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Higher Order Lifts of Complex Structures

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Summary. - *We study about lifts on extended complex manifold. More clearly, we will obtain higher order vertical and complete lifts of differentiable elements on complex manifold M to the extended complex manifold ${}^kM$.*

1. Introduction

Lift method has an important role in differentiable geometry. Because, using lift function it is possible to generalize to differentiable structures on any manifold the extended manifold. The structure of extended manifolds has been obtained, especially the canonical extended manifold ${}^kM$ of order k of the manifold M [1, 4]. Author obtained the higher order vertical and complete lifts of functions, vector fields and 1-forms on vector bundle to extended vector bundle [3]. In this study we define the extended complex manifold ${}^kM$ of a complex manifold M and obtain higher order vertical and complete lifts of functions, vector fields and 1-forms on M to ${}^kM$.

Throughout the paper, all mappings and manifolds are assumed to be differentiable of class C^∞ and the sum is taken over repeated indices.

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2. Extended Complex Manifold (ECM)

DEFINITION 2.1. *Let M $2m$-real dimensional manifold and ${}^kM$ its k order extended manifold. A tensor field J_k on ${}^kM$ is called an* extended almost complex structure *on ${}^kM$ if at every point p of ${}^kM$, J_k is endomorphism of the tangent space $T_p({}^kM)$ such that $(J_k)^2 = -I$. An extended manifold ${}^kM$ with fixed extended almost complex structure J_k is called an* extended almost complex manifold. *If $k = 0$, J_0 is called* almost complex structure; *a manifold ${}^0M = M$ with fixed almost complex structure J_0 is called an almost complex manifold.*

Let $(x^{ri}, y^{ri}), 0 \leq r \leq 1, 1 \leq i \leq m$ be a real coordinate system on a neighborhood ${}^kU$ of any point p of ${}^kM$. In this situation, it is respectively defined by

$$\left\{ \left(\frac{\partial}{\partial x^{ri}} \right) \bigg|_p , \left(\frac{\partial}{\partial y^{ri}} \right) \bigg|_p \right\} \quad \text{and} \quad \left\{ (dx^{ri})|_p, (dy^{ri})|_p \right\}$$

natural bases over R of tangent space $T_p({}^kM)$ and cotangent space $T_p^*({}^kM)$ of ${}^kM$.

DEFINITION 2.2. *Let ${}^kM$ be extended almost complex manifold with fixed extended almost complex structure J_k. ${}^kM$ is called* extended complex manifold *if there exists an open covering $\left\{ {}^kU \right\}$ of ${}^kM$ satisfying the following condition: there is a local coordinate system $(x^{ri}, y^{ri}), 0 \leq r \leq 1, 1 \leq i \leq m$ on each ${}^kU$ such that for each point of ${}^kU$,*

$$J_k(\frac{\partial}{\partial x^{ri}}) = \frac{\partial}{\partial y^{ri}}, \qquad J_k(\frac{\partial}{\partial y^{ri}}) = -\frac{\partial}{\partial x^{ri}}. \tag{1}$$

If $k = 0$, then a manifold ${}^0M = M$ with fixed almost complex structure J_0 is called complex manifold.

Let $z^{ri} = x^{ri} + \mathrm{i}\, y^{ri}, \mathrm{i} = \sqrt{-1}, 0 \leq r \leq k, 1 \leq i \leq m$, be an extended complex local coordinate system on a neighborhood ${}^kU$ of any point p of ${}^kM$. If it is defined by equalities

$$\begin{aligned}\left(\frac{\partial}{\partial z^{ri}}\right)\Big|_p &= \frac{1}{2}\left\{\left(\frac{\partial}{\partial x^{ri}}\right)\Big|_p - \mathrm{i}\left(\frac{\partial}{\partial y^{ri}}\right)\Big|_p\right\} \\ \left(\frac{\partial}{\partial \bar{z}^{ri}}\right)\Big|_p &= \frac{1}{2}\left\{\left(\frac{\partial}{\partial x^{ri}}\right)\Big|_p + \mathrm{i}\left(\frac{\partial}{\partial y^{ri}}\right)\Big|_p\right\}\end{aligned} \tag{2}$$

and

$$\begin{aligned}\left(dz^{ri}\right)\big|_p &= \left(dx^{ri}\right)\big|_p + \mathrm{i}\left(dy^{ri}\right)\big|_p \\ \left(d\bar{z}^{ri}\right)\big|_p &= \left(dx^{ri}\right)\big|_p - \mathrm{i}\left(dy^{ri}\right)\big|_p\end{aligned} \tag{3}$$

then, it has respectively obtained by

$$\left\{\left(\frac{\partial}{\partial z^{ri}}\right)\Big|_p, \left(\frac{\partial}{\partial \bar{z}^{ri}}\right)\Big|_p\right\} \quad \text{and} \quad \left\{(dz^{ri})|_p, (d\bar{z}^{ri})|_p\right\}$$

bases over complex number C of tangent space $T_p({}^kM)$ and cotangent space $T_p^*({}^kM)$ of ${}^kM$. Then endomorphism J_k with respect to base over complex number C of tangent space $T_p({}^kM)$ of ${}^kM$ is defined by

$$J_k(\frac{\partial}{\partial z^{ri}}) = \mathrm{i}\frac{\partial}{\partial z^{ri}}, \qquad J_k(\frac{\partial}{\partial \bar{z}^{ri}}) = -\mathrm{i}\frac{\partial}{\partial \bar{z}^{ri}}. \tag{4}$$

If J_k^* is an endomorphism of the cotangent space $T_p^*({}^kM)$ such that $J_k^{*2} = -I$ on any point p of extended manifold ${}^kM$, then it is defined by

$$J_k^*(dz^{ri}) = \mathrm{i}dz^{ri}, \qquad J_k^*(d\bar{z}^{ri}) = -\mathrm{i}d\bar{z}^{ri}. \tag{5}$$

Let M be a differentiable manifold and $\mathcal{F}(M)$ be the set of functions of class C^∞. A complex valuable function is the element of complexification $(\mathcal{F}(M))^C$ of $\mathcal{F}(M)$. Now, we shall define differential of function f defined on M. Let f be a complex valuable function defined on any complex manifold M and $(z^{0i}, \bar{z}^{0i})$, $1 \le i \le m$ be extended complex coordinates of M. Therefore; the differential of f is complex 1-form given by equality

$$df = \frac{\partial f}{\partial z^{0i}}dz^{0i} + \frac{\partial f}{\partial \bar{z}^{0i}}d\bar{z}^{0i}. \tag{6}$$

Let M be a differentiable manifold and $\chi(M)$ be the set of vector fields on M. A complex vector field is the element of complexification $(\chi(M))^C$ of $\chi(M)$. It is determined by

$$Z = Z^{0i}\frac{\partial}{\partial z^{0i}} + \bar{Z}^{0i}\frac{\partial}{\partial \bar{z}^{0i}} \tag{7}$$

complex vector field Z with respect to complex coordinate system $(z^{0i}, \bar{z}^{0i})$ such that $Z^{0i} \in (\mathcal{F}(M))^C$. Let M be a differentiable manifold and $(\chi(M))^C$ be the set of complex vector fields on M. A complex 1-form is the element of algebra dual $(\chi^*(M))^C$ of $(\chi(M))^C$. It is stated by

$$\omega = \omega_{0i}dz^{0i} + \bar{\omega}_{0i}d\bar{z}^{0i} \tag{8}$$

complex 1-form ω with respect to complex coordinate system $(z^{0i}, \bar{z}^{0i})$ such that $\omega_{0i} \in (\mathcal{F}(M))^C$.

REMARK 2.3. *Now then, in the other sections it will accept*

$$\mathcal{F}(^kM), \quad \chi(^kM), \quad \chi^*(^kM),$$

instead of

$$(\mathcal{F}(^kM))^C, \quad (\chi(^kM))^C, \quad (\chi^*(^kM))^C,$$

respectively.

3. Higher Order Lifts of Complex Functions to ECM

In this section, we extend definitions and properties about vertical and complete lifts of complex valuable functions defined on any complex manifold M to extended complex manifold ^kM.

DEFINITION 3.1. *Let M be any complex manifold and ^{k-1}M its $(k-1)$ order extended complex manifold. Let $\widetilde{f}$ be a complex valuable function defined on ^{k-1}M. Let us denote by $\tau_{^{k-1}M} : {}^kM \to {}^{k-1}M$ canonical projection and by*

$$\begin{array}{rlcl} v: & \mathcal{F}(^{k-1}M) & \longrightarrow & \mathcal{F}(^kM) \\ & \widetilde{f} & \longrightarrow & v(\widetilde{f}) = \widetilde{f}^v \end{array}$$

linear isomorphism. Then the vertical lift *of function $\widetilde{f} \in \mathcal{F}(^{k-1}M)$ to ^kM is the function $\widetilde{f}^v \in \mathcal{F}(^kM)$ given by*

$$\widetilde{f}^v = \widetilde{f} \circ \tau_{^{k-1}M}. \tag{9}$$

Now, let $f^{v^{k-1}}$ be vertical lift of a complex valuable function $f \in \mathcal{F}(M)$ to ${}^{k-1}M$. In (9), if $\widetilde{f} = f^{v^{k-1}}$, the *vertical lift* of function $f \in \mathcal{F}(M)$ to ${}^{k}M$ is the function $f^{v^k} \in \mathcal{F}({}^{k}M)$ given by equality

$$f^{v^k} = f \circ \tau_M \circ \tau_{{}^2M} \circ \ldots \circ \tau_{{}^{k-1}M}. \tag{10}$$

Now, similarly the differential of function $f \in \mathcal{F}(M)$ we shall give the following as the differential of $\widetilde{f} \in \mathcal{F}({}^{k-1}M)$. $(z^{ri}, \bar{z}^{ri})$, $0 \leq r \leq k-1$ be the extended complex coordinates of ${}^{k-1}M$. Then the differential of $\widetilde{f}$ is the complex 1-form given by equality

$$d\widetilde{f} = \frac{\partial \widetilde{f}}{\partial z^{ri}} dz^{ri} + \frac{\partial \widetilde{f}}{\partial \bar{z}^{ri}} d\bar{z}^{ri}. \tag{11}$$

Definition 3.2. *Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Consider the linear isomorphism given by*

$$\begin{array}{rcl} \iota_k : \chi^*({}^{k-1}M) & \longrightarrow & \mathcal{F}({}^{k}M) \\ \iota_k(dz^{ri}) & = & \dot{z}^{ri} \\ \iota_k(d\bar{z}^{ri}) & = & \dot{\bar{z}}^{ri} \end{array} \tag{12}$$

such that $Sp\left\{dz^{ri}, d\bar{z}^{ri} : 0 \leq r \leq k-1, 1 \leq i \leq m\right\} = \chi^*({}^{k-1}M)$. Given by (11) the differential of a complex valuable function $\widetilde{f} \in \mathcal{F}({}^{k-1}M)$. Then the complete lift *of function $\widetilde{f} \in \mathcal{F}({}^{k-1}M)$ to ${}^{k}M$ is the function $\widetilde{f}^c \in \mathcal{F}({}^{k}M)$ determined by equality*

$$\widetilde{f}^c = \iota_k(d\widetilde{f}) = \dot{z}^{ri}(\frac{\partial \widetilde{f}}{\partial z^{ri}})^v + \dot{\bar{z}}^{ri}(\frac{\partial \widetilde{f}}{\partial \bar{z}^{ri}})^v. \tag{13}$$

Now, let $f^{c^{k-1}}$ be complete lift of a complex valuable function $f \in \mathcal{F}(M)$ to ${}^{k-1}M$. In (13), if $\widetilde{f} = f^{c^{k-1}}$, then the *complete lift* of function $f \in \mathcal{F}(M)$ to ${}^{k}M$ is the function $f^{c^k} \in \mathcal{F}({}^{k}M)$ given by

$$f^{c^k} = \dot{z}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial z^{ri}}\right)^v + \dot{\bar{z}}^{ri}\left(\frac{\partial f^{c^{k-1}}}{\partial \bar{z}^{ri}}\right)^v. \tag{14}$$

The general properties of higher order vertical and complete lifts of complex valuable functions on complex manifold M are

$$
\begin{array}{rrcl}
i) & (f+g)^{v^r} & = & f^{v^r} + g^{v^r} \\
ii) & (f\cdot g)^{v^r} & = & f^{v^r} \cdot g^{v^r} \\
iii) & (f+g)^{c^r} & = & f^{c^r} + g^{c^r} \\
iv) & (f\cdot g)^{c^r} & = & \sum_{j=0}^{r} \binom{r}{j} f^{c^{r-j}v^j} \cdot g^{c^j v^{r-j}}
\end{array}
$$

$$
\begin{array}{rcccccc}
v) & \left(\frac{\partial f}{\partial z^{0i}}\right)^{v^r} & = & \frac{\partial f^{c^r}}{\partial z^{ri}}, & \left(\frac{\partial f}{\partial \bar{z}^{0i}}\right)^{v^r} & = & \frac{\partial f^{c^r}}{\partial \bar{z}^{ri}}, \\
vi) & \left(\frac{\partial f}{\partial z^{0i}}\right)^{c^r} & = & \frac{\partial f^{c^r}}{\partial z^{0i}}, & \left(\frac{\partial f}{\partial \bar{z}^{0i}}\right)^{c^r} & = & \frac{\partial f^{c^r}}{\partial \bar{z}^{0i}},
\end{array}
$$

for all $f, g \in \mathcal{F}(M)$.

4. Higher Order Lifts of Complex Vector Fields to ECM

In this section, we derive definitions and propositions about vertical and complete lifts of complex vector fields defined on any complex manifold M to extended complex manifold ${}^kM$.

DEFINITION 4.1. *Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Denote by $\widetilde{Z}$ a complex vector field and by $\widetilde{f}$ a complex valuable function defined on ${}^{k-1}M$. Then the* vertical lift *of $\widetilde{Z} \in \chi({}^{k-1}M)$ to ${}^kM$ is the complex vector field $\widetilde{Z}^v \in \chi({}^kM)$ given by equality*

$$\widetilde{Z}^v(\widetilde{f}^c) = (\widetilde{Z}\widetilde{f})^v. \tag{15}$$

Now, let $f^{c^{k-1}}$ and $Z^{v^{k-1}}$be respectively complete and vertical lifts of a complex valuable function $f \in \mathcal{F}(M)$ and a complex vector field $Z \in \chi(M)$ to ${}^{k-1}M$. In (15), if $\widetilde{f} = f^{c^{k-1}}$ and $\widetilde{Z} = Z^{v^{k-1}}$, then the *vertical lift* of $Z \in \chi(M)$ to ${}^kM$ is the complex vector field $Z^{v^k} \in \chi({}^kM)$ given by

$$Z^{v^k}(f^{c^k}) = (Zf)^{v^k}. \tag{16}$$

PROPOSITION 4.2. *Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (7) the complex vector* *field $Z \in \chi(M)$. Then the vertical lift of $Z \in \chi(M)$ to ${}^kM$ is*

$$Z^{v^k} : \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ (Z^{0i})^{v^k} \\ (\overline{Z}^{0i})^{v^k} \end{pmatrix}, \quad 1 \leq i \leq m. \tag{17}$$

Proof. Let $Z^{v^k} = Z^{ri}\frac{\partial}{\partial z^{ri}} + \bar{Z}^{ri}\frac{\partial}{\partial \bar{z}^{ri}}$ be a vector field on ${}^kM$ such that a complex coordinate system $(z^{ri}, \bar{z}^{ri})$ on a neighborhood ${}^kU$ of any point p of ${}^kM$. Let f^{c^k} be complete lift of function f to extended complex manifold ${}^kM$. Then, from vertical lift properties it is

$$Z^{v^k}(f^{c^k}) = Z^{ri}\frac{\partial f^{c^k}}{\partial z^{ri}} + \bar{Z}^{ri}\frac{\partial f^{c^k}}{\partial \bar{z}^{ri}}$$

and

$$\begin{aligned} (Zf)^{v^k} &= \left(Z^{0i}\frac{\partial f}{\partial z^{0i}} + \bar{Z}^{0i}\frac{\partial f}{\partial \bar{z}^{0i}}\right)^{v^k} \\ &= (Z^{0i})^{v^k}\frac{\partial f^{c^k}}{\partial z^{ki}} + (\bar{Z}^{0i})^{v^k}\frac{\partial f^{c^k}}{\partial \bar{z}^{ki}}. \end{aligned}$$

If the above two equalities are equaled according to (16), we have

$$\begin{array}{lllllll} Z^{ri} & = & 0, & \bar{Z}^{ri} & = & 0, & 0 \leq r \leq k-1 \\ Z^{ki} & = & (Z^{0i})^{v^k}, & \bar{Z}^{ki} & = & (\bar{Z}^{0i})^{v^k}. & \end{array}$$

Hence, the proof is finish. □

DEFINITION 4.3. *Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Let us denote by $\widetilde{Z}$ a complex vector field and by $\widetilde{f}$ a complex valuable function defined on ${}^{k-1}M$. Then the* complete lift *of $\widetilde{Z} \in \chi({}^{k-1}M)$ to ${}^kM$ is the complex vector field $\widetilde{Z}^c \in \chi({}^kM)$ given by equality*

$$\widetilde{Z}^c(\widetilde{f}^c) = (\widetilde{Z}\widetilde{f})^c. \tag{18}$$

Now, let $f^{c^{k-1}}$ and $Z^{c^{k-1}}$ be respectively complete lifts of a complex valuable function $f \in \mathcal{F}(M)$ and a complex vector field $Z \in \chi(M)$ to ${}^{k-1}M$. In (18), if $\widetilde{f} = f^{c^{k-1}}$ and $\widetilde{Z} = Z^{c^{k-1}}$, then the *complete lift* of $Z \in \chi(M)$ to ${}^{k}M$ is the complex vector field $Z^{c^k} \in \chi({}^kM)$ given by

$$Z^{c^k}(f^{c^k}) = (Zf)^{c^k}. \tag{19}$$

PROPOSITION 4.4. *Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (7) the complex vector field $Z \in \chi(M)$. Then the complete lift of $Z \in \chi(M)$ to ${}^kM$ is*

$$Z^{c^k} : \begin{pmatrix} \binom{k}{0}(Z^{0i})^{v^k c^0} \\ \binom{k}{0}(\overline{Z}^{0i})^{v^k c^0} \\ . \\ . \\ . \\ \binom{k}{k}(Z^{0i})^{v^0 c^k} \\ \binom{k}{k}(\overline{Z}^{0i})^{v^0 c^k} \end{pmatrix}, \quad 1 \leq i \leq m. \tag{20}$$

Proof. Similar to previous proposition. Shortly, let

$$Z^{c^k} = \binom{k}{r} Z^{ri}\frac{\partial}{\partial z^{ri}} + \binom{k}{r}\bar{Z}^{ri}\frac{\partial}{\partial \bar{z}^{ri}}.$$

If Z^{c^k}, (7), (14) and the function $f \in \mathcal{F}(M)$ is calculated with respect to (19), the proof is end. □

The higher order vertical and complete lifts of complex vector fields on any complex manifold M obey the following generic properties

$$\begin{array}{rrcl} i) & (Z+U)^{v^r} & = & Z^{v^r} + U^{v^r}, \\ & (Z+U)^{c^r} & = & Z^{c^r} + U^{c^r}, \\ ii) & (fZ)^{v^r} & = & f^{v^r} Z^{v^r}, \\ & (fZ)^{c^r} & = & \sum_{j=0}^{r}\binom{r}{j} f^{c^{r-j}v^j} Z^{c^j v^{r-j}}, \end{array}$$

$$\begin{array}{rrcl rcl} iii) & Z^{v^k}[f^{v^k}] & = & 0, & Z^{c^k}[f^{c^k}] & = & (Zf)^{c^k}, \\ & Z^{c^k}[f^{v^k}] & = & (Zf)^{v^k}, & Z^{v^k}[f^{c^k}] & = & (Zf)^{v^k}, \\ iv) & [Z^{v^k}, U^{v^k}] & = & 0, & [Z^{c^k}, U^{c^k}] & = & [Z,U]^{c^k}, \\ & [Z^{v^k}, U^{c^k}] & = & [Z,U]^{v^k}, & [Z^{c^k}, U^{v^k}] & = & [Z,U]^{v^k}, \\ v) & \left(\frac{\partial}{\partial z^{0i}}\right)^{v^r} & = & \frac{\partial}{\partial z^{ri}}, & \left(\frac{\partial}{\partial \bar{z}^{0i}}\right)^{v^r} & = & \frac{\partial}{\partial \bar{z}^{ri}}, \\ & \left(\frac{\partial}{\partial z^{0i}}\right)^{c^r} & = & \frac{\partial}{\partial z^{0i}}, & \left(\frac{\partial}{\partial \bar{z}^{0i}}\right)^{c^r} & = & \frac{\partial}{\partial \bar{z}^{0i}}, \\ & \chi(M) & = & Sp\left\{\frac{\partial}{\partial z^{0i}}, \frac{\partial}{\partial \bar{z}^{0i}}\right\}, & & & \\ & \chi({}^kM) & = & Sp\left\{\frac{\partial}{\partial z^{ri}}, \frac{\partial}{\partial \bar{z}^{ri}}\right\}, & & & \end{array}$$

for all $Z, U \in \chi(M)$ and $f \in \mathcal{F}(M), 0 \leq r \leq k, 1 \leq i \leq m$.

5. Higher Order Lifts of Complex 1-Forms to ECM

In this section, we extend definitions and propositions about vertical and complete lifts of complex 1-forms defined on any complex manifold M to extended complex manifold ${}^kM$.

DEFINITION 5.1. *Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Denote by $\widetilde{\omega}$ a complex 1-form and by $\widetilde{Z}$ be a complex vector field defined on ${}^{k-1}M$. Then the* vertical lift *of complex 1-form $\widetilde{\omega} \in \chi^*({}^{k-1}M)$ to ${}^kM$ is complex 1-form $\widetilde{\omega}^v \in \chi^*({}^kM)$ given by equality*

$$\widetilde{\omega}^v(\widetilde{Z}^c) = (\widetilde{\omega}\widetilde{Z})^v. \tag{21}$$

Now, let $Z^{c^{k-1}}$ and $\omega^{v^{k-1}}$be respectively complete and vertical lifts of a complex vector field $Z \in \chi(M)$ and a complex 1-form $\omega \in \chi^*(M)$ to ${}^{k-1}M$. In (21), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{\omega} = \omega^{v^{k-1}}$, then the *vertical lift* of $\omega \in \chi^*(M)$ to ${}^kM$ is the complex 1-form $\omega^{v^k} \in \chi^*({}^kM)$ given by

$$\omega^{v^k}(Z^{c^k}) = (\omega Z)^{v^k}. \tag{22}$$

PROPOSITION 5.2. *Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (8) the complex 1-form $\omega \in \chi^*(M)$. Then the vertical lift of $\omega \in \chi^*(M)$ to ${}^kM$ is*

$$\omega^{v^k} : ((\omega_{0i})^{v^k}, (\bar{\omega}_{0i})^{v^k}, 0, \ldots, 0), \quad 1 \leq i \leq m. \tag{23}$$

Proof. Let $\omega^{v^k} = \omega_{ri}dz^{ri} + \bar{\omega}_{ri}d\bar{z}^{ri}$ be a 1-form on ${}^kM$ such that a complex coordinate system $(z^{ri}, \bar{z}^{ri})$ on a neighborhood ${}^kU$ of any point p of ${}^kM$. Let Z^{c^k} be complete lift of a vector field Z to extended complex manifold ${}^kM$. Then, from vertical lift properties it is

$$\begin{aligned} \omega^{v^k}(Z^{c^k}) &= (\omega_{ri}dz^{ri} + \bar{\omega}_{ri}d\bar{z}^{ri})(Z^{c^k}) \\ &= \omega_{ri}\binom{k}{r}(Z^{0i})^{v^{k-r}c^r} + \bar{\omega}_{ri}\binom{k}{r}(\bar{Z}^{0i})^{v^{k-r}c^r} \end{aligned}$$

and

$$\begin{aligned} (\omega Z)^{v^k} &= (\omega_{0i}Z^{0i} + \bar{\omega}_{0i}\bar{Z}^{0i})^{v^k} \\ &= (\omega_{0i})^{v^k}(Z^{0i})^{v^k} + (\bar{\omega}_{0i})^{v^k}(\bar{Z}^{0i})^{v^k}. \end{aligned}$$

If the above two equalities are equaled in respect of (22), we have

$$\begin{aligned} \omega_{ri} &= 0, & \bar{\omega}_{ri} &= 0, & 1 \le r \le k \\ \omega_{0i} &= (\omega_{0i})^{v^k}, & \bar{\omega}_{0i} &= (\bar{\omega}_{0i})^{v^k}. \end{aligned}$$

Hence, the proof is finish. □

Definition 5.3. *Let M be any complex manifold and ${}^{k-1}M$ its $(k-1)$ order extended complex manifold. Denote by $\widetilde{\omega}$ a complex 1-form and by $\widetilde{Z}$ a complex vector field defined on ${}^{k-1}M$. Then the* complete lift *of $\widetilde{\omega} \in \chi^*({}^{k-1}M)$ to ${}^kM$ is the complex 1-form $\widetilde{\omega}^c \in \chi^*({}^kM)$ given by*

$$\widetilde{\omega}^c(\widetilde{Z}^c) = (\widetilde{\omega}\widetilde{Z})^c. \tag{24}$$

Now, let $Z^{c^{k-1}}$ and $\omega^{c^{k-1}}$ be respectively complete lifts of a complex vector field Z and a complex 1-form ω defined on M to ${}^{k-1}M$. In (24), if $\widetilde{Z} = Z^{c^{k-1}}$ and $\widetilde{\omega} = \omega^{c^{k-1}}$, then the *complete lift* of $\omega \in \chi^*(M)$ to ${}^kM$ is the complex 1-form $\omega^{c^k} \in \chi^*({}^kM)$ given by

$$\omega^{c^k}(Z^{c^k}) = (\omega Z)^{c^k}. \tag{25}$$

Proposition 5.4. *Let M be any complex manifold and ${}^kM$ its k order extended complex manifold. Given by (8) components structure of a complex 1-form $\omega \in \chi^*(M)$. Then the complete lift of $\omega \in \chi^*(M)$ to ${}^kM$ is*

$$\omega^{c^k} : ((\omega_{0i})^{c^k v^0}, (\overline{\omega}_{0i})^{c^k v^0}, ..., (\omega_{0i})^{c^0 v^k}, (\overline{\omega}_{0i})^{c^0 v^k}),\ 1 \le i \le m. \tag{26}$$

Proof. Similar to the previous proposition. Briefly, given

$$\omega^{c^k} = Z_{ri}dz^{ri} + \bar{Z}_{ri}d\bar{z}^{ri}.$$

If ω^{c^k}, (7), (8) and (20) is equalized with respect to (25), the proof is complete. □

The properties of higher order vertical and complete lifts of complex 1-forms on complex manifold M are

$$\begin{array}{rrcl} i) & (\omega+\lambda)^{v^r} & = & \omega^{v^r} + \lambda^{v^r}, \\ & (\omega+\lambda)^{c^r} & = & \omega^{c^r} + \lambda^{c^r}, \\ ii) & (f\omega)^{v^r} & = & f^{v^r}\omega^{v^r}, \\ & (f\omega)^{c^r} & = & \sum_{j=0}^{r}\binom{r}{j} f^{c^{r-j}v^j}\omega^{c^j v^{r-j}}, \end{array}$$

$$\begin{array}{rrclrcl} iii) & (dz^{0i})^{c^r} & = & dz^{ri}, & (d\bar{z}^{0i})^{c^r} & = & d\bar{z}^{ri} \\ & (dz^{0i})^{v^r} & = & dz^{0i}, & (d\bar{z}^{0i})^{v^r} & = & d\bar{z}^{0i} \\ & \chi^*(M) & = & Sp\left\{dz^{0i}, d\bar{z}^{0i}\right\}, & \chi^*({}^kM) & = & Sp\left\{dz^{ri}, d\bar{z}^{ri}\right\} \end{array}$$

for all $\omega, \lambda \in \chi^*(M)$ and $f \in \mathcal{F}(M), 0 \le r \le k, 1 \le i \le m$.

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