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On Higher Order Complete-Vertical and Horizontal Lifts of Complex Structures

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Summary. - *In this paper, we will obtain the (r,s) order complete-vertical lifts and horizontal lifts of order higher of the complex structures on complex manifold M to the canonical extensions.*

1. Introduction

In modern differentiable geometry, the readers know that lift method has an important role. Because, it is possible to generalize to the structures on any manifold to the extensions using lift function. The structure of extended manifolds has been obtained, especially the canonical extended manifold ${}^kM$ of order k of the manifold M [1, 3]. It has been founded the higher order vertical and complete lifts of functions, vector fields and 1-forms on vector bundle (resp. complex manifold) to extended vector bundle (resp. extended complex manifold) [2, 6]. In this study we obtain (r, s) order complete-vertical lifts and higher order horizontal lifts of complex structures using the above studies.

In the paper, all mappings and manifolds are assumed to be of class C^∞ and the sum is taken over repeated indices. Also we accept

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$0 \leq r \leq k, 1 \leq i \leq m.$

2. Background

2.1. Extended Complex Manifold

DEFINITION 2.1. *[6] Let M $2m$-real dimensional manifold and ${}^kM$ its k order extended manifold. A tensor field J_k on ${}^kM$ is called an* extended almost complex structure *on ${}^kM$ if at every point p of ${}^kM, J_k$ is endomorphism of the tangent space $T_p({}^kM)$ such that $(J_k)^2 = -I$. An extended manifold ${}^kM$ with fixed extended almost complex structure J_k is called an* extended almost complex manifold. *If $k = 0, J_0$ is called* almost complex structure; *a manifold ${}^0M = M$ with fixed almost complex structure J_0 is called an almost complex manifold.*

Let (x^{ri}, y^{ri}) be a real coordinate system on a neighborhood ${}^kU$ of any point p of ${}^kM$. In this situation, it is respectively defined by $\left\{ \frac{\partial}{\partial x^{ri}}\big|_p , \frac{\partial}{\partial y^{ri}}\Big|_p \right\}$ and $\left\{dx^{ri}|_p, dy^{ri}|_p\right\}$ natural bases over IR of tangent space $T_p({}^kM)$ and cotangent space $T_p^*({}^kM)$ of ${}^kM$.

Let ${}^kM$ be extended almost complex manifold with fixed extended almost complex structure J_k. ${}^kM$ is called *extended complex manifold* if there exists an open covering $\left\{{}^kU\right\}$ of ${}^kM$ satisfying the following condition: there is a local coordinate system (x^{ri}, y^{ri}) on each ${}^kU$ such that for each point of ${}^kU$,

$$J_k(\frac{\partial}{\partial x^{ri}}) = \frac{\partial}{\partial y^{ri}}, \qquad J_k(\frac{\partial}{\partial y^{ri}}) = -\frac{\partial}{\partial x^{ri}}. \tag{1}$$

If $k = 0$, then a manifold ${}^0M = M$ with fixed almost complex structure J_0 is called *complex manifold*. Let $z^{ri} = x^{ri}+\mathrm{i}\,y^{ri}, \mathrm{i} = \sqrt{-1}$, be an extended complex local coordinate system on a neighborhood ${}^kU$ of any point p of ${}^kM$. If it is defined by equalities

$$\begin{aligned} \frac{\partial}{\partial z^{ri}}\Big|_p &= \frac{1}{2}\left\{ \frac{\partial}{\partial x^{ri}}\Big|_p - \mathrm{i}\,\frac{\partial}{\partial y^{ri}}\Big|_p \right\} \\ \frac{\partial}{\partial \bar{z}^{ri}}\Big|_p &= \frac{1}{2}\left\{ \frac{\partial}{\partial x^{ri}}\Big|_p + \mathrm{i}\,\frac{\partial}{\partial y^{ri}}\Big|_p \right\} \end{aligned} \tag{2}$$

and

$$\begin{aligned} dz^{ri}\big|_p &= dx^{ri}\big|_p + \mathrm{i}\, dy^{ri}\big|_p \\ d\bar{z}^{ri}\big|_p &= dx^{ri}\big|_p - \mathrm{i}\, dy^{ri}\big|_p \end{aligned} \tag{3}$$

then, it has respectively obtained by

$$\left\{ \frac{\partial}{\partial z^{ri}}\bigg|_p , \frac{\partial}{\partial \bar{z}^{ri}}\bigg|_p \right\} \quad \text{and} \quad \left\{ dz^{ri}|_p, d\bar{z}^{ri}|_p \right\}$$

bases over complex number C of tangent space $T_p({}^kM)$ and cotangent space $T_p^*({}^kM)$ of ${}^kM$. Then endomorphism J_k with respect to base over complex number C of tangent space $T_p({}^kM)$ of ${}^kM$ is defined by

$$J_k(\frac{\partial}{\partial z^{ri}}) = \mathrm{i}\frac{\partial}{\partial z^{ri}}, \qquad J_k(\frac{\partial}{\partial \bar{z}^{ri}}) = -\mathrm{i}\frac{\partial}{\partial \bar{z}^{ri}} \tag{4}$$

If J_k^* is an endomorphism of the cotangent space $T_p^*({}^kM)$ such that $J_k^{*2} = -I$ on any point p of extended manifold ${}^kM$, then it is defined by

$$J_k^*(dz^{ri}) = \mathrm{i}dz^{ri}, \qquad J_k^*(d\bar{z}^{ri}) = -\mathrm{i}d\bar{z}^{ri}. \tag{5}$$

Let M be a differentiable manifold and $\mathcal{F}(M)$ be the set of functions on M. A complex valuable function is the element of complexification $(\mathcal{F}(M))^C$ of $\mathcal{F}(M)$. Now, we shall define differential of function f defined on M. Let f be a complex valuable function defined on any complex manifold M and $(z^{0i}, \bar{z}^{0i})$, $1 \le i \le m$ be extended complex coordinates of M. Therefore; the differential of f is complex 1-form given by equality

$$df = \frac{\partial f}{\partial z^{0i}} dz^{0i} + \frac{\partial f}{\partial \bar{z}^{0i}} d\bar{z}^{0i} \tag{6}$$

Let M be a differentiable manifold and $\chi(M)$ be the set of vector fields on M. A complex vector field is the element of complexification $(\chi(M))^C$ of $\chi(M)$. It is determined by

$$Z = Z^{0i}\frac{\partial}{\partial z^{0i}} + \bar{Z}^{0i}\frac{\partial}{\partial \bar{z}^{0i}} \tag{7}$$

complex vector field Z with respect to complex coordinate system $(z^{0i}, \bar{z}^{0i})$ such that $Z^{0i} \in (\mathcal{F}(M))^C$. Let M be a differentiable manifold and $(\chi(M))^C$ be the set of complex vector fields on M. A complex 1-form is the element of algebra dual $(\chi^*(M))^C$ of $(\chi(M))^C$. It is stated by

$$\omega = \omega_{0i} dz^{0i} + \bar{\omega}_{0i} d\bar{z}^{0i} \tag{8}$$

complex 1-form ω with respect to complex coordinate system $(z^{0i}, \bar{z}^{0i})$ such that $\omega_{0i} \in (\mathcal{F}(M))^C$.

REMARK 2.2. *Now then, in the other sections it will accept*

$$\mathcal{F}({}^kM), \quad \chi({}^kM), \quad \chi^*({}^kM),$$

instead of

$$(\mathcal{F}({}^kM))^C, \quad (\chi({}^kM))^C, \quad (\chi^*({}^kM))^C,$$

respectively.

2.2. Higher Order Lifts of Complex Functions

In this section, we extend definitions and properties about vertical and complete lifts of complex valuable functions defined on any complex manifold M to extended complex manifold ${}^kM$.

DEFINITION 2.3. *[6] Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Let $\widetilde{f}$ be a complex valuable function defined on ${}^{k-1}M$. Let us denote by $\tau_{{}^{k-1}M} : {}^kM \to {}^{k-1}M$ canonical projection and by*

$$\begin{array}{rlcl} v: & \mathcal{F}({}^{k-1}M) & \longrightarrow & \mathcal{F}({}^kM) \\ & \widetilde{f} & \longrightarrow & v(\widetilde{f}) = \widetilde{f}^v \end{array}$$

linear isomorphism. Then the vertical lift *of function $\widetilde{f} \in \mathcal{F}({}^{k-1}M)$ to ${}^kM$ is the function $\widetilde{f}^v \in \mathcal{F}({}^kM)$ given by*

$$\widetilde{f}^v = \widetilde{f} \circ \tau_{{}^{k-1}M}. \tag{9}$$

Now, let $f^{v^{k-1}}$ be vertical lift of a complex valuable function $f \in \mathcal{F}(M)$ to ${}^{k-1}M$. In (9), if $\widetilde{f} = f^{v^{k-1}}$, the *vertical lift* of function $f \in \mathcal{F}(M)$ to ${}^kM$ is the function $f^{v^k} \in \mathcal{F}({}^kM)$ given by equality

$$f^{v^k} = f \circ \tau_M \circ \tau_{{}^2M} \circ \ldots \circ \tau_{{}^{k-1}M}. \tag{10}$$

Now, similarly the differential of function $f \in \mathcal{F}(M)$ we shall give the following as the differential of $\widetilde{f} \in \mathcal{F}({}^{k-1}M)$. $(z^{ri}, \bar{z}^{ri})$, $0 \leq r \leq k-1$ be the extended complex coordinates of ${}^{k-1}M$. Then the differential of $\widetilde{f}$ is the complex 1-form given by equality

$$d\widetilde{f} = \frac{\partial \widetilde{f}}{\partial z^{ri}} dz^{ri} + \frac{\partial \widetilde{f}}{\partial \bar{z}^{ri}} d\bar{z}^{ri}. \tag{11}$$

Definition 2.4. *[6] Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Consider the linear isomorphism given by*

$$\begin{array}{rcl} \iota_k : \chi^*({}^{k-1}M) & \longrightarrow & \mathcal{F}({}^{k}M) \\ \iota_k(dz^{ri}) & = & \dot{z}^{ri} \\ \iota_k(d\bar{z}^{ri}) & = & \dot{\bar{z}}^{ri} \end{array} \tag{12}$$

such that $Sp\left\{dz^{ri}, d\bar{z}^{ri} : 0 \leq r \leq k-1\right\} = \chi^*({}^{k-1}M)$. Given by (11) the differential of a complex valuable function $\widetilde{f} \in \mathcal{F}({}^{k-1}M)$. Then the complete lift *of function $\widetilde{f} \in \mathcal{F}({}^{k-1}M)$ to ${}^{k}M$ is the function $\widetilde{f}^c \in \mathcal{F}({}^{k}M)$ determined by equality*

$$\widetilde{f}^c = \iota_k(d\widetilde{f}) = \dot{z}^{ri}(\frac{\partial \widetilde{f}}{\partial z^{ri}})^v + \dot{\bar{z}}^{ri}(\frac{\partial \widetilde{f}}{\partial \bar{z}^{ri}})^v. \tag{13}$$

Now, let $f^{c^{k-1}}$ be complete lift of a complex valuable function $f \in \mathcal{F}(M)$ to ${}^{k-1}M$. In (13), if $\widetilde{f} = f^{c^{k-1}}$, then the *complete lift* of function $f \in \mathcal{F}(M)$ to ${}^{k}M$ is the function $f^{c^k} \in \mathcal{F}({}^{k}M)$ given by

$$f^{c^k} = \dot{z}^{ri} \left(\frac{\partial f^{c^{k-1}}}{\partial z^{ri}} \right)^v + \dot{\bar{z}}^{ri} \left(\frac{\partial f^{c^{k-1}}}{\partial \bar{z}^{ri}} \right)^v. \tag{14}$$

The general properties of higher order vertical and complete lifts of complex valuable functions on complex manifold M are

$$\begin{array}{rrcl} i) & (f+g)^{v^r} & = & f^{v^r} + g^{v^r} \\ ii) & (f \cdot g)^{v^r} & = & f^{v^r} \cdot g^{v^r} \\ iii) & (f+g)^{c^r} & = & f^{c^r} + g^{c^r} \\ iv) & (f \cdot g)^{c^r} & = & \sum_{j=0}^{r} \binom{r}{j} f^{c^{r-j} v^j} \cdot g^{c^j v^{r-j}} \end{array}$$

$$v)\quad \left(\frac{\partial f}{\partial z^{0i}}\right)^{v^r} = \frac{\partial f^{c^r}}{\partial z^{ri}},\quad \left(\frac{\partial f}{\partial \bar{z}^{0i}}\right)^{v^r} = \frac{\partial f^{c^r}}{\partial \bar{z}^{ri}},$$
$$\left(\frac{\partial f}{\partial z^{0i}}\right)^{c^r} = \frac{\partial f^{c^r}}{\partial z^{0i}},\quad \left(\frac{\partial f}{\partial \bar{z}^{0i}}\right)^{c^r} = \frac{\partial f^{c^r}}{\partial \bar{z}^{0i}},$$

for all $f, g \in \mathcal{F}(M)$.

2.3. Higher Order Lifts of Complex Vector Fields

In this section, we derive definitions and propositions about vertical and complete lifts of complex vector fields defined on any complex manifold M to extended complex manifold ${}^kM$.

DEFINITION 2.5. *[6] Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Denote by $\widetilde{Z}$ a complex vector field and by $\widetilde{f}$ a complex valuable function defined on ${}^{k-1}M$. Then the* vertical lift *of $\widetilde{Z} \in \chi({}^{k-1}M)$ to ${}^kM$ is the complex vector field $\widetilde{Z}^v \in \chi({}^kM)$ given by equality*

$$\widetilde{Z}^v(\widetilde{f}^c) = (\widetilde{Z}\widetilde{f})^v. \tag{15}$$

Now, let $f^{c^{k-1}}$ and $Z^{v^{k-1}}$be respectively complete and vertical lifts of a complex valuable function $f \in \mathcal{F}(M)$ and a complex vector field $Z \in \chi(M)$ to ${}^{k-1}M$. In (15), if $\widetilde{f} = f^{c^{k-1}}$ and $\widetilde{Z} = Z^{v^{k-1}}$, then the *vertical lift* of $Z \in \chi(M)$ to ${}^kM$ is the complex vector field $Z^{v^k} \in \chi({}^kM)$ given by

$$Z^{v^k}(f^{c^k}) = (Zf)^{v^k}. \tag{16}$$

PROPOSITION 2.6. *[6] Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (7) the complex vector field $Z \in \chi(M)$. Then the vertical lift of $Z \in \chi(M)$ to ${}^kM$ is*

$$Z^{v^k} = (Z^{0i})^{v^k}\frac{\partial}{\partial z^{ki}} + (\bar{Z}^{0i})^{v^k}\frac{\partial}{\partial \bar{z}^{ki}}. \tag{17}$$

DEFINITION 2.7. *[6] Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Let us denote by $\widetilde{Z}$ a complex vector field and by $\widetilde{f}$ a complex valuable function defined on ${}^{k-1}M$. Then the* complete lift *of $\widetilde{Z} \in \chi({}^{k-1}M)$ to ${}^kM$ is the complex vector field $\widetilde{Z}^c \in \chi({}^kM)$ given by equality*

$$\widetilde{Z}^c(\widetilde{f}^c) = (\widetilde{Z}\widetilde{f})^c. \tag{18}$$

Now, let $f^{c^{k-1}}$ and $Z^{c^{k-1}}$ be respectively complete lifts of a complex valuable function $f \in \mathcal{F}(M)$ and a complex vector field $Z \in \chi(M)$ to ${}^{k-1}M$. In (18), if $\widetilde{f} = f^{c^{k-1}}$ and $\widetilde{Z} = Z^{c^{k-1}}$, then the complete lift of $Z \in \chi(M)$ to ${}^{k}M$ is the complex vector field $Z^{c^k} \in \chi({}^kM)$ given by

$$Z^{c^k}(f^{c^k}) = (Zf)^{c^k}. \tag{19}$$

Proposition 2.8. *[6] Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (7) the complex vector field $Z \in \chi(M)$. Then the complete lift of $Z \in \chi(M)$ to ${}^kM$ is*

$$Z^{c^k} = \binom{k}{r}(Z^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial z^{ki}} + \frac{\partial}{\partial \bar{z}^{ki}}\binom{k}{r}(\bar{Z}^{0i})^{v^{k-r}c^r}\frac{\partial}{\partial \bar{z}^{ki}}. \tag{20}$$

The higher order vertical and complete lifts of complex vector fields on any complex manifold M obey the following generic properties

$$\begin{array}{rrcl} i) & (Z+U)^{v^r} & = & Z^{v^r} + U^{v^r}, \\ & (Z+U)^{c^r} & = & Z^{c^r} + U^{c^r}, \\ ii) & (fZ)^{v^r} & = & f^{v^r}Z^{v^r}, \\ & (fZ)^{c^r} & = & \sum_{j=0}^{r}\binom{r}{j} f^{c^{r-j}v^j} Z^{c^j v^{r-j}}, \end{array}$$

$$\begin{array}{rrcllrcl} iii) & Z^{v^k}[f^{v^k}] & = & 0, & & Z^{c^k}[f^{c^k}] & = & (Zf)^{c^k}, \\ & Z^{c^k}[f^{v^k}] & = & (Zf)^{v^k}, & & Z^{v^k}[f^{c^k}] & = & (Zf)^{v^k}, \\ iv) & [Z^{v^k}, U^{v^k}] & = & 0, & & [Z^{c^k}, U^{c^k}] & = & [Z,U]^{c^k}, \\ & [Z^{v^k}, U^{c^k}] & = & [Z,U]^{v^k}, & & [Z^{c^k}, U^{v^k}] & = & [Z,U]^{v^k}, \\ v) & \left(\frac{\partial}{\partial z^{0i}}\right)^{v^r} & = & \frac{\partial}{\partial z^{ri}}, & & \left(\frac{\partial}{\partial \bar{z}^{0i}}\right)^{v^r} & = & \frac{\partial}{\partial \bar{z}^{ri}}, \\ & \left(\frac{\partial}{\partial z^{0i}}\right)^{c^r} & = & \frac{\partial}{\partial z^{0i}}, & & \left(\frac{\partial}{\partial \bar{z}^{0i}}\right)^{c^r} & = & \frac{\partial}{\partial \bar{z}^{0i}}, \\ & \chi(M) & = & Sp\left\{\frac{\partial}{\partial z^{0i}}, \frac{\partial}{\partial \bar{z}^{0i}}\right\}, & & & & \\ & \chi({}^kM) & = & Sp\left\{\frac{\partial}{\partial z^{ri}}, \frac{\partial}{\partial \bar{z}^{ri}}\right\}, & & & & \end{array}$$

for all $Z, U \in \chi(M)$ and $f \in \mathcal{F}(M)$.

2.4. Higher Order Lifts of Complex 1- Forms

In this section, we extend definitions and propositions about vertical and complete lifts of complex 1-forms defined on any complex manifold M to extended complex manifold ${}^kM$.

Definition 2.9. *[6] Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Denote by $\widetilde{\omega}$ a complex 1-form and by $\widetilde{Z}$ be a complex vector field defined on ${}^{k-1}M$. Then the* vertical lift *of complex 1-form $\widetilde{\omega} \in \chi^*({}^{k-1}M)$ to ${}^kM$ is complex 1-form $\widetilde{\omega}^v \in \chi^*({}^kM)$ given by equality*

$$\widetilde{\omega}^v(\widetilde{Z}^c) = (\widetilde{\omega}\widetilde{Z})^v. \tag{21}$$

Now, let $Z^{c^{k-1}}$ and $\omega^{v^{k-1}}$be respectively complete and vertical lifts of a complex vector field $Z \in \chi(M)$ and a complex 1-form $\omega \in \chi^*(M)$ to ${}^{k-1}M$. In (21), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{\omega} = \omega^{v^{k-1}}$, then the *vertical lift* of $\omega \in \chi^*(M)$ to ${}^kM$ is the complex 1-form $\omega^{v^k} \in \chi^*({}^kM)$ given by

$$\omega^{v^k}(Z^{c^k}) = (\omega Z)^{v^k}. \tag{22}$$

Proposition 2.10. *[6] Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (8) the complex 1-form $\omega \in \chi^*(M)$. Then the vertical lift of $\omega \in \chi^*(M)$ to ${}^kM$ is*

$$\omega^{v^k} = (\omega_{0i})^{v^k} dz^{0i} + (\bar{\omega}_{0i})^{v^k} d\bar{z}^{0i}. \tag{23}$$

Definition 2.11. *[6] Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Denote by $\widetilde{\omega}$ a complex 1-form and by $\widetilde{Z}$ a complex vector field defined on ${}^{k-1}M$. Then the* complete lift *of $\widetilde{\omega} \in \chi^*({}^{k-1}M)$ to ${}^kM$ is the complex 1-form $\widetilde{\omega}^c \in \chi^*({}^kM)$ given by*

$$\widetilde{\omega}^c(\widetilde{Z}^c) = (\widetilde{\omega}\widetilde{Z})^c. \tag{24}$$

Now, let $Z^{c^{k-1}}$ and $\omega^{c^{k-1}}$be respectively complete lifts of a complex vector field Z and a complex 1-form ω defined on M to ${}^{k-1}M$. In (24), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{\omega} = \omega^{c^{k-1}}$, then the *complete lift* of $\omega \in \chi^*(M)$ to ${}^kM$ is the complex 1-form $\omega^{c^k} \in \chi^*({}^kM)$ given by

$$\omega^{c^k}(Z^{c^k}) = (\omega Z)^{c^k}. \tag{25}$$

Proposition 2.12. *[6] Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (8) components structure of a complex 1-form $\omega \in \chi^*(M)$. Then the complete lift of $\omega \in \chi^*(M)$ to ${}^kM$ is*

$$\omega^{c^k} = (\omega_{0i})^{c^{k-r}v^r} dz^{ri} + (\bar{\omega}_{0i})^{c^{k-r}v^r} d\bar{z}^{ri}. \tag{26}$$

The properties of higher order vertical and complete lifts of complex 1-forms on complex manifold M are

$$\begin{array}{rrcl} i) & (\omega+\lambda)^{v^r} & = & \omega^{v^r} + \lambda^{v^r}, \\ & (\omega+\lambda)^{c^r} & = & \omega^{c^r} + \lambda^{c^r}, \\ ii) & (f\omega)^{v^r} & = & f^{v^r}\omega^{v^r}, \\ & (f\omega)^{c^r} & = & \sum_{j=0}^{r} \binom{r}{j} f^{c^{r-j}v^j} \omega^{c^j v^{r-j}}, \end{array}$$

$$\begin{array}{rrclrcl} iii) & (dz^{0i})^{c^r} & = & dz^{ri}, & (d\bar{z}^{0i})^{c^r} & = & d\bar{z}^{ri} \\ & (dz^{0i})^{v^r} & = & dz^{0i}, & (d\bar{z}^{0i})^{v^r} & = & d\bar{z}^{0i} \\ & \chi^*(M) & = & Sp\left\{dz^{0i}, d\bar{z}^{0i}\right\}, & \chi^*({}^kM) & = & Sp\left\{dz^{ri}, d\bar{z}^{ri}\right\} \end{array}$$

for all $\omega, \lambda \in \chi^*(M)$ and $f \in \mathcal{F}(M)$.

3. (r, s) Order Lifts of Complex Tensor Fields

In this section, using expressions determined the above we give the definitions and propositions about (r,s) order complete-vertical lifts of functions, vector fields and 1-forms on complex manifold M. We accept $0 \leq r, s \leq k$ and $r + s = k$.

3.1. (r, s) Order Lifts of Functions

DEFINITION 3.1. *Let f^{c^r} be r order complete lift of a complex valuable function $f \in \mathcal{F}(M)$ to ${}^kM$. Then if it is taken s order vertical lift of complex function $f^{c^r} \in \mathcal{F}({}^kM)$ to ${}^kM$, we call* complete-vertical lift of order (r,s) *of $f \in \mathcal{F}(M)$ to ${}^kM$ the function $f^{c^r v^s}$ determined by*

$$(f^{c^r})^{v^s} = f^{c^r v^s} = f^{c^r} \circ \tau_{{}^rM} \circ \ldots \circ \tau_{{}^{r+s-1}M}. \tag{27}$$

There exists chance property taking complete-vertical lift of functions. i.e., It means the same complete-vertical lifts of order (r, s) with complete-vertical lifts of order (s, r) of functions on complex manifold to extended complex manifolds.

3.2. (r, s) Order Lifts of Vector Fields

DEFINITION 3.2. *Let Z be a vector field on complex manifold M. Then the complete-vertical lift of order (r,s) of $Z \in \chi(M)$ to ${}^kM$ is the complex vector field $Z^{c^r v^s} \in \chi({}^kM)$ given by equality*

$$Z^{c^r v^s}(f^{c^k}) = (Zf)^{c^r v^s}. \tag{28}$$

PROPOSITION 3.3. *Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (7) the complex vector field $Z \in \chi(M)$. Then the complete-vertical lift of order (r, s) of $Z \in \chi(M)$ to ${}^kM$ is*

$$Z^{c^r v^s} : \begin{pmatrix} \binom{r}{k-t}(Z^{0i})^{v^{s+k-t}c^{t-s}} \\ \binom{r}{k-t}(\bar{Z}^{0i})^{v^{s+k-t}c^{t-s}} \end{pmatrix}, \quad 0 \leq t \leq k.$$

Proof. Let $Z^{c^r v^s} = Z^{ti}\frac{\partial}{\partial z^{ti}} + \bar{Z}^{ti}\frac{\partial}{\partial \bar{z}^{ti}}$ be a vector field on ${}^kM$ such that a complex coordinate system $(z^{ti}, \bar{z}^{ti})$ on a neighborhood ${}^kU$ of any point p of ${}^kM$. Let f^{c^k} be complete lift of order k of function f to extended complex manifold ${}^kM$. Then, from complete and vertical lift properties it is

$$Z^{c^r v^s}(f^{c^k}) = Z^{hi}\frac{\partial f^{c^k}}{\partial z^{ti}} + \bar{Z}^{hi}\frac{\partial f^{c^k}}{\partial \bar{z}^{ti}}$$

and

$$\begin{aligned} (Zf)^{c^r v^s} &= \left(Z^{0i}\frac{\partial f}{\partial z^{0i}} + \bar{Z}^{0i}\frac{\partial f}{\partial \bar{z}^{0i}}\right)^{c^r v^s} \\ &= \left\{\binom{r}{h}(Z^{0i})^{v^{s+h}c^{r-h}}\frac{\partial f^{c^k}}{\partial z^{k-hi}} + \right. \\ &\qquad \left. +\binom{r}{h}(\bar{Z}^{0i})^{v^{s+h}c^{r-h}}\frac{\partial f^{c^k}}{\partial \bar{z}^{k-hi}}\right\}. \end{aligned}$$

If the above two equalities are equaled according to (28), being $t = k - h$ from the following equalities

$$\frac{\partial f^{c^k}}{\partial z^{ti}} = \frac{\partial f^{c^k}}{\partial z^{k-hi}} \quad \text{and} \quad \frac{\partial f^{c^k}}{\partial \bar{z}^{ti}} = \frac{\partial f^{c^k}}{\partial \bar{z}^{k-hi}}$$

we have for $0 \leq t \leq k$:

$$Z^{ti} = \binom{r}{k-t} (Z^{0i})^{v^{s+k-t}c^{t-s}}, \quad \bar{Z}^{ri} = \binom{r}{k-t} (\bar{Z}^{0i})^{v^{s+k-t}c^{t-s}}.$$

Hence, the proof is finish. □

There exists commutative property taking complete-vertical lift of vector fields. It means the same complete-vertical lifts of order (r, s) with complete-vertical lifts of order (s, r) of vector fields on complex manifold to extended complex manifolds. The complete-vertical lifts of order (r, s) of complex vector fields on any complex manifold M obey the following generic property

$$(fZ)^{c^r v^s} = \binom{r}{h} f^{v^{s+h}c^{r-h}} Z^{c^h v^{k-h}}, \quad 0 \leq r, s \leq k \quad (r + s = k).$$

3.3. (r, s) Order Lifts of 1-Forms

DEFINITION 3.4. *Let Z be a vector field on complex manifold M. Then the* complete-vertical lift of order (r, s) *of $\omega \in \chi^*(M)$ to ${}^kM$ is the complex 1-form $\omega^{c^r v^s} \in \chi^*({}^kM)$ given by equality*

$$\omega^{c^r v^s}(Z^{c^k}) = (\omega Z)^{c^r v^s}. \tag{29}$$

PROPOSITION 3.5. *Let M be any complex manifold and ${}^kM$ its extended complex manifold of order k. Given by (8) the complex 1-form $\omega \in \chi^*(M)$. Then the complete-vertical lift of order (r, s) of $\omega \in \chi^*(M)$ to ${}^kM$ is*

$$\omega^{c^r v^s} : \left(\frac{\binom{r}{t}}{\binom{r}{t}} (\omega_{0i})^{v^{s+t}c^{r-t}}, \frac{\binom{r}{t}}{\binom{r}{t}} (\bar{\omega}_{0i})^{v^{s+t}c^{r-t}} \right), \quad 0 \leq t \leq k.$$

Proof. Let $\omega^{c^r v^s} = \omega_{ti} dz^{ti} + \bar{\omega}_{ti} d\bar{z}^{ti}$ be a 1-form on ${}^kM$ such that a complex coordinate system $(z^{ti}, \bar{z}^{ti})$ on a neighborhood ${}^kU$ of any point p of ${}^kM$. Let Z^{c^k} be complete lift of order k of vector field Z to extended complex manifold ${}^kM$. Then, from complete and vertical lift properties it is

$$\omega^{c^r v^s}(Z^{c^k}) = \left\{ \binom{r}{t} \omega_{ti} (Z^{0i})^{v^{k-t}c^t} + \binom{r}{t} \bar{\omega}_{ti} (\bar{Z}^{0i})^{v^{k-t}c^t} \right\}$$

and

$$\begin{aligned}(\omega Z)^{c^r v^s} &= (\omega_{0i} Z^{0i} + \bar{\omega}_{0i} \bar{Z}^{0i})^{c^r v^s} \\ &= \binom{r}{h} (\omega_{0i})^{v^{s+h} c^{r-h}} (Z^{0i})^{v^{k-h} c^h} + \\ &\qquad + \binom{r}{h} (\bar{\omega}_{0i})^{v^{s+h} c^{r-h}} (\bar{Z}^{0i})^{v^{k-h} c^h}.\end{aligned}$$

If the above two equalities are equaled according to (29), being $t = h$ from the following equalities

$$(Z^{0i})^{v^{k-t} c^t} = (Z^{0i})^{v^{k-h} c^h} \quad \text{and} \quad (\bar{Z}^{0i})^{v^{k-t} c^t} = (\bar{Z}^{0i})^{v^{k-h} c^h}$$

we have

$$\omega_{ti} = \frac{\binom{r}{t}}{\binom{k}{t}} (\omega_{0i})^{v^{s+t} c^{r-t}}, \quad \bar{\omega}_{ti} = \frac{\binom{r}{t}}{\binom{k}{t}} (\bar{\omega}_{0i})^{v^{s+t} c^{r-t}}, \quad \leq t \leq k$$

Hence, the proof is end. □

There exists commutative property taking complete-vertical lift of 1-forms. Clearly, It means the same complete-vertical lifts of order (r, s) with complete-vertical lifts of order (s, r) of 1-forms on complex manifold to extended complex manifolds.

The general property of complete-vertical lifts of order (r, s) of complex 1-forms on any complex manifold M is

$$(f\omega)^{c^r v^s} = \binom{r}{h} f^{v^{s+h} c^{r-h}} \omega^{c^h v^{k-h}}, \quad 0 \leq r, s \leq k \quad (r + s = k).$$

4. Higher Order Horizontal Lifts of Complex Tensor Fields

4.1. The Higher Order Horizontal Lifts of Complex Functions.

The *horizontal lift* of $f \in \Im_0^0(M) = \mathcal{F}(M)$ to ${}^kM$ is the function $f^{H^k} \in \mathcal{F}({}^kM)$ given by

$$f^{H^k} = f^{c^k} - \gamma(\nabla f^{c^{k-1}}), \quad (\gamma(\nabla f^{c^{k-1}}) = \nabla_\gamma f^{c^{k-1}}),$$

where ∇ is an affine linear connection on ${}^{k-1}M$ with local components $\Gamma^{ri}_{rj}, 1 \leq i,j \leq m, \nabla f^{c^{k-1}}$ is gradient of $f^{c^{k-1}}$ and γ is an operator given by

$$\gamma : \Im^r_s({}^{k-1}M) \longrightarrow \Im^r_{s-1}({}^kM).$$

Thus, it is $f^{H^k} = 0$ since

$$\nabla_\gamma f^{c^{k-1}} = \dot{z}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial z^{ri}}\right)^v + \dot{\bar{z}}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial \bar{z}^{ri}}\right)^v.$$

The higher order horizontal lifts of complex functions obey the generic properties

$$\begin{array}{lrcl} i) & (f \cdot g)^{H^k} & = & 0 \\ ii) & (f + g)^{H^k} & = & 0 \end{array}$$

for all $f, g \in \mathcal{F}(M)$.

4.2. The Higher Order Horizontal Lifts of Complex Vector Fields.

The *horizontal lift* of a vector field $Z \in \chi(M)$ to ${}^kM$ is the vector field $Z^{H^k} \in \chi({}^kM)$ given by

$$Z^{H^k} f^{v^k} = (Zf)^{v^k}.$$

Obviously, we have

$$Z^{H^k} = Z^{ri} D_{ri} + \bar{Z}^{ri} \bar{D}_{ri}$$

such that for $1 \leq i,j \leq m$:

$$D_{ri} = \frac{\partial}{\partial z^{ri}} - \Gamma^{ri}_{rj}\frac{\partial}{\partial z^{r+1i}} \quad \text{and} \quad \bar{D}_{ri} = \frac{\partial}{\partial \bar{z}^{ri}} - \bar{\Gamma}^{ri}_{rj}\frac{\partial}{\partial \bar{z}^{r+1i}}.$$

The higher order horizontal lifts of complex vector fields have the general properties

$$\begin{array}{lrcl} i) & (Z+W)^{H^k} & = & Z^{H^k} + W^{H^k}, \\ ii) & Z^{H^k}(f^{v^k}) & = & (Zf)^{v^k}, \\ iii) & \chi(U) & = & Sp\left\{\frac{\partial}{\partial z^{0i}}, \frac{\partial}{\partial \bar{z}^{0i}} : 1 \leq i \leq m\right\}, \\ & \left(\frac{\partial}{\partial z^{0i}}\right)^{H^k} & = & D_{ri}, \\ & \left(\frac{\partial}{\partial \bar{z}^{0i}}\right)^{H^k} & = & \bar{D}_{ri}, \\ & \chi({}^kU) & = & Sp\left\{\frac{\partial}{\partial z^{ri}}, \frac{\partial}{\partial \bar{z}^{ri}} : 0 \leq r \leq k, 1 \leq i \leq m\right\}, \end{array}$$

for all $f \in \mathcal{F}(M)$ and $Z, W \in \chi(M)$. The set of local vector fields

$$\left\{ D_{ri}, \bar{D}_{ri}, V_{ri} = \frac{\partial}{\partial z^{r+1i}}, \bar{V}_{ri} = \frac{\partial}{\partial \bar{z}^{r+1i}} \right\}$$

is called *adapted frame* to ∇.

4.3. The Higher Order Horizontal Lifts of Complex 1-Forms.

The *horizontal lift* of a 1-form $\omega \in \chi^*(M)$ to ${}^kM$ is the 1-form $\omega^{H^k} \in \chi^*({}^kM)$ given by

$$\omega^{H^k}(Z^{H^k}) = 0, \quad \omega^{H^k}(Z^{v^k}) = (\omega Z)^{v^k}.$$

If $\omega = \omega_{0i} dz^{0i} + \bar{\omega}_{0i} d\bar{z}^{0i}$ we obtain

$$\omega^{H^k} = \omega_{ri}\eta^{ri} + \bar{\omega}_{ri}\bar{\eta}^{ri},$$

such that

$$\eta^{ri} = \bar{d}z^{r+1i} + \Gamma^{ri}_{rj}\bar{d}z^{ri}, \quad \bar{\eta}^{ri} = \bar{d}\bar{z}^{r+1i} + \bar{\Gamma}^{ri}_{rj}\bar{d}\bar{z}^{ri}, \quad 1 \leq i, j \leq m.$$

The general properties of higher order horizontal lifts of complex 1-forms are

$$\begin{array}{lrcl}
i) & (\omega + \theta)^{H^k} & = & \omega^{H^k} + \theta^{H^k}, \\
ii) & \omega^{H^k}(Z^{H^k}) & = & 0, \\
 & \omega^{H^k}(Z^{v^k}) & = & (\omega Z)^{v^k}, \\
iii) & \chi^*(U) & = & Sp\left\{dz^{0i}, d\bar{z}^{0i} : 1 \leq i \leq m\right\}, \\
 & (dz^{0i})^{H^k} & = & \eta^{0i}, \\
 & (d\bar{z}^{0i})^{H^k} & = & \bar{\eta}^{0i}, \\
 & \chi^*({}^kU) & = & Sp\left\{dz^{ri}, d\bar{z}^{ri} : 0 \leq r \leq k, 1 \leq i \leq m\right\},
\end{array}$$

for all $Z \in \chi(M)$, $f \in \mathcal{F}(M)$, and $\omega, \theta \in \chi^*(M)$. The dual coframe

$$\left\{\theta^{ri} = dz^{ri}, \bar{\theta}^{\alpha ri} = d\bar{z}^{ri}, \eta^{ri}, \bar{\eta}^{ri}\right\}$$

is called *adapted coframe* to ∇.

References

[1] R.H. Bowman, *On differentiable extensions*, Tensor N.S. **21** (1970), 139–150.

[2] Ş. Civelek, *Higher order lifts on the extended vector bundles*, Ph.D. thesis, Gazi University, Ankara, 1993.

[3] Ş. Civelek, *The extended vector bundles*, Journal of Geometry **62** (1998), 40–47.

[4] C.T.J. Dodson and T. Poston, *Tensor geometry*, Pitman XIII, London, 1979, 598pp.

[5] S. Ishihara and K. Yano, *Tangent and cotangent bundles*, Marcel Dekker Inc., New York, 1973.

[6] M. Tekkoyun, *Higher order lifts of Euler-Lagrange and hamiltonian equations to the extended Kähler manifold*, Ph.D. thesis, Osmangazi University, Eskişehir, 2002.

[7] R.O. Wells, *Differential analysis on complex manifolds*, GTM, vol. 65, Springer-Verlag, New York, 1980.

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