Rend. Istit. Mat. Univ. Trieste
Vol. XXXVI, 27–47 (2004)

Quantum Connections and Quantum Fields

Daniel Canarutto [(*)]

Summary. - *Recent developments in the geometry of distributional bundles yield a natural way of describing quantum fields on a curved spacetime background.*

1. Introduction

This work is addressed mainly to mathematicians and mathematical physicists having a background in differential geometry, who wish to understand fundamental notions of quantum field theory on curved spacetime in a rigorous geometric framework. The attention here is focused on the general notion of a quantum field rather than on particular instances, though in the last section a basic example is given of how the described ideas can be put to work in practice. Note, however, that other pieces need to be added in order to obtain a complete geometrical QFT framework; above all, the still open question of the description of particle interactions is left untouched here. I plan to address at least some of the remaining pieces in forthcoming papers.

Basically, quantum fields are certain geometric structures naturally arising on quantum bundles; these, on turn, are functional bundles derived from the 'classical' finite-dimensional bundles where the corresponding classical field theory is formulated. The method

---

[(*)] Author's address: Daniel Canarutto, Dipartimento di Matematica Applicata "G. Sansone", via S. Marta 3, 50139 Firenze, Italia, e-mail: daniel.canarutto@unifi.it
Keywords: Quantum bundles, quantum connections, quantum fields.
AMS Subject Classification: 53C05, 81T20.

used for studying the geometry of functional bundles is based on the notion of smoothness introduced by Frölicher (or *F-smoothness*) and studied by him and several authors [6, 7, 9, 10]. This method was enlarged by myself [3, 4] to include a treatment of *distributional bundles*, namely bundles over classical (i.e. finite-dimensional, Hausdorff) manifolds, whose fibres are distributional spaces.

In order to recover the usual notion of a quantum field, namely that of a distribution (on configuration or phase space) valued into a space of operators and obeying a classical field equation, one has to introduce the notion of a *quantum connection*. One already finds a notion of quantum connection in geometric formulations of Quantum Mechanics, in particular in the standard geometric quantization approach [12], as well as in developments such as the 'covariant quantization' approach [5, 8]. There, the term under consideration refers to a connection, on a finite dimensional ('classical') bundle, related to the PDE obeyed by wave functions. This equation, however, can be reinterpreted as the equation of motion for 'quantum histories', sections of a 'Hilbert functional bundle' over time describing the evolution of a quantum state; on turn, one can view such sections as covariantly constant relatively to a connection on the functional bundle.

Now the method of F-smoothness allows to introduce and study, in the context of functional bundles, several usual notions of differential geometry. In the distributional case, a connection in the underlying finite-dimensional structure determines a distributional connection, while other interesting distributional connections do not arise from classical ones. In this context, a quantum connection on a distributional bundle $\boldsymbol{\mathcal{V}} \to \boldsymbol{M}$ (where $\boldsymbol{M}$ is the classical spacetime manifold) is defined to be an F-smooth linear connection such that horizontal transport along any timelike curve determines continuous isomorphisms among the fibres. Then, a geometric formulation of the basic notions of quantum field theory can be achieved by starting from certain classical structures, which naturally yield quantum bundles and various connections on them. The usual notion of a quantum field, in the form of a section of the quantum state bundle valued into a space of operators, can be recovered from the above said quantum structures through certain bundle splittings; so called *free fields* and *interpolating fields* are also recovered.

The plan of the paper can be summarized as follows: after a short review of the F-smooth geometry of distributional bundles, I introduce quantum bundles, quantum connections and briefly discuss the traditional quantum 'pictures' from this point of view. Next, the geometrical structures corresponding to the objects traditionally called 'quantum fields' are introduced, and their main properties discussed. Finally, I describe a possible practical implementation of the above ideas on a curved spacetime, by using the notion of 'detector'; a new characterization of the connections naturally induced on quantum phase bundles is also provided.

2. F-smooth geometry on distributional bundles

For details about the ideas reviewed in this section, see [1, 3, 4].

Let $\mathsf{p} : \boldsymbol{Y} \to \underline{\boldsymbol{Y}}$ be a real or complex classical vector bundle, namely a finite-dimensional vector bundle over the Hausdorff paracompact smooth real manifold $\underline{\boldsymbol{Y}}$. Moreover assume that $\underline{\boldsymbol{Y}}$ is oriented, let $n := \dim \underline{\boldsymbol{Y}}$, and consider the positive component $\mathbb{V}^*\underline{\boldsymbol{Y}} := (\wedge^n \mathrm{T}^*\underline{\boldsymbol{Y}})^+ \to \underline{\boldsymbol{Y}}$, called the bundle of *positive densities* on $\underline{\boldsymbol{Y}}$.

Let $\boldsymbol{\mathcal{Y}}_\circ \equiv \boldsymbol{\mathcal{D}}_\circ(\underline{\boldsymbol{Y}}, \mathbb{V}^*\underline{\boldsymbol{Y}} \otimes_{\underline{\boldsymbol{Y}}} \boldsymbol{Y}^*)$ be the vector space of all 'test sections', namely smooth sections $\underline{\boldsymbol{Y}} \to \mathbb{V}^*\underline{\boldsymbol{Y}} \otimes_{\underline{\boldsymbol{Y}}} \boldsymbol{Y}^*$ which have compact support. A topology on this space can be introduced by a standard procedure [11]; its topological dual will be denoted as $\boldsymbol{\mathcal{Y}} \equiv \boldsymbol{\mathcal{D}}(\underline{\boldsymbol{Y}}, \boldsymbol{Y})$ and called the space of *generalized sections*, or *distribution-sections* of the given classical bundle. Some particular cases of generalized sections are that of r*-currents* ($\boldsymbol{Y} \equiv \wedge^r \mathrm{T}^*\underline{\boldsymbol{Y}}$, $r \in \mathbb{N}$) and that of *half-densities* ($\boldsymbol{Y} \equiv (\mathbb{V}^*\underline{\boldsymbol{Y}})^{1/2} \equiv \mathbb{V}^{-1/2}\underline{\boldsymbol{Y}}$).

A curve $\alpha : \mathbb{R} \to \boldsymbol{\mathcal{Y}}$ is said to be *F-smooth* if the map

$$\langle \alpha, u \rangle : \mathbb{R} \to \mathbb{C} : t \mapsto \langle \alpha(t), u \rangle$$

is smooth for every $u \in \boldsymbol{\mathcal{Y}}_\circ$. Accordingly, a function $\phi : \boldsymbol{\mathcal{Y}} \to \mathbb{C}$ is called F-smooth if $\phi \circ \alpha : \mathbb{R} \to \mathbb{C}$ is smooth for all F-smooth curve α. The general notion of F-smoothness, for any mapping involving distributional spaces, is introduced in terms of the standard smoothness of all maps, between finite-dimensional manifolds, which can be defined through compositions with F-smooth curves and functions. Moreover, it can be proved that a function $f:\boldsymbol{M}\to\mathbb{R}$, where $\boldsymbol{M}$ is a classical manifold, is smooth (in the standard sense) iff the composition $f\circ c$ is a smooth function of one variable for any smooth curve $c:\mathbb{R}\to\boldsymbol{M}$. Thus one has a unique notion of smoothness based on smooth curves, including both classical manifolds and distributional spaces.

In the basic classical geometric setting underlying distributional bundles one considers a classical 2-fibred bundle

$$\boldsymbol{V}\xrightarrow{\;\mathsf{q}\;}\boldsymbol{E}\xrightarrow{\;\underline{\mathsf{q}}\;}\boldsymbol{B}\;,$$

where $\mathsf{q}:\boldsymbol{V}\to\boldsymbol{E}$ is a vector bundle, and the fibres of the bundle $\boldsymbol{E}\to\boldsymbol{B}$ are smoothly oriented. Moreover, one assumes that $\mathsf{q}\circ\underline{\mathsf{q}}:\boldsymbol{V}\to\boldsymbol{B}$ is also a bundle, and that for any sufficiently small open subset $\boldsymbol{X}\subset\boldsymbol{B}$ there are bundle trivializations

$$(\underline{\mathsf{q}}\,,\underline{\mathsf{y}}):\boldsymbol{E}_{\boldsymbol{X}}\to\boldsymbol{X}\times\underline{\boldsymbol{Y}}\;,\quad(\mathsf{q}\circ\underline{\mathsf{q}}\,,\mathsf{y}):\boldsymbol{V}_{\boldsymbol{X}}\to\boldsymbol{X}\times\boldsymbol{Y}$$

with the following projectability property: there exists a surjective submersion $\mathsf{p}:\boldsymbol{Y}\to\underline{\boldsymbol{Y}}$ such that the diagram

$$\begin{array}{ccc}\boldsymbol{V}_{\boldsymbol{X}} & \xrightarrow{(\mathsf{q}\circ\underline{\mathsf{q}}\,,\mathsf{y})} & \boldsymbol{X}\times\boldsymbol{Y}\\ {\scriptstyle\mathsf{q}}\big\downarrow & & \big\downarrow{\scriptstyle 1\!\!1_{\boldsymbol{X}}\times\mathsf{p}}\\ \boldsymbol{E}_{\boldsymbol{X}} & \xrightarrow[(\underline{\mathsf{q}},\underline{\mathsf{y}})]{} & \boldsymbol{X}\times\underline{\boldsymbol{Y}}\end{array}$$

commutes; this implies that $\boldsymbol{Y}\to\underline{\boldsymbol{Y}}$ is a vector bundle, not trivial in general.

The above conditions are easily checked to hold in many cases which are relevant for physical applications, and in particular when $\boldsymbol{V}=\boldsymbol{E}\times_{\boldsymbol{B}}\boldsymbol{W}$ where $\boldsymbol{W}\to\boldsymbol{B}$ is a vector bundle, when $\boldsymbol{V}=\mathrm{V}\boldsymbol{E}$ (the vertical bundle of $\boldsymbol{E}\to\boldsymbol{B}$) and when $\boldsymbol{V}$ is any component of the tensor algebra of $\mathrm{V}\boldsymbol{E}\to\boldsymbol{E}$.

For each $x\in\boldsymbol{B}$ one considers the distributional space $\boldsymbol{\mathcal{V}}_x:=\boldsymbol{\mathcal{D}}(\boldsymbol{E}_x\,,\,\boldsymbol{V}_x)$, and obtains the fibred set

$$\wp:\boldsymbol{\mathcal{V}}\equiv\boldsymbol{\mathcal{D}}_{\boldsymbol{B}}(\boldsymbol{E},\boldsymbol{V}):=\bigsqcup_{x\in\boldsymbol{B}}\boldsymbol{\mathcal{V}}_x\to\boldsymbol{B}\;.$$

An isomorphism of vector bundles yields an isomorphism of the corresponding spaces of generalized sections; hence, a local trivialization of the underlying classical 2-bundle, as above, yields a local bundle trivialization

$$(\wp, \mathsf{Y}) : \boldsymbol{\mathcal{V}}_{\!X} \to \boldsymbol{X} \times \boldsymbol{\mathcal{Y}}\ , \quad \boldsymbol{\mathcal{Y}} \equiv \boldsymbol{\mathcal{D}}(\underline{\boldsymbol{Y}}\,, \boldsymbol{Y})$$

of $\boldsymbol{\mathcal{V}} \to \boldsymbol{B}$. Moreover, a smooth atlas of 2-bundle trivializations determines a linear F-smooth bundle atlas on $\boldsymbol{\mathcal{V}} \to \boldsymbol{B}$, which is said to be an *F-smooth distributional bundle.* In general, the F-smoothness of any map from or to $\boldsymbol{\mathcal{V}}$ is equivalent to the F-smoothness of its local trivialized expression.

One defines the *tangent space* of any F-smooth space through equivalence classes of F-smooth curves; tangent prolongations of any F-smooth mappings can also be shown to exist. Thus one gets, in particular, the tangent space $\mathrm{T}\boldsymbol{\mathcal{V}}$, which has local trivializations as $\mathrm{T}\boldsymbol{X} \times \mathrm{T}\boldsymbol{\mathcal{Y}}$, its *vertical subspace* and the *first jet bundle* $\mathrm{J}\boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}$. A *connection* is defined to be an F-smooth section $\mathfrak{G} : \boldsymbol{\mathcal{V}} \to \mathrm{J}\boldsymbol{\mathcal{V}}$.

With some care, many of the usual chart expressions of finite-dimensional differential geometry can be extended to the distributional case. In particular, let $\sigma : \boldsymbol{B} \to \boldsymbol{\mathcal{V}}$ be an F-smooth section and $\sigma^{\mathsf{Y}} := \mathsf{Y} \circ \sigma : \boldsymbol{B} \to \boldsymbol{\mathcal{Y}}$ its 'chart expression'. Then its covariant derivative has the chart expression

$$(\nabla\sigma)^{\mathsf{Y}} = \dot{\mathsf{x}}^a\,(\partial_a\sigma^{\mathsf{Y}} - \mathfrak{G}_{a}{}^{\mathsf{Y}}{}_{\mathsf{Y}}\,\sigma^{\mathsf{Y}})\ ,$$

where (x^a) is a chart on $\boldsymbol{X} \subset \boldsymbol{B}$ and $\mathfrak{G}_{a}{}^{\mathsf{Y}}{}_{\mathsf{Y}} : \boldsymbol{X} \to \mathrm{End}(\boldsymbol{\mathcal{Y}})$.

The notions of *curvature* and of *adjoint connection* can also be introduced. Furthermore, it can be shown that a projectable connection on the underlying classical 2-bundle determines a distributional connection; however, not all distributional connections arise from classical ones.

3. Quantum bundles

The formulation of quantum field theory, in a standard sense, requires time (though eventually the scattering matrix in flat spacetime turns out to be an invariant quantity). On a curved spacetime $\boldsymbol{M}$ one could fix a *time map*, namely a fibration $\mathsf{t} : \boldsymbol{M} \to \boldsymbol{T}$, where

$\boldsymbol{T}$ is a 1-dimensional manifold, such that the fibres $\boldsymbol{M}_t \subset \boldsymbol{M}$, $t \in \boldsymbol{T}$, are spacelike submanifolds. Then one gets a distributional bundle $\boldsymbol{\mathcal{V}} \to \boldsymbol{T}$ by taking the fibre $\boldsymbol{\mathcal{V}}_t$ to be a space of distributions on $\boldsymbol{M}_t$. The classical structure yields a connection on this bundle.

Another approach considers a 'detector', represented by a 1-dimensional timelike submanifold $\boldsymbol{T} \subset \boldsymbol{M}$. The fibre of the quantum bundle over any $t \in \boldsymbol{T}$ can be taken to be a space of distributions over the phase-space, at t, of the considered particle. The phase-bundle of a particle is the subbundle $\boldsymbol{P} \subset \mathrm{T}^*\boldsymbol{M}$ over $\boldsymbol{M}$ whose fibres are the hyperboloids constituted of future-pointing covectors of given length, determined by the particle's mass; for a particle of vanishing mass, the phase-bundle is the bundle of *celestial spheres*, whose fibres are the projective spaces of the null cones in $\mathrm{T}^*\boldsymbol{M}$. It can be shown [4] that the classical structures naturally determine a connection on the distributional bundle $\boldsymbol{\mathcal{P}} := \boldsymbol{\mathcal{D}}_{\boldsymbol{M}}(\boldsymbol{P}, \boldsymbol{V})$, where $\boldsymbol{V} \to \boldsymbol{M}$ is the bundle describing the particle's internal structure. On turn, this yield a connection on the restricted bundle over $\boldsymbol{T}$, $\boldsymbol{\mathcal{P}}_{\boldsymbol{T}} = \boldsymbol{\mathcal{D}}_{\boldsymbol{T}}(\boldsymbol{P}, \boldsymbol{V})$.

I'll discuss the detector approach further in §8. In the next sections, however, we won't be strictly concerned with a particular quantum bundle, but rather with a general argument, which will enable us to recover a notion of quantum field in a precise geometrical setting. So, we'll consider a distributional bundle $\boldsymbol{\mathcal{V}} \to \boldsymbol{T}$; it is convenient to assume that this is a bundle of $\boldsymbol{V}$*-valued generalized half-densities*, $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{D}}_{\boldsymbol{T}}(\boldsymbol{E}, \mathbb{V}^{-1/2}\boldsymbol{E} \otimes \boldsymbol{V})$, which implies that the topological dual bundle $\boldsymbol{\mathcal{V}}_\circ \to \boldsymbol{T}$ of test elements is a subbundle of the 'adjoint' bundle $\boldsymbol{\mathcal{V}}^\star = \boldsymbol{\mathcal{D}}_{\boldsymbol{T}}(\boldsymbol{E}, \mathbb{V}^{-1/2}\boldsymbol{E} \otimes \boldsymbol{V}^\star)$. One also has the conjugate bundle $\bar{\boldsymbol{\mathcal{V}}} = \boldsymbol{\mathcal{D}}_{\boldsymbol{T}}(\boldsymbol{E}, \overline{\boldsymbol{V}})$; usually, a Hermitian structure on $\boldsymbol{V}$ is also assumed, yielding an isomorphism $\bar{\boldsymbol{\mathcal{V}}} \leftrightarrow \boldsymbol{\mathcal{V}}^\star$.

Next, one considers the corresponding *Fock bundle* to be

$$\boldsymbol{\mathcal{W}} = \bigoplus_{r=0}^{\infty} \boldsymbol{\mathcal{V}}^r = \boldsymbol{\mathcal{V}}^0 \underset{\boldsymbol{T}}{\oplus} \boldsymbol{\mathcal{V}}^1 \underset{\boldsymbol{T}}{\oplus} \boldsymbol{\mathcal{V}}^2 \underset{\boldsymbol{T}}{\oplus} \cdots \ ,$$

where either $\boldsymbol{\mathcal{V}}^r = \vee^r \boldsymbol{\mathcal{V}}$ (bosons) or $\boldsymbol{\mathcal{V}}^r = \wedge^r \boldsymbol{\mathcal{V}}$ (fermions). If $\psi \in \boldsymbol{\mathcal{V}}^r$ and $u \in \boldsymbol{\mathcal{V}}_\circ$ are in the fibres over the same $t \in \boldsymbol{T}$, then their contraction $u|\psi$ is in $\boldsymbol{\mathcal{V}}^{r-1}$ over t. Then one has a morphism over $\boldsymbol{T}$

$$\mathrm{a} : \boldsymbol{\mathcal{V}}_\circ \to \mathrm{End}(\boldsymbol{\mathcal{W}}) : u \mapsto \mathrm{a}[u] \ , \quad \mathrm{a}[u]\psi := u|\psi \ .$$

Its Hermitian adjoint is an antilinear morphsim $\mathrm{a}^\dagger : \boldsymbol{\mathcal{V}}_\circ \to \mathrm{End}(\boldsymbol{\mathcal{W}})$, given by

$$\mathrm{a}^\dagger[u]\psi = \begin{cases} u^\dagger \vee \psi & \text{(bosons)} \ , \\ u^\dagger \wedge \psi & \text{(fermions)} \ . \end{cases}$$

The combination

$$\Psi := \mathrm{a} + \mathrm{a}^\dagger$$

is a real linear morphism $\boldsymbol{\mathcal{V}}_\circ \to \mathrm{End}(\boldsymbol{\mathcal{W}})$, called the *quantum field* of the considered distributional bundle.

Let $u, v \in \boldsymbol{\mathcal{V}}_\circ$ be in the same fibre. Then, by elementary tensor algebra, one finds the boson case commutators

$$\begin{aligned}
&\mathrm{a}[u]\mathrm{a}^\dagger[v] - \mathrm{a}^\dagger[v] \circ \mathrm{a}[u] = \langle u|v^\dagger\rangle \, 1\!\!1 \ , \\
&\mathrm{a}[u] \circ \mathrm{a}[v] - \mathrm{a}[v] \circ \mathrm{a}[u] = 0 \ , \\
&\mathrm{a}^\dagger[u] \circ \mathrm{a}^\dagger[v] - \mathrm{a}^\dagger[v] \circ \mathrm{a}^\dagger[u] = 0 \ , \\
&\Psi[u] \circ \Psi[v] - \Psi[v] \circ \Psi[u] = 2\,\mathrm{i}\,\Im\langle u|v^\dagger\rangle \, 1\!\!1 \ .
\end{aligned}$$

Note that $\mathrm{a}[u]$ and $\mathrm{a}^\dagger[v]$ commute iff $\langle u|v^\dagger\rangle = 0$; in particular, this is true if u and v are different from 0 only in non-intersecting domains.

In the Fermion case one has the anticommutators

$$\begin{aligned}
&\mathrm{a}[u] \circ \mathrm{a}^\dagger[v] + \mathrm{a}^\dagger[v] \circ \mathrm{a}[u] = \langle u|v^\dagger\rangle \, 1\!\!1 \ , \\
&\mathrm{a}[u] \circ \mathrm{a}[v] + \mathrm{a}[v] \circ \mathrm{a}[u] = 0 \ , \\
&\mathrm{a}^\dagger[u] \circ \mathrm{a}^\dagger[v] + \mathrm{a}^\dagger[v] \circ \mathrm{a}^\dagger[u] = 0 \ , \\
&\Psi[u] \circ \Psi[v] + \Psi[v] \circ \Psi[u] = 2\,\Re\langle u|v^\dagger\rangle \, 1\!\!1 \ .
\end{aligned}$$

Note that $\mathrm{a}[u]$ and $\mathrm{a}^\dagger[v]$ anticommute iff $\langle u|v^\dagger\rangle = 0$.

4. Quantum connections

Let $\mathfrak{G}$ be an F-smooth linear connection on the Fock bundle $\boldsymbol{\mathcal{W}} \rightarrowtail \boldsymbol{T}$. Since the base is 1-dimensional, the curvature of $\mathfrak{G}$ vanishes. The corresponding covariant derivative will be denoted by $\eth[\mathfrak{G}]$, or simply by $\eth$ if no confusion arises. Possibly, $\mathfrak{G}$ may arise from a linear connection on $\boldsymbol{\mathcal{V}} \rightarrowtail \boldsymbol{T}$.

Given an F-smooth linear trivialization $(\mathsf{t},\mathsf{Y}):\boldsymbol{\mathcal{W}}\to\boldsymbol{T}\times\boldsymbol{\mathcal{Y}}$, for any F-smooth section $\psi:\boldsymbol{T}\to\boldsymbol{\mathcal{W}}$ one has the trivialization expression

$$(\eth\psi)^{\mathsf{Y}}=\partial(\psi^{\mathsf{Y}})-\mathfrak{G}^{\mathsf{Y}}{}_{\mathsf{Y}}\psi^{\mathsf{Y}}\ ,$$

where $\psi^{\mathsf{Y}}:=\mathsf{Y}\circ\psi:\boldsymbol{T}\to\boldsymbol{\mathcal{Y}}$ and the like, and

$$\mathfrak{G}^{\mathsf{Y}}{}_{\mathsf{Y}}:\mathrm{T}\boldsymbol{T}\to\mathrm{End}(\boldsymbol{\mathcal{Y}})$$

is the Y-expression of $\mathfrak{G}:\boldsymbol{\mathcal{W}}\to\mathrm{J}\boldsymbol{\mathcal{W}}$. If $(\mathsf{t},\mathsf{Y}'):\boldsymbol{\mathcal{V}}\to\boldsymbol{T}\times\boldsymbol{\mathcal{Y}}'$ is another trivialization, then I denote the transition map by

$$\begin{aligned}&\mathcal{A}^{\mathsf{Y}'}{}_{\mathsf{Y}}:=(\mathsf{t},\mathsf{Y}')\circ(\mathsf{t},\mathsf{Y})^{-1}:\boldsymbol{T}\times\boldsymbol{\mathcal{Y}}\to\boldsymbol{T}\times\boldsymbol{\mathcal{Y}}'\ ,\\ \text{or}\quad&\mathcal{A}^{\mathsf{Y}'}{}_{\mathsf{Y}}(t)=\mathsf{Y}'\circ\mathsf{Y}_t^{-1}:\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{Y}}'\ .\end{aligned}$$

In particular the connection $\mathfrak{G}$ itself, being flat, determines a bundle trivialization $(\mathsf{t},\mathsf{G}):\boldsymbol{\mathcal{V}}\to\boldsymbol{T}\times\boldsymbol{\mathcal{G}}$, characterized by $\mathfrak{G}^{\mathsf{G}}{}_{\mathsf{G}}=0\in\mathrm{End}(\boldsymbol{\mathcal{G}})$. Thus

$$\mathfrak{G}^{\mathsf{Y}}{}_{\mathsf{Y}}=(\partial\mathcal{A}^{\mathsf{Y}}{}_{\mathsf{G}})\,\mathcal{A}^{\mathsf{G}}{}_{\mathsf{Y}}=-\mathcal{A}^{\mathsf{Y}}{}_{\mathsf{G}}\,\partial\mathcal{A}^{\mathsf{G}}{}_{\mathsf{Y}}\ .$$

Definition 4.1. *A* quantum connection *is an F-smooth linear connection on a Fock bundle $\boldsymbol{\mathcal{W}}\rightarrowtail\boldsymbol{T}$, such that parallel transport along $\boldsymbol{T}$ determines continuous isomorphisms among the fibres.*

Namely, if $\mathfrak{G}$ is a quantum connection, then for each $t,t'\in\boldsymbol{T}$ one has an isomorphism

$$\mathcal{G}_{(t',t)}:\boldsymbol{\mathcal{W}}_t\to\boldsymbol{\mathcal{W}}_{t'}\ ;$$

this family of isomorphisms fulfils the natural group properties

$$(\mathcal{G}_{(t',t)})^{-1}=\mathcal{G}_{(t,t')}\ ,\quad\mathcal{G}_{(t'',t')}\circ\mathcal{G}_{(t',t)}=\mathcal{G}_{(t'',t)}\ ,\quad\mathcal{G}_{(t,t)}=1\!\!1\ ,$$

and a section $\psi:\boldsymbol{T}\to\boldsymbol{\mathcal{W}}$ is parallely transported, namely $\psi(t')=\mathcal{G}_{(t',t)}\,\psi(t)$, if and only if it is covariantly constant, that is $\eth\psi=0$. With regard to the trivialization of $\boldsymbol{\mathcal{W}}\rightarrowtail\boldsymbol{T}$ determined by $\mathfrak{G}$, observe that the space of covariantly constant states can be identified with any fibre $\boldsymbol{\mathcal{W}}_t$, $t\in\boldsymbol{T}$; but there is no distinguished choice of such reference fibre.

A quantum connection yields a dual connection on the bundle $\boldsymbol{\mathcal{W}}_\circ \to \boldsymbol{T}$ of test elements, generated by $\boldsymbol{\mathcal{V}}_\circ$: a covariantly constant section $u : \boldsymbol{T} \to \boldsymbol{\mathcal{W}}_\circ$ fulfills $u(t') = u(t) \circ \mathcal{G}_{(t,t')}$, namely the contraction between covariantly constant sections is constant. Possibly, this may extend to an *adjoint* connection on $\boldsymbol{\mathcal{W}}^\star$.

The two basic ingredients of a quantum field theory are the *free particle states* and the *interactions*. Free particle states are described as sections $\boldsymbol{T} \to \boldsymbol{\mathcal{W}}$ which are covariantly constant relatively to a *free-field connection* $\underline{\mathfrak{G}}$ induced by a connection $\underline{\mathfrak{G}}^1$ on $\boldsymbol{\mathcal{W}}^1 \equiv \boldsymbol{\mathcal{V}}$. Then, the free-field connection transport preserves the particle number.

Interactions are described by a section

$$\mathfrak{I} : \boldsymbol{T} \to \mathrm{T}^*\boldsymbol{T} \underset{\boldsymbol{T}}{\otimes} \mathrm{End}(\boldsymbol{\mathcal{V}}) \ ,$$

namely one has an *interaction connection* $\mathfrak{G} := \underline{\mathfrak{G}} + \mathfrak{I}$ which describes the full dynamics of the particle system under consideration. Typically, $\mathfrak{I}$ changes the particle number and mixes different types of particles. Its nature and existence pose complex questions, which won't be addressed in detail in this paper.

In a typical QFT setting one has, besides the free-field and the interaction connections, a further linear connection $\mathfrak{Y}$ on $\boldsymbol{\mathcal{W}} \rightarrowtail \boldsymbol{T}$. In the time-map $\mathtt{t} : \boldsymbol{M} \to \boldsymbol{T}$ scheme, for example, this is related to the flux associated with the vector field orthogonal to the fixed time hypersurfaces; for an inertial observer in flat spacetime, this is just the distinguished family of isomorphisms among space slices at different times. In general, it can be thought of as associated with the choice of some suitable charts on the underlying classical bundles.

Consider the parallel transports, among the fibres of $\boldsymbol{\mathcal{W}} \rightarrowtail \boldsymbol{T}$, determined by the above connections $\mathfrak{Y}$, $\underline{\mathfrak{G}}$ and $\mathfrak{G}$. These will be respectively indicated by $\mathcal{Y}$, $\underline{\mathcal{G}}$, $\mathcal{G}$. Namely, for each $t, t' \in \boldsymbol{T}$ one has linear maps

$$\mathcal{Y}_{(t',t)} \ , \ \underline{\mathcal{G}}_{(t',t)} \ , \ \mathcal{G}_{(t',t)} : \boldsymbol{\mathcal{W}}_t \to \boldsymbol{\mathcal{W}}_{t'} \ ,$$

fulfilling the natural group properties. The maps $\underline{\mathcal{G}}_{(t',t)}$ and $\mathcal{G}_{(t',t)}$, restricted to the appropriate Hilbert subspaces, will turn out to be unitary (i.e. isometries). In general, this is *not* true for $\mathcal{Y}$, but for the flat inertial case.

The three quantum connections $\mathfrak{Y}$, $\underline{\mathfrak{G}}$ and $\mathfrak{G}$ respectively determine trivializations

$$(\mathsf{t},\mathsf{Y}):\boldsymbol{\mathcal{W}}\to\boldsymbol{T}\times\boldsymbol{\mathcal{Y}}\;,\quad(\mathsf{t},\underline{\mathsf{G}}):\boldsymbol{\mathcal{W}}\to\boldsymbol{T}\times\underline{\boldsymbol{\mathcal{G}}}\;,\quad(\mathsf{t},\mathsf{G}):\boldsymbol{\mathcal{W}}\to\boldsymbol{T}\times\boldsymbol{\mathcal{G}}\;.$$

In order to recover the traditional quantum mechanical *pictures* in the curved spacetime case, it is convenient to fix an element $t_0\in\boldsymbol{T}$. Then, using $\boldsymbol{\mathcal{Y}}\equiv\boldsymbol{\mathcal{W}}_{t_0}$ as the fibre type, one obtains bundle trivializations

$$(\mathsf{t},\mathsf{Y}),\;(\mathsf{t},\underline{\mathsf{G}}),\;(\mathsf{t},\mathsf{G}):\boldsymbol{\mathcal{W}}\to\boldsymbol{T}\times\boldsymbol{\mathcal{Y}}\;,$$

where

$$\mathsf{Y}(\psi_t):=\mathcal{Y}_{(t_0,t)}(\psi_t)\;,\quad\psi_t\in\boldsymbol{\mathcal{V}}_t\;,\quad t\in\boldsymbol{T}\;,$$

and the like. Moreover, some notations can be simplified by setting

$$\mathcal{Y}_t:=\mathcal{Y}_{(t,t_0)}\quad\text{i.e.}\quad\mathsf{Y}_t=(\mathcal{Y}_t)^{-1}=\mathcal{Y}_{(t_0,t)}$$

and the like, where Y_t denotes the restriction of Y to $\boldsymbol{\mathcal{W}}_t$.

5. Quantum pictures

A section $\psi:\boldsymbol{T}\to\boldsymbol{\mathcal{V}}$ is said to represent a 'quantum history' if it is $\mathfrak{G}$-constant, namely if $\mathsf{G}\circ\psi=$ constant, or, still equivalently, if it is parallely transported through $\mathcal{G}_t$, that is

$$\psi(t)=\mathcal{G}_t\,\psi_0\;,\quad\psi_0:=\psi(t_0)\in\boldsymbol{\mathcal{V}}_{t_0}\;.$$

A $\mathfrak{G}$-constant ψ can be characterized by the map

$$\psi_{\mathrm{S}}:\boldsymbol{T}\to\boldsymbol{\mathcal{Y}}:t\mapsto\psi_{\mathrm{S}}(t):=\mathsf{Y}\big(\psi(t)\big)=\mathcal{Y}_t^{-1}\circ\mathcal{G}_t(\psi_0)\;.$$

This 'time dependent state vector' describes the quantum history in the *Schrödinger picture.* In this picture, a physical observable is represented by some operator $A_{\mathrm{S}}\equiv A_0\in\mathrm{End}(\boldsymbol{\mathcal{W}}_{t_0})$; a measurement made at time $t\in\boldsymbol{T}$ is represented by $A_{\mathrm{S}}\psi_{\mathrm{S}}(t)$. One says that state vectors are time-dependent, while operators representing observables are time-independent.

The converse is true in the *Heisenberg picture*: now the state vector is the time-independent object $\psi_{\mathrm{H}} \equiv \psi_0 := \mathsf{G}\big(\psi(t)\big)$ (in other terms, quantum histories are represented by their value at t_0). One introduces the 'time-dependent operator'

$$A_{\mathrm{H}} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Y}}) : t \mapsto A_{\mathrm{H}}(t) := \mathcal{G}_t^{-1} \circ \mathcal{Y}_t \circ A_0 \circ \mathcal{Y}_t^{-1} \circ \mathcal{G}_t \ .$$

Let $u_0 \in (\boldsymbol{\mathcal{W}}_\circ)_{t_0} \equiv \boldsymbol{\mathcal{Y}}_\circ$ and set $u_{\mathrm{S}} : \boldsymbol{T} \to : t \mapsto u_{\mathrm{S}}(t) := u_0 \circ \mathcal{G}_t^{-1} \circ \mathcal{Y}_t$; then

$$\langle u_{\mathrm{H}} \mid A_{\mathrm{H}}(t)\psi_{\mathrm{H}}\rangle = \langle u_{\mathrm{S}}(t) \mid A_{\mathrm{S}}\psi_{\mathrm{S}}(t)\rangle \ .$$

One can consider other pictures, such as for example an *interaction picture*, where both state vectors and operators are time-dependent:

$$\psi_{\mathrm{I}} : \boldsymbol{T} \to \boldsymbol{\mathcal{Y}} : t \mapsto \psi_{\mathrm{I}}(t) := \underline{\mathcal{G}}_t^{-1} \circ \mathcal{G}_t(\psi_0) \ ,$$

$$A_{\mathrm{I}} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Y}}) : t \mapsto A_{\mathrm{I}}(t) := \underline{\mathcal{G}}_t^{-1} \circ \mathcal{Y}_t \circ A_0 \circ \mathcal{Y}_t^{-1} \circ \underline{\mathcal{G}}_t \ ,$$

$$\text{with} \quad \psi_0 \in \boldsymbol{\mathcal{W}}_{t_0} \equiv \boldsymbol{\mathcal{Y}} \ , \quad A_0 \in \mathrm{End}(\boldsymbol{\mathcal{Y}}) \ .$$

Here, one could say that state vectors carry the time-dependency generated by interactions, while the time-dependency of operators comes from the free-field transport.

The Schrödinger and Heisenberg pictures arise from intertwining the transports $\mathcal{G}$ and $\mathcal{Y}$. Similar pictures are obtained by replacing $\mathcal{Y}$ with $\underline{\mathcal{G}}$: time dependence now comes from comparing the full interaction transport $\mathcal{G}$ with the free-field transport $\underline{\mathcal{G}}$. Consider the map

$$\mathcal{U} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Y}}) : t \mapsto \mathcal{U}_t := \underline{\mathcal{G}}_t^{-1} \circ \mathcal{G}_t \equiv \underline{\mathsf{G}} \circ \mathcal{G}_t \equiv \mathcal{A}^{\underline{\mathsf{G}}}_{\mathsf{G}}(t) \ .$$

Then one has the modified Schrödinger and Heisenberg pictures

$$\psi_{\mathrm{S}} : \boldsymbol{T} \to \boldsymbol{\mathcal{Y}} : t \mapsto \psi_{\mathrm{S}}(t) := \mathcal{U}_t(\psi_0) \ , \quad A_{\mathrm{S}} = A_0 \in \mathrm{End}(\boldsymbol{\mathcal{Y}}) \ ,$$

$$\psi_{\mathrm{H}} := \psi_0 \in \boldsymbol{\mathcal{Y}} \ , \quad A_{\mathrm{H}} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Y}}) : t \mapsto A_{\mathrm{H}}(t) := \mathcal{U}_t^{-1} \circ A_0 \circ \mathcal{U}_t \ .$$

On the other hand, consider a section $Z : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{W}})$. Define maps $\underline{Z}_{t_0}\, , Z_{t_0} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Y}})$ by

$$\underline{Z}_{t_0}(t) := \underline{\mathcal{G}}_t^{-1} \circ Z(t) \circ \underline{\mathcal{G}}_t \ , \quad Z_{t_0}(t) := \mathcal{G}_t^{-1} \circ Z(t) \circ \mathcal{G}_t \ ,$$

so that $Z_{t_0}(t) = \mathcal{U}_t^{-1} \circ \underline{Z}_{t_0}(t) \circ \mathcal{U}_t$ (if $\underline{Z}_{t_0}$ is constant in time, then it can be seen as a "Schrödinger picture" operator, and Z_{t_0} is its corresponding "Heisenberg picture" operator). Moreover

$$Z_{t_0}(t)\psi_0 = \mathcal{G}_{(t_0\,,t)} \circ Z(t) \circ \mathcal{G}_{(t,t_0)}\psi_0 = \mathcal{G}_{(t_0\,,t)}\big(Z(t)\psi_t\big)\ .$$

PROPOSITION 5.1. *The map* $\mathcal{U} : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{Y}}) : t \mapsto \mathcal{U}_t := \underline{\mathsf{G}} \circ \mathcal{G}_t$ *fulfils*

$$\partial\mathcal{U} - \mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}} \circ \mathcal{U} = 0\ .$$

Proof. For $\psi_0 \in \boldsymbol{\mathcal{Y}}$ the section $\psi : \boldsymbol{T} \to \boldsymbol{\mathcal{W}}$ given by $\psi(t) := \mathcal{G}_t(\psi_0)$ fulfils $\eth\psi = 0$ and $\psi^{\underline{\mathsf{G}}} = \mathcal{U}\psi_0$. Thus $0 = (\eth\psi)^{\underline{\mathsf{G}}} = \partial(\psi^{\underline{\mathsf{G}}}) - \mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}\,\psi^{\underline{\mathsf{G}}} = (\partial\mathcal{U})\psi_0 - \mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}\mathcal{U}\psi_0 = (\partial\mathcal{U} - \mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}} \circ \mathcal{U})\,\psi_0$. □

Besides the 'reference time' t_0, fix any two other times $t_1\,, t_2 \in \boldsymbol{T}$, $t_1 < t_2$, to be regarded as 'initial' and 'final' times, respectively; namely, one considers the system's evolution from an 'initial' to a 'final' state (usually, one takes the limits $t_1 \to -\infty$ and $t_2 \to \infty$). Let $\zeta : \boldsymbol{T} \to \boldsymbol{\mathcal{W}}$ be constant relatively to the free-field connection $\underline{\mathfrak{G}}$, namely $\zeta(t') = \underline{\mathcal{G}}_{(t',t)}\zeta(t)\,,\ t,t' \in \boldsymbol{T}$. Set

$$|\zeta\rangle_{\mathrm{in}} \equiv \zeta_{\mathrm{in}} \equiv \zeta_{t_1} := \mathcal{G}_{(t_0\,,t_1)}\zeta(t_1) \in \boldsymbol{\mathcal{Y}}\ ,$$
$$|\zeta\rangle_{\mathrm{out}} \equiv \zeta_{\mathrm{out}} \equiv \zeta_{t_2} := \mathcal{G}_{(t_0\,,t_2)}\zeta(t_2) \in \boldsymbol{\mathcal{Y}}\ .$$

The idea behind this definition is that e.g. ζ_{in} represents, in the reference space $\boldsymbol{\mathcal{Y}}$, the state of a system which, at time t_1, was in the same state as $\zeta(t_1)$; namely, ζ_{in} is the element in $\boldsymbol{\mathcal{Y}}$ corresponding to the section $t \mapsto \mathcal{G}_{(t,t_1)}\zeta(t_1)$, which is parallely transported relatively to the full interaction connection. Similarly, if $Z : \boldsymbol{T} \to \mathrm{End}(\boldsymbol{\mathcal{W}})$ is $\underline{\mathfrak{G}}$-constant, then one defines Z_{in} and Z_{out} by $Z_{\mathrm{in}} \equiv Z_{t_1} := \mathcal{G}_{(t_0,t_1)} \circ Z(t_1) \circ \mathcal{G}_{(t_1,t_0)} \in \boldsymbol{\mathcal{O}}_{t_0}$ and the like.

6. Quantum free fields

The fibred morphims $\mathrm{a}, \mathrm{a}^{\dagger}, \Psi : \boldsymbol{\mathcal{V}}_{\circ} \to \mathrm{End}(\boldsymbol{\mathcal{W}})$ introduced in §3 can be seen as sections

$$\begin{aligned}
&\mathrm{a} : \boldsymbol{T} \to \boldsymbol{\mathcal{V}} \underset{\boldsymbol{T}}{\otimes} \mathrm{End}(\boldsymbol{\mathcal{W}}) \ , \\
&\mathrm{a}^{\dagger} : \boldsymbol{T} \to \bar{\boldsymbol{\mathcal{V}}} \underset{\boldsymbol{T}}{\otimes} \mathrm{End}(\boldsymbol{\mathcal{W}}) \ , \\
&\Psi := \mathrm{a} + \mathrm{a}^{\dagger} : \boldsymbol{T} \to (\boldsymbol{\mathcal{V}} \underset{\boldsymbol{T}}{\oplus} \bar{\boldsymbol{\mathcal{V}}}) \ ten\boldsymbol{T} \, \mathrm{End}(\boldsymbol{\mathcal{W}}) \ .
\end{aligned}$$

The usual notion of quantum field[1] is rather that of a distribution valued into the operators of a fixed configuration space, and can be recovered by chosing some $t_0 \in \boldsymbol{T}$ as a reference time, and using the same fibre type $\boldsymbol{\mathcal{Y}} \equiv \boldsymbol{\mathcal{W}}_{t_0}$ for all trivializations.

Definition 6.1. *Let $t_1 \in \boldsymbol{T}$; the* free fields *relative to t_1 are the sections*

$$\underline{\mathrm{a}}_{t_1} \ , \ \underline{\mathrm{a}}^{\dagger}_{t_1} \ , \ \underline{\Psi}_{t_1} : \boldsymbol{T} \to (\boldsymbol{\mathcal{V}} \underset{\boldsymbol{T}}{\oplus} \bar{\boldsymbol{\mathcal{V}}}) \otimes \mathrm{End}(\boldsymbol{\mathcal{Y}})$$

given by

$$\underline{\mathrm{a}}_{t_1}[u_t] := \mathcal{G}_{(t_0,t_1)} \circ \mathrm{a}\big[u_t \circ \underline{\mathcal{G}}_{(t,t_1)}\big] \circ \mathcal{G}_{(t_1,t_0)} \ , \quad u_t \in \boldsymbol{\mathcal{V}}^1_{\circ} \ ,$$

and the like.

One may say that $\underline{\mathrm{a}}_{t_1}[u_t]$ (resp. $\underline{\mathrm{a}}^{\dagger}_{t_1}[u_t]$) annihilates (resp. creates) a particle at time t_1, the particle's state being specified at time t and transported to time t_1 through the free-field connection. So if $\psi_0 \in \boldsymbol{\mathcal{W}}_{t_0} \equiv \boldsymbol{\mathcal{Y}}$ then

$$\begin{aligned}
&\underline{\mathrm{a}}_{t_1}[u_t]\psi_0 = \mathcal{G}_{(t_0,t_1)}(u_{t_1}|\psi_{t_1}) \ , \\
\text{with} \quad &u_{t_1} := u_t \circ \underline{\mathcal{G}}_{(t,t_1)} \ , \quad \psi_{t_1} := \mathcal{G}_{(t_1,t_0)}\psi_0 \ ,
\end{aligned}$$

is the element in $\boldsymbol{\mathcal{Y}}$ which represents the Heisenberg-picture state ψ_0 with a particle annihilated at time t_1. In particular one writes $\underline{\mathrm{a}} \equiv \underline{\mathrm{a}}_{t_0}$ and the like, getting

$$\underline{\mathrm{a}}[u_t] \equiv \underline{\mathrm{a}}_{t_0}[u_t] := \mathrm{a}[\underline{\mathcal{G}}^*_{(t,t_0)}u_t] = \underline{\mathcal{G}}_{(t_0,t)} \circ \mathrm{a}[u_t] \circ \underline{\mathcal{G}}_{(t,t_0)} \in \mathrm{End}(\boldsymbol{\mathcal{Y}})$$

---

[1]Note that these objects are actually fixed structures, rather than 'fields' in the standard sense.

and the like. If $u : \boldsymbol{T} \to \boldsymbol{\mathcal{V}}_\circ$ is a section then, provided that the integral converges,

$$\underline{\mathrm{a}}_{t_1}[u] := \int_{\boldsymbol{T}} \underline{\mathrm{a}}_{t_1}[u_{\mathrm{t}}]\,\mathrm{dt} \in \mathrm{End}(\boldsymbol{\mathcal{Y}})$$

and the like.

Viewing $\underline{\mathrm{a}}_{t_1}$ as a section of $\boldsymbol{\mathcal{W}} \to \boldsymbol{T}$ valued in the vector space $\mathrm{End}(\boldsymbol{\mathcal{Y}})$, one sees that it is covariantly transported through the free-field connection (this explains the term 'free field'). In fact, setting $u_{t_1} := u_t \circ \underline{\mathcal{G}}_{(t,t_1)}$, one may write (for example)

$$\underline{\mathrm{a}}_{t_1}[u_t] \equiv \big\langle u_t \,|\, \underline{\mathrm{a}}_{t_1}(t) \big\rangle = \big\langle u_{t_1} \,|\, \underline{\mathrm{a}}_{t_1}(t_1) \big\rangle = \Big\langle u_t \,\underline{\mathcal{G}}_{(t,t_1)} \,|\, \underline{\mathrm{a}}_{t_1}(t_1) \Big\rangle =$$

$$= \Big\langle u_t \,|\, \underline{\mathcal{G}}_{(t,t_1)} \underline{\mathrm{a}}_{t_1}(t_1) \Big\rangle ,$$

or

$$\underline{\mathrm{a}}_{t_1}(t) = \underline{\mathcal{G}}_{(t,t_1)} \underline{\mathrm{a}}_{t_1}(t_1) ,$$

namely $\underline{\mathrm{a}}_{t_1}$ is a $\underline{\mathfrak{G}}$-horizontal section. A similar observation holds for $\underline{\mathrm{a}}^{\dagger}_{t_1}$ and $ul\Psi_{t_1}$. When $t_1, t_2 \in \boldsymbol{T}$ are chosen, acting as initial and final times for a scattering problem, then $\underline{\Psi}_{t_1}$ and $\underline{\Psi}_{t_2}$ correspond to the free fields which are usually denoted as Ψ_{in} and Ψ_{out}.

Proposition 6.2. *If* $t, t' \in \boldsymbol{T}$, $u_t \in (\boldsymbol{\mathcal{V}}_\circ)_t$, $v_{t'} \in (\boldsymbol{\mathcal{V}}_\circ)_{t'}$, *then one has the* boson case commutators

$$\underline{\mathrm{a}}_{t_1}[u_t] \circ \underline{\mathrm{a}}^{\dagger}_{t_1}[v_{t'}] - \underline{\mathrm{a}}^{\dagger}_{t_1}[v_{t'}] \circ \underline{\mathrm{a}}_{t_1}\ ![u_t] = \Big\langle u_t \,\underline{\mathcal{G}}_{(t,t_1)} \,, (v_{t'} \,\underline{\mathcal{G}}_{(t',t_1)})^{\dagger} \Big\rangle \mathbb{1} ,$$

$$\underline{\Psi}_{t_1}[u_t] \circ \underline{\Psi}_{t_1}[v_{t'}] - \underline{\Psi}_{t_1}[v_{t'}] \circ \underline{\Psi}_{t_1}[u_t] = 2\,\mathrm{i}\,\Im\Big\langle u_t \,\underline{\mathcal{G}}_{(t,t_1)} \,, (v_{t'} \,\underline{\mathcal{G}}_{(t',t_1)})^{\dagger} \Big\rangle \mathbb{1} ,$$

and the fermion case anticommutators

$$\underline{\mathrm{a}}_{t_1}[u_t] \circ \underline{\mathrm{a}}^{\dagger}_{t_1}[v_{t'}] + \underline{\mathrm{a}}^{\dagger}_{t_1}[v_{t'}] \circ \underline{\mathrm{a}}_{t_1}\ ![u_t] = \Big\langle u_t \,\underline{\mathcal{G}}_{(t,t_1)} \,, (v_{t'} \,\underline{\mathcal{G}}_{(t',t_1)})^{\dagger} \Big\rangle \mathbb{1} ,$$

$$\underline{\Psi}_{t_1}[u_t] \circ \underline{\Psi}_{t_1}[v_{t'}] + \underline{\Psi}_{t_1}[v_{t'}] \circ \underline{\Psi}_{t_1}[u_t] = 2\,\Re\Big\langle u_t \,\underline{\mathcal{G}}_{(t,t_1)} \,, (v_{t'} \,\underline{\mathcal{G}}_{(t',t_1)})^{\dagger} \Big\rangle \mathbb{1} ,$$

where $\mathbb{1} \equiv \mathbb{1}_{t_0}$ *is the identity of* $\boldsymbol{\mathcal{V}}_{t_0}$ *.*

Proof. It follows from the commuting and anticommuting formulas of §3, after working out the various compositions involved. □

REMARK 6.3. *In particular, the above commutators (resp. anticommutators) vanish if the supports of the free-field transports of* u_t *and* $v_{t'}$ *at any given time do not intersect.*

One may consider the 'chart expressions'[2]

$$\underline{\Psi}^{\mathsf{Y}}, \underline{\Psi}^{\underline{\mathsf{G}}}, \underline{\Psi}^{\mathsf{G}} : \boldsymbol{T} \to (\boldsymbol{\mathcal{Y}} \oplus \bar{\boldsymbol{\mathcal{Y}}}) \otimes \mathrm{End}(\boldsymbol{\mathcal{Y}})$$

of the free field $\underline{\Psi}$, and the like for $\underline{\mathrm{a}}$ and $\underline{\mathrm{a}}^\dagger$. Namely, one has

$$\underline{\Psi}^{\mathsf{Y}}(t)[u_0] \equiv \langle u_0 \mid \underline{\Psi}^{\mathsf{Y}}(t)\rangle = \underline{\Psi}[u_0 \mathcal{Y}_{(t_0,t)}] \ , \quad t \in \boldsymbol{T}, \ u_0 \in \boldsymbol{\mathcal{Y}}$$

and the like (replace Y with $\underline{\mathsf{G}}$ or G in the above formula), and finds

$$\underline{\Psi}^{\mathsf{Y}}(t)[u_0] = \underline{\Psi}[u_0\, \mathcal{Y}_{(t_0,t)}] = \underline{\mathcal{G}}_{(t_0,t)} \circ \mathcal{Y}_{(t,t_0)} \circ \Psi[u_0] \circ \mathcal{Y}_{(t_0,t)} \circ \underline{\mathcal{G}}_{(t,t_0)} \ ,$$

$$\underline{\Psi}^{\underline{\mathsf{G}}}(t)[u_0] = \Psi[u_0\, \underline{\mathcal{G}}_{(t_0,t)}\, \underline{\mathcal{G}}_{(t,t_0)}] = \Psi[u_0] = \text{constant} \ ,$$

$$\underline{\Psi}^{\mathsf{G}}(t)[u_0] = \Psi[u_0\, \mathcal{G}_{(t_0,t)}\, \underline{\mathcal{G}}_{(t,t_0)}] = \Psi[u_0 \ \ Ucal_t^{-1}] \ ,$$

and the like. Of course, the fact that $\underline{\Psi}^{\underline{\mathsf{G}}}$ is constant is just another way of saying that $\underline{\Psi}$ is covariantly constant relatively to the free field connection $\underline{\mathfrak{G}}$.

PROPOSITION 6.4. *In the boson case one has*

$$\Big[\underline{\Psi}^{\mathsf{Y}}(t)[u_0]\, ,\, \underline{\Psi}^{\mathsf{Y}}(t')[v_0]\Big] = 2\,\mathrm{i}\,\Im\Big\langle u_0\, \mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t)\, ,\, \big(v_0\, \mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t')\big)^\dagger\Big\rangle \mathbb{1} \ ,$$

$$\Big[\tfrac{\partial}{\partial \mathsf{x}^0}\underline{\Psi}^{\mathsf{Y}}(t)[u_0]\, ,\, \underline{\Psi}^{\mathsf{Y}}(t')[v_0]\Big] = 2\,\mathrm{i}\,\Im\Big\langle u_0\, \tfrac{\partial}{\partial \mathsf{x}^0}\mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t)\, ,\, \big(v_0\, \mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t')\big)^\dagger\Big\rangle \mathbb{1} \ ,$$

where $t, t' \in \boldsymbol{T}$, $u^0, v^0 \in \boldsymbol{\mathcal{Y}}_\circ$.

In the fermion case one has

$$\Big\{\underline{\Psi}^{\mathsf{Y}}(t)[u_0]\, ,\, \underline{\Psi}^{\mathsf{Y}}(t')[v_0]\Big\} = 2\,\Re\Big\langle u_0\, \mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t)\, ,\, \big(v_0\, \mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t')\big)^\dagger\Big\rangle \mathbb{1} \ ,$$

$$\Big\{\tfrac{\partial}{\partial \mathsf{x}^0}\underline{\Psi}^{\mathsf{Y}}(t)[u_0]\, ,\, \underline{\Psi}^{\mathsf{Y}}(t')[v_0]\Big\} = 2\,\Re\Big\langle u_0\, \tfrac{\partial}{\partial \mathsf{x}^0}\mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t)\, ,\, \big(v_0\, \mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t')\big)^\dagger\Big\rangle \mathbb{1} \ ,$$

Proof. The first boson relation follows from $\underline{\Psi}^{\mathsf{Y}}(t)[u_0] = \Psi[u_0]\mathcal{A}^{\mathsf{Y}}_{\underline{\mathsf{G}}}(t)$ and from the commutation relation for Ψ; the second follows by keeping t' fixed and deriving with respect to time at t. The argument is similar in the Fermion case. ☐

[2]Note that the free fields themselves are already defined through $\underline{\mathsf{G}}$ and G; these are further compositions with the considered bundle trivializations.

7. Interpolating fields

Next, one may recover the usual notion of *interpolating* quantum field. This object is the morphism $\underset{\sim}{\Psi} : \boldsymbol{\mathcal{V}}_{\circ} \to \mathrm{End}(\boldsymbol{\mathcal{V}}_{t_0})$ over $\boldsymbol{T}$ given by

$$\underset{\sim}{\Psi}[u_t] := \mathcal{G}_{(t_0,t)} \circ \Psi[u_t] \circ \mathcal{G}_{(t,t_0)} = (\Psi[u_t])^{\mathsf{G}} \ .$$

Similarly, one has the interpolating fields $\underset{\sim}{\mathrm{a}}$ and $\underset{\sim}{\mathrm{a}}^{\dagger}$.

The 'interpolating' qualification refers to the fact that, for any $t_1 \in \boldsymbol{T}$, one has

$$\underset{\sim}{\Psi}[u_{t_1}] = \mathcal{G}_{(t_0,t_1)} \circ \Psi[u_{t_1}] \circ \mathcal{G}_{(t_1,t_0)} = \underline{\Psi}_{t_1}[u_{t_1}] \ ,$$

namely $\underset{\sim}{\Psi}$ takes the value $\underline{\Psi}_{t_1} \in \mathrm{End}(\boldsymbol{\mathcal{Y}})$ at time t_1. Moreover one finds

$$\underset{\sim}{\Psi}[u_t] = \mathcal{G}_{(t_0,t)} \circ \Psi[u_t] \circ \mathcal{G}_{(t,t_0)} = (\mathcal{U}_t)^{-1} \circ \underline{\Psi}[u_t] \circ \mathcal{U}_t$$

and the like.

As for the free-fields, one can consider the 'chart expressions'

$$\underset{\sim}{\Psi}^{\mathsf{Y}} \, , \underset{\sim}{\Psi}^{\underline{\mathsf{G}}} \, , \underset{\sim}{\Psi}^{\mathsf{G}} : \boldsymbol{T} \to (\boldsymbol{\mathcal{Y}} \oplus \bar{\boldsymbol{\mathcal{Y}}}) \otimes \mathrm{End}(\boldsymbol{\mathcal{Y}})$$

of the interpolating fields seen as sections $\boldsymbol{T} \to \boldsymbol{\mathcal{V}} \otimes \mathrm{End}(\boldsymbol{\mathcal{V}}_{t_0})$, where $\mathrm{End}(\boldsymbol{\mathcal{V}}_{t_0}) \equiv \mathrm{End}(\boldsymbol{\mathcal{Y}})$ is a fixed vector space. One finds

$$\underset{\sim}{\Psi}^{\mathsf{Y}}(t)[u_0] = \underset{\sim}{\Psi}[u_0\, \mathcal{Y}_{(t_0,t)}] = \mathcal{U}_t^{-1} \circ \underline{\Psi}^{\mathsf{Y}}(t)[u_0] \circ \mathcal{U}_t \ ,$$

$$\underset{\sim}{\Psi}^{\underline{\mathsf{G}}}(t)[u_0] = \underset{\sim}{\Psi}[u_0\, \underline{\mathcal{G}}_{(t_0,t)}] = \mathcal{U}_t^{-1} \circ \Psi[u_0] \circ \mathcal{U}_t \ ,$$

$$\underset{\sim}{\Psi}^{\mathsf{G}}(t)[u_0] = \underset{\sim}{\Psi}[u_0\, \mathcal{G}_{(t_0,t)}] = \mathcal{G}_{(t_0,t)} \circ \Psi[u_0\, \mathcal{G}_{(t_0,t)}] \circ \mathcal{G}_{(t,t_0)} \ ,$$

and the like for $\underset{\sim}{\mathrm{a}}$ and $\underset{\sim}{\mathrm{a}}^{\dagger}$.

Seeing interpolating field as sections $\boldsymbol{T} \to (\boldsymbol{\mathcal{V}} \oplus_{\boldsymbol{T}} \bar{\boldsymbol{\mathcal{V}}}) \otimes \mathrm{End}(\boldsymbol{\mathcal{Y}})$ one may take their covariant derivatives relatively to the various connections. These derivatives can be studied through contractions of the fields with covariantly constant sections $\boldsymbol{T} \to \boldsymbol{\mathcal{W}}_{\circ}$. In particular, the above formula for $\underset{\sim}{\Psi}^{\underline{\mathsf{G}}}(t)$ shows how the interpolating field $\underset{\sim}{\Psi}$ behaves relatively to the free-field connection.

PROPOSITION 7.1. *One has*

$$\eth[\underline{\mathfrak{G}}]\underset{\sim}{\Psi}^{\mathsf{G}} = \mathcal{U}^{-1}\circ[\Psi,\mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}]\circ\mathcal{U}\ .$$

Proof. Since $\eth[\underline{\mathfrak{G}}^*](u_0\,\underline{\mathcal{G}}_{(t_0,t)})=0$ one has

$$\begin{aligned}\eth[\underline{\mathfrak{G}}]\underset{\sim}{\Psi}^{\mathsf{G}}(t)[u_0] &\equiv \Big\langle \eth[\underline{\mathfrak{G}}]\underset{\sim}{\Psi}(t),\,u_0\,\underline{\mathcal{G}}_{(t_0,t)}\Big\rangle = \partial\Big\langle\underset{\sim}{\Psi}(t),\,u_0\,\underline{\mathcal{G}}_{(t_0,t)}\Big\rangle = \\ &= \partial\left(\mathcal{U}_t^{-1}\circ\Psi[u_0]\circ\mathcal{U}_t\right) = \partial\mathcal{U}_t^{-1}\circ\Psi[u_0]\circ\mathcal{U}_t+\mathcal{U}_t^{-1}\circ\Psi[u_0]\circ\partial\mathcal{U}_t = \\ &= -(\mathcal{U}_t^{-1}\circ\partial\mathcal{U}_t\circ\mathcal{U}_t^{-1})\circ\Psi[u_0]\circ\mathcal{U}_t+\mathcal{U}_t^{-1}\circ\Psi[u_0]\circ\partial\mathcal{U}_t = \\ &= -\mathcal{U}_t^{-1}\circ(\mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}\circ\mathcal{U}_t)\circ\mathcal{U}_t^{-1}\circ\Psi[u_0]\circ\mathcal{U}_t+\mathcal{U}_t^{-1}\circ\Psi[u_0]\circ\mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}\circ\mathcal{U}_t = \\ &= \mathcal{U}_t^{-1}\circ\left(-\mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}\circ\Psi[u_0]+\Psi[u_0]\circ\mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}\right)\circ\mathcal{U}_t\ ,\end{aligned}$$

where the identities $\partial\mathcal{U}=\mathfrak{I}^{\mathsf{G}}_{\underline{\mathsf{G}}}\circ\mathcal{U}$ and $\partial\mathcal{U}=-\mathcal{U}^{-1}\circ\partial\mathcal{U}\circ\mathcal{U}^{-1}$ (following from $\partial(\mathcal{U}^{-1}\circ\mathcal{U})=0$) were used. ☐

8. In practice...

Let $\{\mathsf{B}_{\alpha i}\}$, $\alpha\in\boldsymbol{A}$, $1\le i\le n$ be a family of F-smooth sections $\boldsymbol{T}\to\boldsymbol{\mathcal{V}}$ such that, for each $t\in\boldsymbol{T}$, $\{\mathsf{B}_{\alpha i}(t)\}$ constitutes a generalized orthonormal complete set (the index i is relative to the fibres of $\boldsymbol{V}\to\boldsymbol{E}$, while $\boldsymbol{A}$ is a further index set). The assignment of such a 'generalized frame' determines an F-smooth connection on $\boldsymbol{\mathcal{V}}\to\boldsymbol{T}$, characterized by the requirement that each of its elements is covariantly constant. An arbitrary section $\phi:\boldsymbol{T}\to\boldsymbol{\mathcal{V}}$ can be written as

$$\phi=\phi^{\alpha i}\,\mathsf{B}_{\alpha i}\equiv\sum_{i=1}^{n}\int_{\boldsymbol{A}}\phi^{\alpha i}\,\mathsf{B}_{\alpha i}\,\mathrm{d}\alpha\ ,$$

where the integral is to be intended in a generalized sense and the 'components' $\phi^{\alpha i}$ are sections $\boldsymbol{T}\to\boldsymbol{\mathcal{D}}_{\boldsymbol{T}}(\boldsymbol{E},\mathbb{C}\otimes\mathbb{V}^{-1/2}\boldsymbol{E})$. Usually, a free-field connection will be assigned exactly through the choice of a generalized frame.

The definitions of the operators $\mathsf{a}[u]$ and $\mathsf{a}^{\dagger}[u]$, which were given for any test element u, can be extended by continuity to the case when u is a more general object, such as a distribution, provided that the operators' domains are restricted accordingly. In particular, one considers the sets of *annihilation* and *creation* operators defined by

$$\mathsf{a}^{\alpha i} := \mathsf{a}[\mathsf{B}^{\alpha i}] \;, \quad \mathsf{a}^{\dagger}_{\alpha i} := \mathsf{a}^{\dagger}[\mathsf{B}^{\alpha i}] \;,$$

where $\{\mathsf{B}^{\alpha i}\}$ is the generalized dual frame: $\langle \mathsf{B}^{\alpha i}, \mathsf{B}_{\beta j}\rangle = \delta^{\alpha}{}_{\beta}\,\delta^{i}{}_{j}$. Thus one writes

$$\Psi[u] = u_{\alpha A}\,\mathsf{a}^{\alpha A} + (u^{\dagger})^{\alpha A}\,\mathsf{a}^{\dagger}_{\alpha A} \equiv \sum_{A=1}^{n}\int_{\boldsymbol{A}}\Big(u_{\alpha A}\,\mathsf{a}^{\alpha A} + (u^{\dagger})^{\alpha A}\,\mathsf{a}^{\dagger}_{\alpha A}\Big)\,\mathrm{d}\alpha \;.$$

In a typical QFT, particles must be of at least two kinds, classically described by double bundles $\boldsymbol{F} \to \boldsymbol{P} \to \boldsymbol{M}$ and $\boldsymbol{B} \to \boldsymbol{C} \to \boldsymbol{M}$. Here, $\boldsymbol{P}$ and $\boldsymbol{C}$ are the 'phase bundles' of the particles (a 'dual' approach is also possible, replacing these with 'spatial position bundles' over $\boldsymbol{T}$). The one-particle state bundles are

$$\boldsymbol{\mathcal{F}}^1 := \boldsymbol{\mathcal{D}}_{\boldsymbol{T}}(\boldsymbol{P}, \mathbb{V}^{-1/2}\boldsymbol{P}\otimes\boldsymbol{F}) \rightarrowtail \boldsymbol{T} \;,$$

$$\boldsymbol{\mathcal{B}}^1 := \boldsymbol{\mathcal{D}}_{\boldsymbol{T}}(\boldsymbol{C}, \mathbb{V}^{-1/2}\boldsymbol{C}\otimes\boldsymbol{B}) \rightarrowtail \boldsymbol{T} \;.$$

One then has the Fock bundles

$$\boldsymbol{\mathcal{F}} := \bigoplus_{p=0}^{\infty}\boldsymbol{\mathcal{F}}^p \;, \quad \boldsymbol{\mathcal{B}} := \bigoplus_{p=0}^{\infty}\boldsymbol{\mathcal{B}}^p \;,$$

and, correspondingly, one has two kinds of annihilation and creation operators. The total Fock bundle of the theory is defined to be

$$\boldsymbol{\mathcal{V}} = \overline{\boldsymbol{\mathcal{F}}}\underset{\boldsymbol{T}}{\otimes}\boldsymbol{\mathcal{F}}\underset{\boldsymbol{T}}{\otimes}\overline{\boldsymbol{\mathcal{B}}}\underset{\boldsymbol{T}}{\otimes}\boldsymbol{\mathcal{B}} \;.$$

Next, I'll describe a (non-completely standard) way in which the ideas described in this paper can be implemented, in practice, on a curved spacetime. This approach is based on the notion of 'detector', which is basically just a time-like submanifold $\boldsymbol{T}\subset\boldsymbol{M}$. On an open neighbourhood of $\boldsymbol{T}$, one has a time+spacedecomposition determined through the exponential mapping $(\mathrm{T}^{\perp}\boldsymbol{M})_{\boldsymbol{T}} \to \boldsymbol{M}$, where $(\mathrm{T}^{\perp}\boldsymbol{M})_{\boldsymbol{T}} \subset (\mathrm{T}\boldsymbol{M})_{\boldsymbol{T}}$ is the subbundle over $\boldsymbol{T}$ orthogonal to $\boldsymbol{T}$. If this decomposition turns out to be global (which is certainly the case on flat spacetime), then one can implement a QFT approach where particle states, at any time, can be described as generalised sections on a position space; the corresponding implementation on $(\mathrm{T}^{\perp}\boldsymbol{M})_{\boldsymbol{T}}$ can be seen as a 'linearized' version, but is also well-defined when no global spacetime decomposition can be obtained.

A modification of the above scheme can be formulated by replacing $(\mathrm{T}^{\perp}\boldsymbol{M})_{\boldsymbol{T}}$ with a more interesting phase bundle; I'll need to set this somewhat more precisely than anticipated in §3.

Let $\mathbb{L}$ be the semi-vector space of *length units* (see [5, 2] for a review of unit spaces). The spacetime metric g has 'conformal weight' $\mathbb{L}^2 \cong \mathbb{L}\otimes\mathbb{L}$, i.e. it is a bilinear map $\mathrm{T}\boldsymbol{M}\times_{\boldsymbol{M}}\mathrm{T}\boldsymbol{M}\to\mathbb{L}^2$, while its inverse $g^{\#}$ has conformal weight $\mathbb{L}^{-2}\cong\mathbb{L}^*\otimes\mathbb{L}^*$. For $\mu\in\mathbb{L}^{-1}\cong\mathbb{L}^*$ let $\boldsymbol{P}\cong\boldsymbol{K}^+_{\mu}\subset\mathrm{T}^*\boldsymbol{M}$ be the subbundle over $\boldsymbol{M}$ of all future-pointing $p\in\mathrm{T}^*\boldsymbol{M}$ such that[3] $g^{\#}(p,p)=\mu^2$. This is the classical *phase bundle* for a particle of mass $m=\mu\,\hbar\in\mathbb{M}$, where $\mathbb{M}$ is the unit space of masses and $\hbar\in\mathbb{M}\otimes\mathbb{L}$ (here the speed of light is taken equal to 1, namely proper time is measured in $\mathbb{L}$-units as the g-length of a timelike curve). The phase bundle for a massless particle is the bundle $\boldsymbol{C}\to\boldsymbol{M}$ of *celestial spheres*, whose fibres are the projective spaces of the null cones $\boldsymbol{K}_0\subset\mathrm{T}^*\boldsymbol{M}$.

The distributional bundles of generalized half-densities

$$\boldsymbol{\mathcal{P}}:=\boldsymbol{\mathcal{D}}_{\boldsymbol{M}}\big(\boldsymbol{P},\mathbb{C}\otimes\mathbb{V}^{-1/2}\boldsymbol{P}\big)\rightarrowtail\boldsymbol{M}\ ,$$

$$\boldsymbol{\mathcal{C}}:=\boldsymbol{\mathcal{D}}_{\boldsymbol{M}}\big(\boldsymbol{C},\mathbb{C}\otimes\mathbb{V}^{-1/2}\boldsymbol{C}\big)\rightarrowtail\boldsymbol{M}\ ,$$

will be called the *quantum phase-bundles* of the two kinds of particles. It was proved [4] that the spacetime connection yields quantum connections on each of these bundles; a new, equivalent characterization of those connections is the following.

Consider the massive case first. There is a canonical section $\boldsymbol{M}\to\mathbb{L}^{-1}\otimes\boldsymbol{\mathcal{P}}$, which at each spacetime point is the square root of the naturally induced volume form ω on the fibres of $\boldsymbol{P}$. If $p\in\boldsymbol{P}$, then the Dirac density δ_p is an element in $\boldsymbol{\mathcal{D}}_{\boldsymbol{M}}(\boldsymbol{P},\mathbb{C}\otimes\mathbb{V}^*\boldsymbol{P})$ over the same spacetime point, and $\omega^{-1/2}\otimes\delta_p\in\mathbb{L}\otimes\boldsymbol{\mathcal{P}}$. Next, for any 1-dimensional submanifold $\boldsymbol{I}\subset\boldsymbol{M}$ consider the family of all horizontal sections $p:\boldsymbol{I}\to\boldsymbol{P}_{\boldsymbol{I}}$, relatively to the spacetime connection (which

[3] The signature of the metric is assumed to be $(1,3)$.

can be actually proved to be reducible to P). Then, a connection on $\boldsymbol{\mathcal{P}} \to \boldsymbol{M}$ is defined by the condition that all sections

$$\mathsf{A}_p := l^{-1} \otimes \omega^{-1/2} \otimes \delta_p : \boldsymbol{I} \to \boldsymbol{\mathcal{P}}_{\boldsymbol{I}}$$

be horizontal (where $l \in \mathbb{L}$ is any chosen length unit) for all 1-submanifolds $\boldsymbol{I} \subset \boldsymbol{M}$.

The massless case is slightly more complicated. One has the canonical Leray form on the fibres of $\boldsymbol{K}_0 \to \boldsymbol{M}$, but this does not 'pass to the quotient' to yield a distinguished volume form ξ on the fibres of $\boldsymbol{C} \to \boldsymbol{M}$. One natural way to fixing such a form is by chosing a unit future-pointing vector field—namely an 'observer', according to a certain meaning of the word; spherical coordinates associated with the observer yield coordinates on the fibres of $\boldsymbol{C}$, and one has $\xi = \sin\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi$. If one is only interested to the restriction of $\boldsymbol{C}$ over a detector $\boldsymbol{T}$, then one can simply take the construction induced by the unit vector field tangent to $\boldsymbol{T}$; hence $\sqrt{\xi} : \boldsymbol{T} \to \boldsymbol{\mathcal{C}}$.

It easy to show that the spacetime connection is reducible to a connection on $\boldsymbol{K}_0 \to \boldsymbol{M}$, which on turn yields a connection on $\boldsymbol{C} \to \boldsymbol{M}$. Then, for each horizontal section $c : \boldsymbol{I} \to \boldsymbol{C}_{\boldsymbol{I}}$ one considers

$$\mathsf{B}_c := \xi^{-1/2} \otimes \delta_c : \boldsymbol{I} \to \boldsymbol{\mathcal{C}}_{\boldsymbol{I}} \ ,$$

and the argument proceeds as before. Note that a few, still natural variations of this procedure can also be devised.

If (a_i) is a classical frame in the fibres of $\boldsymbol{F} \to \boldsymbol{P}$, then one gets a generalized frame $\{\mathsf{A}_{pi}\} := \{\mathsf{A}_p \otimes \mathsf{a}_i\}$ in $\boldsymbol{\mathcal{F}}^1$. In the practical cases, a projectable connection on $\boldsymbol{F} \to \boldsymbol{P} \to \boldsymbol{M}$ is given (a horizontal curve in $\boldsymbol{F}$ projects over a horizontal curve in $\boldsymbol{P}$), and one can find a classical frame horizontally transported along any given curve; thus the free-field quantum connection can be viewed as defined by the condition that such generalized frames are horizontally transported.

A similar argument holds in the massless case.

Finally, one can take horizontal transport of these generalized frames along a given detector $\boldsymbol{T}$ in order to implement quantum fields according to the procedure described in the previous sections.

References

[1] D. Canarutto, *Smooth bundles of generalized half-densities*, Archivum Mathematicum **36** (2000), 111–124.

[2] D. Canarutto, *Two-spinors, field theories and geometric optics in curved spacetime*, Acta Appl. Math. **62** (2000), no. 2, 187–224.

[3] D. Canarutto, *Generalized densities and distributional adjoints of natural operators*, Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino **59** (2001), no. 4, 27–36.

[4] D. Canarutto, *Connections on distributional bundles*, Rend. Semin. Mat. Univ. Padova **111** (2004), 71–97.

[5] D. Canarutto, A. Jadczyk, and M. Modugno, *Quantum mechanics of a spin particle in a curved spacetime with absolute time*, Rep. Math. Phys. **36** (1995), 95–140.

[6] A. Frölicher, *Smooth structures*, LNM **962** (1982), 69–81.

[7] A. Frölicher and A. Kriegl, *Linear spaces and differentiation theory*, John Wiley & sons, 1988.

[8] J. Janyška and M. Modugno, *Covariant Schrödinger operator*, J. Phys. A **35** (2002), 8407–8434.

[9] I. Kolář, P. Michor, and J. Slovák, *Natural Operations in Differential Geometry*, Springer, 1993.

[10] A. Kriegl and P. Michor, *The convenient setting of global analysis*, American Mathematical Society, 1997.

[11] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[12] N. Woodhouse, *Geometric Quantization*, Clarendon Press, Oxford, 1992.

Received May 31, 2004.