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Petri Maps for Very Ample Line Bundles

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Summary. - *Here we construct several smooth curves C and several very ample and special line bundles L on C such that the multiplication map $H^0(C,L)\otimes H^0(C,\omega_C\otimes L^*)\to H^0(C,\omega_C)$ (i.e. the Petri map) is surjective.*

1. Introduction

Let C be a smooth and connected projective curve and $L\in \mathrm{Pic}(C)$ such that $h^0(C,L)\ge 2$ and $h^1(C,L)\ge 2$. A key role in the classical theory of special divisors on C is played by the Petri map, i.e. by the multiplication map $\mu_L : H^0(C,L)\otimes H^0(C,\omega_C\otimes L^*)\to H^0(C,\omega_C)$ ([2] or [3], Ch. II). The best situation occurs when μ_L is injective, but this is a rather restrictive assumption for the pair (C,L), because it implies $h^0(C,L)\cdot h^0(C,\omega_C\otimes L^*)\le h^0(C,\omega_C)$, i.e. $h^0(X,L)\cdot (h^0(X,L)+p_a(C)-1-\deg(L))\le p_a(C)$. This condition is not satisfied when L induces an embedding of C in $\mathbf{P}^n$ as a curve of high genus with respect to its degree and in particular it is never satisfied for many pairs (C,L) which arise quite often in the Castelnuovo theory of curves in $\mathbf{P}^n$ ([3], Ch. III). The next best thing would be the surjectivity of μ_L, because it would give the key

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information dim(Ker(μ_L)). Notice that the surjectivity of μ_L is an open condition for a flat family of pairs (C, L) for which $h^0(C, L)$ is constant. Following [1] we use the following classical definition.

DEFINITION 1.1. *Fix an integer $k > 0$ and curve $C \subset \mathbf{P}^n$ (i.e. a locally closed equidimensional one-dimensional suscheme of $\mathbf{P}^n$). C is said to be k-normal if the restriction map $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k)) \to H^0(C, \mathcal{O}_C(k))$ is surjective. C is said to be strongly k-normal if it is t-normal for every integer t such that $1 \le t \le k$.*

The definition of μ_L makes sense even if C is singular and we will get for free the surjectivity of the Petri map for pairs pairs (C, L) with C Gorenstein (but perhaps with multiple components) and C any member of a suitable linear system on a certain degree $2n - 2$ linearly normal $K3$ surface $S \subset \mathbf{P}^n$.

THEOREM 1.2. *Fix integers d, g, n, x such that $n \ge 3$ and $0 \le d - n < g < d^2/(4n-4) - (n-1)/4$. Set $r := \lfloor (d - \sqrt{d^2 - (4n-4)g}/(2n-2) \rfloor$, $d_0 := d - (2n-2)r$ and $g_0 := (n-1)r^2 - dr + g$; notice that $r \ge 1$ and $0 \le g_0 \le d - n$. Assume $1 \le 3x \le r$. Then there exists a smooth linearly normal $K3$ surface $S \subset \mathbf{P}^n$ with degree $2n - 2$ and the following properties. Set $H := \mathcal{O}_S(1)$. There is a smooth and connected curve $C_0 \subset S$ such that $deg(C_0) = d_0$, $p_a(C_0) = g_0$, $h^0(S, \mathcal{O}_S(H - C_0)) = h^0(S, \mathcal{O}_S(C_0 - H)) = 0$, $Pic(S)$ is freely generated by H and C_0 and a general element of $|C_0 + rH|$ is smooth. Fix any $C \in |C_0 + rH|$. C is strictly r-normal. C is $(r+1)$-normal if and only if C_0 is linearly normal, i.e. if and only if $n + g_0 = d_0 + h^1(C_0, \mathcal{O}_{C_0}(1))$. Set $L := \mathcal{O}_C(x)$. Then μ_L is surjective.*

We work over an algebraically closed field $\mathbb{K}$ such that char$(\mathbb{K}) = 0$. Our proofs use an existence theorem for curves in certain $K3$ surfaces proved only in characteristic zero ([4], [5]).

2. Proof of Theorem 1.2

All the results, except the surjectivity of μ_L were proved in [1]. More precisely, the results were proved in [1], Prop. 2.2 and proof of Th. 1.4, but only stated when C is smooth; however, no difference is for the case in which C is an arbitrary element of $|C_0 + rH|$. A key tool for the quoted results in [1] was [4], Th. 4.6. Let $\eta_t : H^0(S, \mathcal{O}_S(tH)) \otimes H^0(S, \mathcal{O}_S(C_0 + (r-t)H) \to H^0(S, \mathcal{O}_S(C_0 + rH))$ denote the multiplication map. Since C is strictly r-normal, the restriction map $\rho_t : H^0(S, \mathcal{O}_S(t)) \to H^0(C, \mathcal{O}_C(t))$ is surjective for every integer t such that $1 \le t \le r$. Since $h^0(S, \mathcal{O}_S((t-r)H - C_0)) = 0$ if $t \le r$, we get the bijectivity of ρ_t for $1 \le t \le r$. Since $\omega_S \cong \mathcal{O}_S$, we have $\omega_C \cong \mathcal{O}_C(C)$ (adjunction formula). Hence for any integer z we have the exact sequence

$$0 \to \mathcal{O}_S((z-r)H) \to \mathcal{O}_S(C_0 + zH) \to \omega_C((z-r)) \to 0 \qquad (1)$$

Since S is arithmetically Cohen - Macaulay (or use Kodaira vanishing and duality), we have $h^1(S, \mathcal{O}_S(yH)) = 0$ for every integer y. Hence the restriction map $\tau_z : H^0(S, \mathcal{O}_S(C_0 + zH)) \to H^0(C, \omega_C((z-r))$ is bijective for every integer $z < r$ and surjective with one-dimensional kernel for $z = r$. Hence to show that μ_L is surjective, it is sufficient to show that η_x is surjective. We fix a general linear subspace W of $H^0(S, \mathcal{O}_S(xH))$ such that $\dim(W) = 3$. Since $\mathcal{O}_S(H)$ is very ample, W spans $\mathcal{O}_S(xH)$. Hence for all integers y the vector space W induces the following Koszul complex type exact sequence

$$0 \to \mathcal{O}_S(C_0 + (r - 3x + y)H) \to \mathcal{O}_S(C_0 + (r - 2x + y)H)^{\oplus 3} \qquad (2)$$
$$\xrightarrow{\alpha_y} \mathcal{O}_S(C_0 + (r - x + y)H)^{\oplus 3} \xrightarrow{\beta_y} \mathcal{O}_S(C_0 + (r + y)H) \to 0$$

Set $A_y := \mathrm{Im}(\alpha_y) = \mathrm{Ker}(\beta_y)$. Since $W \subseteq H^0(S, \mathcal{O}_S(xH))$ and ρ_x and τ_r are surjective, to prove the surjectivity of μ_L it is sufficient to prove $h^1(S, A_0) = 0$. Since $3x \le r$, then $h^2(S, \mathcal{O}_S(C_0 + (r-3x)H)) = h^0(S, \mathcal{O}_S(-C_0 + (3x - r)H)) = 0$. Hence we would get $h^1(S, A_0) = 0$ if $h^1(S, \mathcal{O}_S(C_0 + (r - 2x)H)) = 0$. By [1], proof of part (iiib) of Proposition 2.2, the linear system $|C_0|$ has no base point. Since C_0 is irreducible and H ample, we get $h^1(S, \mathcal{O}_S(C_0 + (r - 2x)H)) = 0$ by Kodaira vanishing if either $2x < r$ or $2x = r$ and $C_0 \cdot C_0 > 0$, concluding the proof.

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