^k}) = ((\nabla_X\Phi)(Y,Z))^{c^k}$ (which follows from iv) in Section 5, using eq. (39), we get:

$$d\Phi^{c^k}(X^{c^k}, Y^{c^k}, Z^{c^k}) = (d\Phi(X,Y,Z))^{c^k}.$$

Then one proves the statement, since Φ is closed, M being almost Kähler. □

PROPOSITION 6.4. *Let M be a quasi Kähler manifold with almost complex structure J_0. Then the extended manifold $({}^kM, J_0^{c^k})$ is a quasi Kähler manifold.*

Proof. Given by Z, W complex vector fields and by J_0 almost complex structure on any quasi Kähler manifold M. Also, let Z^{c^k}, W^{c^k}, and $J_0^{c^k}$ be respectively complete lifts of this tensor fields to extended complex manifold ${}^kM$. Then we have

$$\begin{aligned}
\left[\nabla^{c^k}_{J_0^{c^k}Z^{c^k}}, J_0^{c^k}\right] W^{c^k} &= \nabla^{c^k}_{J_0^{c^k}Z^{c^k}} J_0^{c^k} W^{c^k} - J_0^{c^k}\, \nabla^{c^k}_{J_0^{c^k}Z^{c^k}} W^{c^k} \\
&= (\nabla_{J_0Z} J_0 W - J_0 \nabla_{J_0Z} W)^{c^k} \\
&= ([\nabla_{J_0Z}, J_0]\, W)^{c^k} \\
&= (-J_0\, [\nabla_Z, J_0]\, W)^{c^k} \\
&= -J_0^{c^k}\left[\nabla^{c^k}_{Z^{c^k}}, J_0^{c^k}\right].
\end{aligned}$$

Hence, the proof is finished. □

PROPOSITION 6.5. *Let M be a Hermitian manifold with almost complex structure J_0. Then the extended manifold $({}^kM, J_0^{c^k})$ is a Hermitian manifold.*

Proof. We easily make as similar the proof of proposition 6.4. □

PROPOSITION 6.6. *Let M be a Semi Kähler manifold with almost complex structure J_0. Then the extended manifold $({}^kM, J_0^{c^k})$ is a Semi Kähler manifold.*

Proof. Given a local orthonormal frame $\{E_i, J_0E_i\}_{1\leq i\leq m}$, one has:

$$\nabla^{c^k}_{E_i^{c^k}}(J_0^{c^k})E_i^{c^k} + \nabla^{c^k}_{J_0^{c^k}E_i^{c^k}}(J_0^{c^k})J_0^{c^k}E_i^{c^k} = (\nabla_{E_i}(J_0)E_i + \nabla_{J_0E_i}(J_0)J_0E_i),$$

for any $i \in \{1, ..., m\}$.
Therefore, if M is Semi Kähler , ${}^kM$ is Semi Kähler. □

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