

# Propagation versus Constancy of Support in the Degenerate Parabolic Equation $u_t = f(u)\Delta u$

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SUMMARY. - *A weak solution concept for the Dirichlet problem in bounded domains for the degenerate parabolic equation*

$$u_t = f(u)\Delta u$$

*is presented. It is shown that if  $\int_0^1 \frac{ds}{f(s)} < \infty$  then each nontrivial nonnegative weak solution eventually becomes positive, while if  $\int_0^1 \frac{ds}{f(s)} = \infty$  then all weak solutions have their support constant in time.*

## 1. Introduction

For parabolic problems of the form

$$\begin{aligned} u_t &= f(u)\Delta u && \text{in } \Omega \times (0, T), \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0, \end{aligned} \tag{1.1}$$

with  $\Omega \subset \mathbb{R}^n$  a smoothly bounded domain,  $0 \not\equiv u_0$  nonnegative and continuous in  $\bar{\Omega}$  and  $0 \leq f \in C^0([0, \infty)) \cap C^1((0, \infty))$ , it is well

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Keywords: Degenerate diffusion, support evolution.

AMS Subject Classification: 35K65, 35K55

known that if  $\inf_{s>0} f(s) > 0$  then any solution becomes positive instantaneously, i.e.  $u(t) > 0$  in  $\Omega$  for all  $t > 0$ . Thus, a diffusion model of type (1.1) which aims at describing the propagation of the support of  $u(t)$  correctly should be *degenerate* at  $s = 0$ , that is,  $f(0) = 0$ . Among these, the one probably best understood is obtained when  $f(s) = s^p$  for some  $0 < p < 1$ ; in this case, namely, the substitution  $v := (1-p)^{\frac{1-p}{p}} u^{1-p}$  transfers (1.1) to the corresponding Dirichlet problem for the porous medium equation

$$v_t = \Delta v^m, \quad m = \frac{1}{1-p} > 1.$$

According to results given e.g. in [7] and in [2], solutions to the porous medium equation have support propagating with finite speed and eventually reaching all of  $\bar{\Omega}$ . In Theorem 5 in [4] it is shown that if merely  $f'(s) \geq 0$  for small positive  $s$  and  $\int_0^1 \frac{ds}{f(s)} < \infty$  then  $u(t) > 0$  in  $\Omega$  for  $t$  sufficiently large. However, the question is left open whether the same is true if  $f$  is not necessarily monotonic near  $s = 0$ .

On the other hand, an analysis of the asymptotic behavior of solutions to the porous medium equation ([7], p.634) reveals that  $\text{supp } u(t) \subset B_{r(t)}(0)$  where  $r(t) = ct^{\frac{1}{2+(m-1)n}}$ . Taking  $m \nearrow \infty$  or, equivalently,  $p \nearrow 1$ , one is led to the conjecture that  $\text{supp } u(t)$  grows very slowly or even remains constant if  $f(s) = s$ . Indeed, constancy of the support has been proved in [5] and in [9] for all “weak solutions” to (1.1) with  $f(s) = s$ . We do not see, however, an obvious way how to extend the weak solution concept used there to “arbitrary” – e.g. non-monotonic –  $f$ .

The present paper thus aims at

- presenting a unified concept of weak solutions fo (1.1) which in the porous medium case coincides with the “traditional” one (Section 2), and
- finding a necessary and sufficient condition on  $f$  distinguishing between eventual positivity on the one hand and constant

support on the other.

More precisely, we shall see in Section 3 that

- if  $f$  is *weakly degenerate* at  $s = 0$  in the sense that

$$\int_0^1 \frac{ds}{f(s)} < \infty,$$

then there is  $T_0 > 0$  such that

$$\text{supp } u(t) = \bar{\Omega} \quad \forall t \geq T_0,$$

while

- if  $f$  is *strongly degenerate* at  $s = 0$ , that is,

$$\int_0^1 \frac{ds}{f(s)} = \infty,$$

then

$$\text{supp } u(t) = \text{supp } u_0 \quad \forall t \geq 0.$$

## 2. Existence and uniqueness of weak solutions

For the sake of convenient notation, let us define  $H : [0, \infty) \rightarrow [-\infty, \infty)$  by

$$Hs := \begin{cases} \int_0^s \frac{d\sigma}{f(\sigma)} & \text{if } \int_0^1 \frac{d\sigma}{f(\sigma)} < \infty, \\ \int_1^s \frac{d\sigma}{f(\sigma)} & \text{if } \int_0^1 \frac{d\sigma}{f(\sigma)} = \infty. \end{cases}$$

Then via  $v = Hu$  the first in (1.1) formally transforms into  $v_t = \Delta H^{-1}v$ . In the weakly degenerate case  $\int_0^1 \frac{ds}{f(s)} < \infty$ , the data are translated into  $v|_{\partial\Omega} = 0$  and  $v|_{t=0} = Hu_0$ . For this type of problem, a successful weak solution concept has been used for a long time and was described in [1], for example. It is obtained in a rather natural way by multiplying the equation for  $v$  by a smooth function  $\varphi(x, t)$  and integrating. Due to the positivity of  $f$  for  $s > 0$ ,  $H^{-1}$

is strictly increasing, which implies uniqueness of weak solutions in this situation ([1], Thm.12). If  $\int_0^1 \frac{ds}{f(s)} = \infty$ , however, (1.1) cannot be transformed into a “nice” parabolic problem of the same type since e.g.  $v$  would attain the boundary values  $H(0) = -\infty$ . Thus, a term like  $\int_0^t \int_{\Omega} Hu \cdot \varphi_t$  in general is meaningless since it seems unnatural to make assumptions on the sign of  $\varphi_t$ .

On the other hand, testing the original equation (1.1) directly by  $\varphi$  will involve the squared gradient term  $\int_0^t \int_{\Omega} f'(u)|\nabla u|^2 \varphi$  which turns out to produce serious difficulties in the corresponding existence theory unless  $f(s) = s$  or, a bit more generally,  $\liminf_{s \rightarrow 0} \frac{sf'(s)}{f(s)} > 0$  (see [9] for  $f(s) = s$ ). Furthermore, even in the particular case  $f(s) = s$  it is shown in [9] that weak solutions obtained in this way are never unique.

To avoid all these problems, we test the equation  $\partial_t Hu = \Delta u$  with a function  $\psi \geq 0$  depending on  $x$  only, and integrate on  $\Omega \times (t_1, t_2)$ , so that all resulting integrals exist at least if we admit them to have values in  $[-\infty, \infty)$ .

DEFINITION 2.1. *A nonnegative function  $u$  from the space*

$$V := L^\infty(\Omega \times (0, T)) \cap C^0([0, T]; L^1(\Omega)) \cap L^2((0, T); W_0^{1,2}(\Omega))$$

*is called a weak solution of (1.1) in  $\Omega \times (0, T)$  if  $u|_{t=0} = u_0$  and for all  $0 \leq t_1 < t_2 < T$  and all  $0 \leq \psi \in W_0^{1,2}(\Omega)$ , the identity*

$$\int_{\Omega} Hu(t_2) \cdot \psi + \int_{t_1}^{t_2} \int_{\Omega} \nabla u \cdot \nabla \psi = \int_{\Omega} Hu(t_1) \cdot \psi \quad (2.1)$$

*holds in  $[-\infty, \infty)$ .*

*A global weak solution is a nonnegative function  $u$  defined in  $\Omega \times [0, \infty)$  which is a weak solution in  $\Omega \times (0, T)$  for all  $T > 0$ .*

We now prove existence of solutions by a standard regularization procedure. It turns out that in the weakly degenerate situation  $\int_0^1 \frac{ds}{f(s)} < \infty$  our concept coincides with the familiar one in [1] and thus solutions are unique. In the delicate case  $\int_0^1 \frac{ds}{f(s)} = \infty$ , however, we gain uniqueness only for “nicely supported”  $u_0$ , but within a smaller solution class.

In the sequel, let us assume that

- (H1)  $\Omega \subset \mathbb{R}^n$  is a bounded domain of class  $C^3$ ,  
(H2)  $u_0 \in C^0(\bar{\Omega}) \cap W_0^{1,2}(\Omega)$  is nonnegative, and  
(H3)  $f \in C^0([0, \infty)) \cap C^1((0, \infty))$  fulfils  $f(0) = 0$  and  $f(s) > 0$  for  $s > 0$ .

**THEOREM 2.1.** *i) Under conditions (H1)-(H3), (1.1) has a global weak solution.*

*ii) If  $\int_0^1 \frac{ds}{f(s)} < \infty$  then the solution is unique.*

*iii) If  $\int_0^1 \frac{ds}{f(s)} = \infty$  and*

(H4) each component of  $\{u_0 > 0\}$  has Lipschitz boundary

*then (1.1) has a weak solution  $u \in V \cap C^0(\bar{\Omega} \times [0, \infty))$  which is unique within this class.*

*Proof.* i) For a sequence  $\varepsilon = \varepsilon_j \searrow 0$ , let  $u_{0,\varepsilon} \in C^3(\bar{\Omega})$  satisfy  $u_{0,\varepsilon} \geq \varepsilon$ ,  $u_0 + \frac{\varepsilon}{2} \leq u_{0,\varepsilon} \leq u_0 + 2\varepsilon$ ,  $u_{0,\varepsilon} \equiv \varepsilon$  in a neighborhood of  $\partial\Omega$ ,  $u_{0,\varepsilon} \searrow u_0$  in  $\Omega$  as  $\varepsilon = \varepsilon_j \searrow 0$  and  $u_{0,\varepsilon} \rightarrow u_0$  in  $W^{1,2}(\Omega)$ . Then the problems

$$\begin{aligned} u_{\varepsilon t} &= f(u_{\varepsilon})\Delta u_{\varepsilon} && \text{in } \Omega \times (0, \infty), \\ u_{\varepsilon}|_{\partial\Omega} &= \varepsilon, \\ u_{\varepsilon}|_{t=0} &= u_{0,\varepsilon}, \end{aligned} \tag{2.2}$$

are actually nondegenerate and thus ([8]) have (unique) classical solutions  $u_{\varepsilon} \in C^{2,1}(\bar{\Omega} \times [0, \infty))$  which by comparison fulfil  $\varepsilon \leq u_{\varepsilon} \leq \|u_0\|_{L^\infty(\Omega)} + 2\varepsilon$  and  $u_{\varepsilon} \geq u_{\eta}$  for  $\varepsilon > \eta$ . Multiplying (2.2) by  $\frac{\psi(x)}{f(u_{\varepsilon})}$  and  $\frac{u_{\varepsilon t}}{f(u_{\varepsilon})}$ , respectively, we obtain for  $0 \leq t_1 < t_2$  and  $t > 0$

$$\int_{\Omega} H u_{\varepsilon}(t_2) \cdot \psi + \int_{t_1}^{t_2} \int_{\Omega} \nabla u_{\varepsilon} \cdot \nabla \psi = \int_{\Omega} H u_{\varepsilon}(t_1) \cdot \psi \quad \forall 0 \leq \psi \in W_0^{1,2}(\Omega) \tag{2.3}$$

and

$$\int_0^t \int_{\Omega} \frac{u_{\varepsilon t}^2}{f(u_{\varepsilon})} + \frac{1}{2} \int_{\Omega} |\nabla u_{\varepsilon}(t)|^2 = \frac{1}{2} \int_{\Omega} |\nabla u_{0,\varepsilon}|^2.$$

Thus, as  $\varepsilon = \varepsilon_j \searrow 0$ ,  $u_{\varepsilon} \searrow u$  in  $\Omega \times [0, \infty)$  as well as

$$\begin{aligned} u_{\varepsilon} - \varepsilon &\rightharpoonup u && \text{in } L_{loc}^2([0, \infty); W_0^{1,2}(\Omega)), \\ u_{\varepsilon t} &\rightharpoonup u_t && \text{in } L_{loc}^2(\bar{\Omega} \times [0, \infty)), \\ u_{\varepsilon} &\rightarrow u && \text{in } L_{loc}^{\infty}([0, \infty); L^p(\Omega)) \quad \forall p \in [1, \infty). \end{aligned}$$

Since by the Beppo-Levi theorem

$$\int_{\Omega} H u_{\varepsilon}(t) \cdot \psi \searrow \int_{\Omega} H u(t) \cdot \psi \in [-\infty, \infty)$$

for all  $0 \leq \psi \in W_0^{1,2}(\Omega)$  and all  $t \geq 0$ , it follows that  $u$  is a global weak solution of (1.1).

ii) In order to prove uniqueness in the case  $\int_0^1 \frac{ds}{f(s)} < \infty$ , we claim that if  $u$  is a weak solution of (1.1) in  $\Omega \times (0, T)$  then for all  $0 \leq \varphi \in W^{1,\infty}(\Omega \times (0, T))$  with  $\varphi|_{\partial\Omega} = 0$ , we have

$$\begin{aligned} \int_{\Omega} H u(t) \cdot \varphi(t) - \int_0^t \int_{\Omega} H u \cdot \varphi_t + \int_0^t \int_{\Omega} \nabla u \cdot \nabla \varphi &= \\ &= \int_{\Omega} H u_0 \cdot \varphi(0) \quad \forall t \in (0, T). \end{aligned}$$

It then follows from [1], Sects. 3 and 4, that  $u$  must be unique.

Indeed, inserting  $\psi := \varphi(s)$  in (2.1) for  $t_1 := s, t_2 := s + h, h > 0$  small, we obtain

$$\begin{aligned} 0 &= \int_0^t \int_{\Omega} \frac{H u(s+h) - H u(s)}{h} \varphi(s) dx ds \\ &\quad + \int_0^t \int_{\Omega} \frac{1}{h} \int_s^{s+h} \nabla u(\sigma) \cdot \nabla \varphi(s) d\sigma dx ds \\ &=: I_1 + I_2. \end{aligned}$$

By [6], Lemma I.3.2 on Steklov averages,

$$\frac{1}{h} \int_s^{s+h} \nabla u(\sigma) d\sigma \rightarrow \nabla u(s) \quad \text{in } L^2(\Omega \times (0, t))$$

as  $h \rightarrow 0$ , while

$$\begin{aligned} I_1 &= \int_0^t \int_{\Omega} \frac{Hu(s+h)\varphi(s+h) - Hu(s)\varphi(s)}{h} dx ds \\ &\quad - \int_0^t \int_{\Omega} Hu(s+h) \frac{\varphi(s+h) - \varphi(s)}{h} dx ds \\ &=: I_{11} + I_{12}. \end{aligned}$$

As  $s \mapsto \int_{\Omega} Hu(s)\varphi(s)$  is continuous in  $[0, t]$ , we have  $I_{11} \rightarrow \int_{\Omega} Hu(t) \cdot \varphi(t) - \int_{\Omega} Hu_0 \cdot \varphi(0)$  as  $h \rightarrow 0$ . Again by [6], Lemma I.3.2,  $\frac{\varphi(\cdot+h) - \varphi(\cdot)}{h} \rightarrow \varphi_t$  in  $L^2(\Omega \times (0, t))$ , whence  $I_{1,2} \rightarrow \int_0^t \int_{\Omega} Hu \cdot \varphi_t$ , from which the claim follows.

iii) For the – rather involved – proof of existence and uniqueness of a *continuous* weak solution under condition (H4), we refer to [10], Thm.1.2.4.  $\square$

REMARK 2.1. *i) It is an open question whether in the strongly degenerate case weak solutions in the sense of Definition 2.1 continue to be unique in  $V$  for general  $u_0$ .*

*ii) In [5] it is demonstrated that indeed the regularity of the boundary of  $\{u_0 > 0\}$  is in close relation to the equicontinuity properties of the sequence  $(u_{\varepsilon_j})_{j \in \mathbb{N}}$ . It is not clear whether (1.1) possesses a continuous weak solution if e.g.  $\{u_0 = 0\}$  contains isolated points.*

### 3. Positivity versus localization

Evidently, if existing at all, conditions on  $f$  ensuring either propagation or constancy of  $\text{supp } u(t)$  should reflect in some sense the growth of  $f$  near zero since an  $f$  that is very small near  $s = 0$  should

be able to inhibit diffusion very effectively near the boundary of the support. Fortunately, our choice of measure of degeneracy distinguishing between weak and strong degeneracies (that is, the value  $\int_0^1 \frac{ds}{f(s)}$ ) turns out to be the appropriate one for this purpose.

As a technical premise, let us define the support of an a.e. defined function  $v : \Omega \rightarrow \mathbb{R}$  by

$$\text{supp } v := \bigcap_{N \subset \Omega, |N|=0} \overline{\{x \in \Omega \setminus N \mid v(x) \neq 0\}}.$$

### 3.1. Eventual positivity in presence of weak degeneracies

As already mentioned before, eventual positivity of  $u$  was proved in [4] under the additional assumption that  $f'$  be nonnegative near  $s = 0$  (which is equivalent to  $H^{-1}$  being convex there). The authors in this reference essentially make use of a semi-convexity estimate  $\Delta u(t) \geq -C$  for  $t$  large which holds for such  $f$  in case of suitable initial data. From this, the main part of their result follows upon an elliptic comparison argument. Being similar in spirit through large passages, our proof, as compared to theirs, does not involve an estimate for  $\Delta u$  and hence relies on no monotonicity hypothesis on  $f$ .

The most important step is done in

**LEMMA 3.1.** *Suppose  $\int_0^1 \frac{ds}{f(s)} < \infty$  and  $u$  is the weak solution to (1.1). Then for all  $R \in (0, 1)$  and  $\nu > 0$  there are constants  $T_1 = T_1(R, \nu)$  and  $c = c(R, \nu)$  such that*

$$\begin{aligned} & u(t_0) \geq \nu \text{ a.e. in } B_{\frac{R}{4}}(x_0) \text{ for some } (x_0, t_0) \in \Omega \times (0, \infty) \\ \Rightarrow & u(x, t_0 + T_1) \geq c(\text{dist}(x, \partial B_R(x_0)))^2 \quad \forall x \in B_R(x_0) \text{ and} \\ & u(t) > 0 \quad \text{in } B_R(x_0) \quad \forall t \geq t_0 + T_1. \end{aligned}$$

*Proof.* We may assume without loss of generality that  $(x_0, t_0) = (0, 0)$  and abbreviate  $B_\rho := B_\rho(0)$ . Fix  $0 \leq \varphi \in C_0^\infty([0, \frac{R}{4}])$  with  $\varphi' \leq 0$  and  $\varphi \equiv \nu$  in  $[0, \frac{R}{8}]$ . Let  $\phi(x) := \varphi(|x|)$  for  $x \in B_R$  and



$$c_1 := \int_{B_{\frac{R}{4}}} H\phi. \quad (3.1)$$

By a standard comparison argument, it suffices to prove the claim for the case  $\Omega = B_R$  and  $u_0 = \phi$ , in which the solution clearly will be radially symmetric and decreasing w.r. to  $|x|$  for all  $t$ . In this case, dividing (2.2) by  $f(u_\varepsilon)$  and integrating, for  $T > 0$  and  $r \in (\frac{R}{2}, R)$  we find

$$\int_{B_r} H u_\varepsilon(T) - \int_{B_r} H u_{0,\varepsilon} = \int_0^T \int_{\partial B_r} \partial_N u_\varepsilon. \quad (3.2)$$

Now fix  $\varepsilon_0 \in (0, 1)$  such that  $|B_R| \cdot H(3\varepsilon_0) < \frac{c_1}{2}$  and let  $v(x, t) := \varepsilon + \varepsilon_0 + y(t)e(x)$ , where  $e$  is the solution of  $-\Delta e = 1$  in  $B_R$ ,  $e|_{\partial B_R} = 1$ , and  $y(t) := \|\phi\|_{L^\infty(B_R)} e^{-\alpha t}$  with  $\alpha := \|e\|_{L^\infty(B_R)}^{-1} \cdot \min\{f(s) \mid s \in [\varepsilon_0, 2 + \|\phi\|_{L^\infty(B_R)} \cdot \|e\|_{L^\infty(B_R)}]\}$ . Then  $v \geq u_\varepsilon$  at  $t = 0$  and on  $\partial B_R$  and

$$\begin{aligned} v_t - f(v)\Delta v &= y'e + f(\varepsilon + \varepsilon_0 + ye)v \\ &\geq \|e\|_{L^\infty(B_R)} \cdot [y' + \alpha y] = 0 \quad \text{in } B_R \times (0, T), \end{aligned}$$

whence

$$u_\varepsilon \leq \varepsilon + \varepsilon_0 + c_2 e^{-\alpha t} \quad \text{in } B_R \times (0, T).$$

Thus, if we choose  $T_1$  large such that  $c_2 e^{-\alpha T_1} < \varepsilon_0$ , we have

$$u_\varepsilon(T_1) < 3\varepsilon_0 \quad \text{in } B_R \quad \forall \varepsilon < \varepsilon_0,$$

so that, by (3.2) and (3.1),

$$\int_0^{T_1} \int_{\partial B_r} \partial_N u_\varepsilon \leq |B_r| \cdot H(3\varepsilon_0) - c_1 \leq -\frac{c_1}{2} \quad \forall \varepsilon < \varepsilon_0.$$

Writing  $z(r) := \int_0^{T_1} \int_{\partial B_r} u_\varepsilon$ , we infer from the radial symmetry of  $u_\varepsilon$  that

$$\begin{aligned} z'(r) &= \int_0^{T_1} \int_{\partial B_r} \partial_N u_\varepsilon + \frac{n-1}{r} z(r) \\ &\leq -\frac{c_1}{2} + c_3 z(r) \quad \forall r \in \left(\frac{R}{2}, R\right). \end{aligned}$$

As  $z(R) \geq 0$ , this implies by ODE comparison

$$\begin{aligned} z(r) &\geq \frac{c_1}{2c_3}(1 - e^{-c_3(R-r)}) \\ &\geq c_4(R-r) \quad \forall r \in \left(\frac{R}{2}, R\right). \end{aligned}$$

Hence, for any  $r \in (\frac{R}{2}, R)$  there exists  $t_r \in (0, T_1)$  such that

$$\int_{\partial B_r} u_\varepsilon(T_r) \geq \frac{c_4(R-r)}{T_1},$$

so that, since  $u_\varepsilon$  decreases in  $|x|$ ,

$$u_\varepsilon(t_r) \geq \frac{c_4}{T_1|\partial B_R|}(R-r) =: c_5(R-r) \quad \text{in } B_r.$$

To see that  $u$  remains positive in  $B_r$ , let  $\Theta_r(x)$  denote the principal eigenfunction of  $-\Delta$  in  $B_r$  with  $\max \Theta_r = 1$ , corresponding to the first eigenvalue  $\lambda_r$  which decreases from  $\lambda_{\frac{R}{2}}$  to  $\lambda_R$  as  $r$  increases from  $\frac{R}{2}$  to  $R$ . Set  $w(x, t) := y_1(t)\Theta_r(x)$  with  $y_1(t) := c_5(R-r)e^{-\beta(t-t_r)}$ , where  $\beta := \lambda_{\frac{R}{2}} \cdot \max\{f(s) \mid s \in [0, c_5(R-r)]\}$ . Then  $u_\varepsilon \geq w$  at  $t = t_r$  and on  $\partial B_r$  and as

$$\begin{aligned} w_t - f(w)\Delta w &= y_1' \Theta_r + f(y_1 \Theta_r) y_1 \lambda_r \Theta_r \\ &\leq [y_1' + \beta y_1] \Theta_r = 0 \quad \text{in } B_r \times (t_r, \infty), \end{aligned}$$

we find  $u_\varepsilon \geq w$  and thus, letting  $\varepsilon \searrow 0$ , that  $u > 0$  in  $B_r \times (T_1, \infty)$  and

$$u(T_1) \geq c_6(R-r)\Theta_r \quad \text{in } B_r$$

with  $c_6 = c_5 e^{-\beta T_1}$ . To conclude, fix  $x \in B_R$  and choose  $r := \frac{R+|x|}{2}$ . As  $\Theta_r(x) \geq c_7(r - |x|)$  for all  $x \in B_r$  (with  $c_7$  independent of  $r \in (\frac{R}{2}, R)$ ), we obtain

$$u(x, T_1) \geq c_6 \frac{R - |x|}{2} \cdot c_7 \frac{R - |x|}{2} \geq c(\text{dist}(x, \partial B_R))^2,$$

which proves the claim, since all constants depend on  $R$  and  $\nu$  only.  $\square$

REMARK 3.2. *If  $f' \geq 0$  near  $s = 0$ , Lemma 4 in [4] gives the stronger estimate  $u(x, t_0 + T_1) \geq c \text{dist}(x, \partial B_R(x_0))$ ; we do not know, whether this holds true in the general situation. For the relevance of this question concerning large time behavior of solutions, see [2].*

Although the proof of the following theorem is a rather straightforward iterated application of Lemma 3.1 as performed in [2], we include a short proof for the sake of completeness.

THEOREM 3.1. *Suppose*

$$\int_0^1 \frac{ds}{f(s)} < \infty.$$

*Then there is  $T_0 > 0$  such that the weak solution of (1.1) satisfies*

$$u(x, t) > 0 \quad \forall x \in \Omega, \quad \forall t \geq T_0.$$

*In particular,*

$$\text{supp } u(t) = \bar{\Omega} \quad \forall t \geq T_0.$$

*Proof.* Let us fix  $\delta \in (0, \frac{1}{4})$  small enough such that

- (i)  $\exists x_0 \in \Omega$  with  $\text{dist}(x_0, \partial\Omega) > 4\delta$  and  $u_0 > 0$  in  $\bar{B}_\delta(x_0)$ ;
- (ii)  $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}$  is connected (see [2], Sect. 1);
- (iii)  $\forall y \in \partial\Omega$  there is  $x_y \in \Omega$  such that  $\bar{B}_{4\delta}(x_y) \cap \partial\Omega = \{y\}$ .

By compactness, there exist  $x_1, \dots, x_N \in \bar{\Omega}_\delta$  such that

$$\bar{\Omega}_\delta \subset \bigcup_{i=0}^N B_\delta(x_i) \subset \Omega,$$

where due to (ii) we may assume that  $x_i \in \bigcup_{j=1}^{i-1} B_\delta(x_j)$  for all  $i = 1, \dots, N$ . From (i) we obtain  $u_0 \geq \nu_0$  in  $B_\delta(x_0)$  for some  $\nu_0 > 0$  and thus Lemma 3.1 provides  $t_1 > 0$  and  $\nu_1 \in (0, \nu_0)$  such that  $u(t_1) \geq \nu_1$

in the ball  $B_{3\delta}(x_0)$  which contains  $B_\delta(x_1)$ , so that induction over  $i = 1, \dots, N$  finally yields  $t_N > 0$  and  $\nu_N > 0$  such that

$$u(t_N) \geq \nu_N \quad \text{in } \Omega_\delta \quad (3.3)$$

and

$$u(t) > 0 \quad \text{in } \Omega_\delta \quad \forall t \geq t_N. \quad (3.4)$$

Next, let us take  $T_1 = T_1(4\delta, \nu_N)$  from Lemma 3.1. Then for any  $y \in \partial\Omega$ , (3.3) implies  $u(t_N) \geq \nu_N$  in  $B_\delta(x_y) \subset \Omega_\delta$ , whence  $u(t) > 0$  in  $B_{4\delta}(x_y)$  for all  $t \geq t_N + T_1$  by Lemma 3.1. As  $\Omega \setminus \Omega_\delta \subset \bigcup_{y \in \partial\Omega} B_{4\delta}(x_y)$ , this gives together with (3.4)

$$u(t) > 0 \quad \text{in } \Omega \quad \forall t \geq t_N + T_1$$

and thereby proves the theorem.  $\square$

**REMARK 3.3.** *The proof shows that actually  $u$  is a classical positive solution of (1.1) in  $\Omega \times (T_0, \infty)$ .*

### 3.2. Constant support in the case of strong degeneracies

In the special case  $f(s) = s$ , the very detailed studies of the positivity properties of  $u = \lim u_\varepsilon$  in [3] and in [5] show that such solutions in fact have their support constant in time, while, roughly speaking, the set  $\{u(t) > 0\}$  may increase with  $t$  if e.g.  $u_0$  has an isolated zero at  $x_0 \in \Omega$  at which  $u$  eventually becomes positive. Some of the proofs given there employ a further semi-convexity estimate  $\Delta u(t) \geq -\frac{\varepsilon}{t}$ . Such a type of estimate, however, seems to be available only for certain  $f$  (satisfying  $\liminf_{s \rightarrow 0} \frac{sf'(s)}{f(s)} > 0$ ) and in addition only for limits  $u$  of positive solutions to (1.1) (cf. [11], Lemma 2.2).

For general  $f$  being strongly degenerate at  $s = 0$ , we do not know whether weak solutions are unique in general and thus find it desirable not to rely on any kind of approximation.

THEOREM 3.2. *Suppose*

$$\int_0^1 \frac{ds}{f(s)} = \infty.$$

*Then any weak solution of (1.1) has constant support:*

$$\text{supp } u(t) = \text{supp } u_0 \quad \forall t > 0. \quad (3.5)$$

*Proof.* i) Fix  $t > 0$  and assume first that  $u(t) \in W_0^{1,2}(\Omega)$ . Then for any  $\delta > 0$ , the function  $\psi(x) := (u(x, t) - \delta)_+$  is in  $W_0^{1,2}(\Omega)$ . Inserting  $\psi$  into (2.1) yields

$$\begin{aligned} \int_{\Omega} Hu(t) \cdot (u(t) - \delta)_+ + \int_0^t \int_{\Omega} \nabla u(s) \cdot \nabla (u(t) - \delta)_+ &= \\ &= \int_{\Omega} Hu_0 \cdot (u(t) - \delta)_+. \end{aligned}$$

In particular, it follows that the term on the right is finite, so that

$$|\{u(t) > \delta\} \cap \{u_0 = 0\}| = 0 \quad \forall \delta > 0,$$

hence also

$$|\{u(t) > 0\} \cap \{u_0 = 0\}| = 0,$$

that is,

$$u(t) = 0 \text{ in } \{u_0 = 0\} \setminus N(t), \quad (3.6)$$

where  $N(t) \subset \Omega$  has measure zero.

We claim that (3.6) is in fact valid for all  $t > 0$ . Indeed, any such  $t$  is the limit of a sequence of times  $t_k > 0$  for which  $u(t_k) \in W_0^{1,2}(\Omega)$ . As  $u$  is continuous as an  $L^1(\Omega)$ -valued function, it follows that  $u(t_k) \rightarrow u(t)$  in  $\Omega \setminus N_0$  for a subsequence and some  $N_0 \subset \Omega$  with  $|N_0| = 0$ , so that (3.6) will be proved if we let  $N(t) := N_0 \cup \bigcup_{k \in \mathbb{N}} N(t_k)$ .

To see that (3.6) implies

$$\text{supp } u(t) \subset \text{supp } u_0, \quad (3.7)$$

fix  $N \subset \Omega$  with  $|N| = 0$ . Then, by complementing (3.6),

$$\{u(t) > 0\} \setminus (N(t) \cup N) \subset \{u_0 > 0\} \setminus N,$$

and we arrive at (3.7) upon taking closures and intersecting over all such  $N$ .

ii) Conversely, if  $0 \leq \psi \in C_0^\infty(\{u_0 > 0\})$  then (2.1) implies for all  $t > 0$

$$\int_{\Omega} Hu(t) \cdot \psi = \int_{\Omega} Hu_0 \cdot \psi + \int_0^t \int_{\Omega} \nabla u \cdot \nabla \psi > -\infty,$$

so that  $u(t) > 0$  a.e. in  $\{\psi > 0\}$  and hence, due to continuity of  $u_0$ , a.e. in  $\{u_0 > 0\}$ , which gives, in much the same manner as in part i),

$$\text{supp } u_0 \subset \text{supp } u(t),$$

and the proof is complete.  $\square$

REMARK 3.4. *By comparison with  $ce^{-\gamma t}\Theta_B$  for eigenfunctions  $\Theta_B$  of  $-\Delta$  in balls  $B \subset\subset \{u_0 > 0\}$  and suitable positive constants  $c$  and  $\gamma$ , it can easily be checked (cf. the proof of Lemma 3.1) that the particular solution  $u = \lim u_\varepsilon$  fulfils (like that in [5] for  $f(s) = s$ )*

$$\{u_0 > 0\} \subset \{u(t) > 0\} \subset \text{supp } u_0 \quad \forall t > 0,$$

*which is in fact sharper than (3.5) and asserts that such  $u$  cannot develop singularities of dead core type.*

*It remains an open question whether  $u = \lim u_\varepsilon$  is continuous in  $\bar{\Omega} \times [0, \infty)$  at least for compactly supported initial data with nice positivity set; in the case that e.g.  $\{u_0 > 0\}$  is a smooth subdomain of  $\Omega$ , continuity would yield the sharp result  $\{u(t) > 0\} = \{u_0 > 0\}$  for all  $t > 0$ . An affirmative answer to this in space dimension one was given in [3], Proposition 2.2.*

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Received January 13, 2003.