

# Blow-up for Semilinear Wave Equations with a Data of the Critical Decay having a Small Loss

YUKI KUROKAWA AND HIROYUKI TAKAMURA (\*)

SUMMARY. - *It is known that we have a global existence for wave equations with super-critical nonlinearities when the data has a critical decay of powers. In this paper, we will see that a blow-up result can be established if the data decays like the critical power with a small loss such as any logarithmic power. This means that there is no relation between the critical decay of the initial data and the integrability of the weight, while the critical power of the nonlinearity is closely related to the integrability. The critical decay of the initial data is determined only by scaling invariance of the equation. We also discuss a nonexistence of local in time solutions for the initial data increasing at infinity.*

## 1. Introduction

We are concerned with classical solutions of the following initial value problem for semilinear wave equations. For a scalar unknown func-

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(\*) Authors' addresses: Yuki Kurokawa, 4th year of the graduate course, Institute of Mathematics, University of Tsukuba, Tsukuba Ibaraki 305-8571, Japan. e-mail: [kurokawa@math.tsukuba.ac.jp](mailto:kurokawa@math.tsukuba.ac.jp)

Hiroyuki Takamura, Institute of Mathematics, University of Tsukuba, Tsukuba Ibaraki 305-8571, Japan. *During 1/7/2002-30/6/2003:* Università di Pisa, Dipartimento di Matematica "L.Tonelli", Via F.Buonarroti 2, Pisa 56127, Italy. e-mail: [takamura@math.tsukuba.ac.jp](mailto:takamura@math.tsukuba.ac.jp).

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tion  $u = u(x, t)$ , we shall investigate

$$\begin{cases} \square u = F(u, \partial_t u, \nabla_x u) & \text{in } \mathbf{R}^n \times [0, \infty), \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \end{cases} \quad (1)$$

where  $\square = \partial^2/\partial t^2 - \Delta_x$  and  $u_0, u_1$  are given smooth functions. The general theory can be discussed only for small amplitude solutions, so that we assume that  $F$  is a power nonlinearity. As for the initial data, most of existence theorems for (1) require the compactly supported data. By finiteness of the propagation speed of solutions, it seems to be removable assumption. But in fact we have the following stories.

First we consider the case where the nonlinearity includes only unknown function itself, for example

$$F = |u|^p, \quad \text{or} \quad F = |u|^{p-1}u \quad (2)$$

with  $p > 1$ . For the compactly supported data, we know the following Strauss' conjecture. See Section 4 in W.A.Strauss[22]. If  $p > p_0(n)$ , (1) of (2) has a global solution for "small" data. If  $1 < p \leq p_0(n)$ , (1) of (2) has no global solution for "positive" data. The critical number  $p_0(n)$  is a positive root of the following quadratic equation;

$$(n-1)p^2 - (n+1)p - 2 = 0 \quad (3)$$

which comes from the integrability in the iteration of a weight  $(1 + |t - |x||)^{(n-1)p/2 - (n+1)/2}$ . This weight also comes from the iteration of the decay of a solution to free equation;  $(1 + t + |x|)^{(n-1)/2}$ . This conjecture was first verified by F.John[11] for  $n = 3$  and by R.T.Glassey[6][5] for  $n = 2$ , except for the critical case. The critical case was studied by J.Schaeffer[19] for  $n = 2, 3$ . For higher dimensions, T.C.Sideris[21] proved the sub-critical case. There are many partial results on the super-critical case, but the final proof was given by V.Georgiev & H.Lindblad & C.Sogge[3]. The critical case for  $n \geq 4$  is open. We note that the non-existence result for  $n \geq 4$  is proved only for a positive nonlinearity  $|u|^p$ , and that we should consider a weak solution near the critical power due to the lack of the differentiability of the nonlinearity.

For the noncompactly supported data, we may have a nonexistence result even for the super-critical case. Actually, for  $p > p_0(n)$ ,

we know that (1) of (2) has no global solution provided the initial data satisfies the following condition.

$$u_0(x) \equiv 0, \quad u_1(x) \geq \frac{\varepsilon}{(1+|x|)^{1+\kappa}} \quad \text{with } 0 < \kappa < \kappa_0 \equiv \frac{2}{p-1}, \quad (4)$$

where  $\varepsilon$  is any positive constant.  $\kappa_0$  is the critical decay in the following sense. (1) of (2) with  $p > p_0(n)$  has a global solution provided

$$(1+|x|)^{\kappa+1} \left( \sum_{|\alpha| \leq [n/2]+2} |\nabla_x^\alpha u_0(x)| + \sum_{|\beta| \leq [n/2]+1} |\nabla_x^\beta u_1(x)| \right) \quad (5)$$

with  $\kappa \geq \kappa_0$

is small enough. This fact was first verified by F.Asakura[2] for  $n = 3$  except for the critical decay. The critical case was studied by K.Kubota[16], or independently by K.Tsutaya[26]. The two dimensional case was verified by R.Agemi & H.Takamura[1] for the nonexistence part and K.Kubota[16] for the existence part, or independently both parts by K.Tsutaya[24][25]. In higher dimensional case, only a radially symmetric solution was studied. But the nonexistence part was verified by H.Takamura[23], and also the existence for odd  $n$  by H.Kubo[15]. Note that, in the nonexistence case, we have an estimate of the lifespan  $T(\varepsilon)$  of a solution by making use of long-time existence under (5) with  $0 < \kappa < \kappa_0$ . More precisely, let  $u_0(x) = \varepsilon f(x), u_1(x) = \varepsilon g(x)$ , where  $\varepsilon$  is a positive parameter and  $f, g$  are given smooth functions. Then there exist positive constants  $c$  and  $C$  independent of any small  $\varepsilon$  such that

$$c\varepsilon^{-1/(\kappa_0-\kappa)} \leq T(\varepsilon) \leq C\varepsilon^{-1/(\kappa_0-\kappa)} \quad \text{when } 0 < \kappa < \kappa_0 \quad (6)$$

holds with arbitrarily fixed  $f, g$  for the estimate from below, and with some special data for the estimate from above.

We find many similarities between compactly supported case and for noncompactly supported case. So naturally one may have that the critical decay of powers is not a real critical decay from the viewpoint of the integrability. More precisely, we may have the critical decay of  $l > 0$  in the following condition which guarantees the global

existence instead of (5).

$$\frac{(1 + |x|)^{\kappa_0+1}}{\log^l(2 + |x|)} \left( \sum_{|\alpha| \leq [n/2]+2} |\nabla_x^\alpha u_0(x)| + \sum_{|\beta| \leq [n/2]+1} |\nabla_x^\beta u_1(x)| \right) \ll 1. \tag{7}$$

But we do have a negative answer for this conjecture.

We now consider the radially symmetric version of the problem.

$$\begin{cases} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r} \right) u(r, t) = F_p(u(r, t)) & \text{in } [0, \infty)^2, \\ u(r, 0) = u_0(r), \quad u_t(r, 0) = u_1(r), \end{cases} \tag{8}$$

where  $F_p \in C^1(\mathbf{R})$  satisfies

$$F_p(u) \geq Au^p \quad \text{with a constant } A > 0 \text{ for } u \geq 0. \tag{9}$$

Then we have the following result.

**THEOREM 1.1.** *Assume that*

$$u_0(r) \equiv 0, \quad u_1(r) \geq \frac{\phi(r)}{(1+r)^{1+\kappa_0}}, \tag{10}$$

where  $\phi$  is a positive and monotonously increasing function in  $[0, \infty)$ . Then (8) with (9) admits no global  $C^2$ -solution if

$$\lim_{r \rightarrow \infty} \phi(r) = \infty \tag{11}$$

**REMARK 1.2.** *One can put*

$$\phi(r) = \varepsilon \log^l(2+r) \quad \text{with arbitrarily fixed } \varepsilon > 0. \tag{12}$$

In this case (11) holds for any  $l > 0$ . Therefore the critical decay  $\kappa_0$  is not related to any integrability. We note that  $\kappa_0$  is the number of the scaling invariance since  $u^R(x, t) = R^{\kappa_0} u(Rx, Rt)$  with  $R > 0$  is a solution of the equation if so is  $u(x, t)$ . This observation is regarded as a self-similarity of solutions. See the existence result in H.Pecher[18], or in K.Hidano[10].

The proof of Theorem1 also gives us the following nonexistence result of local in time solutions for the increasing data. This is closely related to large amplitude solutions which was discussed in R.T.Grassey[4], or in H.Levine[17].

COROLLARY 1.3. *Suppose that the same assumptions as in Theorem1 are fulfilled. Assume that there exists a function  $\phi_0$  such that*

$$\phi(r) = (1+r)^{1+\kappa_0} \phi_0(r), \quad \lim_{r \rightarrow \infty} \phi_0(r) = \infty. \quad (13)$$

*Then (8) with (9) admits no  $C^2$ -solution till any positive time.*

According to Remark1.2, one may have an estimate of the lifespan also in this case. Actually we have the following result for three space dimensions. Let us consider

$$\begin{cases} \square u = G_p(u) & \text{in } \mathbf{R}^3 \times [0, \infty), \\ u|_{t=0} = \varepsilon f, \quad u_t|_{t=0} = \varepsilon g, \end{cases} \quad (14)$$

where  $\varepsilon > 0$  is a parameter,  $f \in C^3(\mathbf{R}^3)$  and  $g \in C^2(\mathbf{R}^3)$  satisfy

$$(1+|x|)^{\kappa_0+1} \left( \sum_{|\alpha| \leq 3} |\nabla_x^\alpha f(x)| + \sum_{|\beta| \leq 2} |\nabla_x^\beta g(x)| \right) \leq \psi(|x|). \quad (15)$$

Here  $\psi \in C^1([0, \infty))$  satisfies the following conditions.

$$\begin{cases} \psi > 0, \quad \psi' > 0, \quad \lim_{r \rightarrow \infty} \psi(r) = \infty. \\ \text{There exist constants } \delta > 0 \text{ and } K > 0 \text{ such that} \\ \quad (1+r)\psi'(r) \leq K\psi(r)^{1-\delta} \text{ for } r \in [0, \infty). \\ \text{For any } K_0 > 0, \text{ there exists } K_1 > 0 \text{ such that} \\ \quad \psi(K_0 r) \leq K_1 \psi(r) \text{ for } r \in [0, \infty). \end{cases} \quad (16)$$

The assumption on the nonlinearity  $G_p \in C^2(\mathbf{R})$  is the following.

There exist  $p > 1 + \sqrt{2}$  and  $A_1 > 0$  such that,

for  $|s|, |s_1|, |s_2| \leq 1$ ,

$$\begin{aligned} & |G_p^{(j)}(s)| \leq A_1 |s|^{p-j} \quad (j = 0, 1, 2), \\ & |G_p''(s_1) - G_p''(s_2)| \\ & \leq A_1 p(p-1) \begin{cases} (\max\{|s_1|, |s_2|\})^{p-3} |s_1 - s_2| & \text{if } p \geq 3, \\ |s_1 - s_2|^{p-2} & \text{if } p \leq 3. \end{cases} \end{aligned} \quad (17)$$

Then we have the long-time existence.

**THEOREM 1.4.** *Let  $p > p_0(3) = 1 + \sqrt{2}$ . Assume that (15) and (16) on the initial data, and that (17) on the nonlinearity. Then there exists  $\varepsilon_0 = \varepsilon_0(p, \psi, A_1) > 0$  such that, for any  $0 < \varepsilon \leq \varepsilon_0$ , the problem (14) admits a unique  $C^2$ -solution  $u(x, t)$  in the time interval  $[0, T]$  as far as  $T$  satisfies*

$$T \leq \psi^{-1}(c\varepsilon^{-1}), \quad (18)$$

where  $c$  is a positive constant depending on  $p, \psi, A_1$ .

Consequently we have an estimate of the lifespan of the solution in the special case.

**COROLLARY 1.5.** *Let  $G_p(u) = A|u|^{p-1}u$ , or  $A|u|^p$  with  $A > 0$ . Assume (15), (16), and that there exists a constant  $g_0 > 0$  such that*

$$f(x) \equiv 0, \quad g(x) = g(|x|) \geq \frac{g_0\psi(|x|)}{(1+|x|)^{1+\kappa_0}}. \quad (19)$$

Then there exists  $\varepsilon_0 = \varepsilon_0(p, \psi, g_0, A) > 0$  such that, for any  $0 < \varepsilon \leq \varepsilon_0$ , the lifespan  $T(\varepsilon)$ , the maximal existence time, of  $C^2$ -solution of (14) satisfies

$$\psi^{-1}(c\varepsilon^{-1}) \leq T(\varepsilon) \leq \psi^{-1}(C\varepsilon^{-1}), \quad (20)$$

where  $c, C$  ( $c < C$ ) are positive constants depending on  $p, g_0, \psi, A$ .

**REMARK 1.6.** *Taking the spherical mean of  $u(x, t)$ , we can remove the assumption of the spherical symmetricity on  $g$  in Corollary 1.5. See F.John[11] for example.*

**REMARK 1.7.** *The second condition on  $\psi$  in (16) implies that*

$$\psi(r) \leq \left[ \psi(0)^\delta + \delta K \log(1+r) \right]^{1/\delta}. \quad (21)$$

Hence  $\psi(r) = \log^l(3+r)$  with  $l > 0$  is admissible. In this case we have  $T(\varepsilon) \sim \exp(Ce^{-1/l})$ . Also  $\psi(r) = \log(3 + \log(3+r))$  is admissible. In this case we have  $T(\varepsilon) \sim \exp(\exp(Ce^{-1}))$ .

REMARK 1.8. *One can check the compatibility between  $\psi$  in (16) and  $\phi$  in Corollary 1.3 as follows. In Corollary 1.3 we assume that  $\phi \in C^1$ . Suppose that we can put  $\psi = \phi$ . Since*

$$\phi'(r) = (1 + \kappa_0)(1 + r)^{\kappa_0} \phi_0(r) + (1 + r)^{1+\kappa_0} \phi_0'(r), \quad (22)$$

*the second line in (16) implies that*

$$1 + \kappa_0 + \frac{(1 + r)\phi_0'(r)}{\phi_0(r)} \leq K\phi(r)^{-\delta}. \quad (23)$$

*But  $\lim_{r \rightarrow \infty} \phi(r) = \infty$  and  $\phi_0 > 0$  show  $\phi_0'(r) < 0$  for large  $r$ . This is a contradiction to  $\lim_{r \rightarrow \infty} \phi_0(r) = \infty$ . Therefore it is impossible to set  $\psi(r) = (1 + r)^{1+\kappa_0} \phi_0(r)$ .*

Next we consider (1) in the case where the nonlinearity includes only time-derivative of unknown functions,

$$F = |u_t|^p, \quad \text{or} \quad F = |u_t|^{p-1} u_t \quad (24)$$

with  $p > 1$ . In this case we also have the similar result to the equation of  $u$  itself, (2). For (24), one can discuss long-time existence in  $L^2$  framework if the nonlinearity is smooth, but the precise behaviour on the support will be lost. So we should outline the radially symmetric case in three dimensions here. We note that the critical decay is clarified only in this situation. The results for other dimensions are cited in Introduction in H.Takamura[23].

For the compactly supported data, F.John[12] proved that (1) with (24) has no non-trivial global solutions provided  $1 < p \leq 2$ . T.C.Sideris[20] proved the counter part, the existence of a unique global  $C^2$ -solution for any small data having compact support provided  $p > 2$ . This critical power is regarded as  $(n + 1)/(n - 1)$  by general theory. For the noncompactly supported case, we have the following results. Even if  $p > 2$ , (1) with (24) in three space dimensions has no global  $C^2$ -solution provided

$$u_0(r) \equiv 0, \quad u_1(r) \geq \frac{\varepsilon}{(1 + r)^{1+\kappa}} \quad \text{with} \quad 0 < \kappa < \kappa_1 \equiv \frac{2 - p}{p - 1}, \quad (25)$$

where  $r = |x|$  and  $\varepsilon$  is any positive constant.  $\kappa_1$  is the critical decay in the following sense. (1) of (24) with  $p > 2$  in three space dimensions has a global  $C^2$ -solution provided

$$(1+r)^{\kappa+1}[|u_0'(r)| + |u_1(r)| + |(ru_0(r))'''| + |(ru_1(r))''| + (1+r)(|u_0''(r)| + |u_1'(r)|)] \quad (26)$$

with  $\kappa \geq \kappa_1$  is small enough. We note again that  $\kappa_1$  is also related to the scaling invariance of the equation. See Remark1.2. Except for the critical case, the results above were proved by K.Hidano[8][9], or independently by H.Kubo[13][14]. The critical case has been studied by K.Hidano[7]. We note that the results were established for more general situation including the nonlinearity of  $|u_r|^p$ .

Taking into account of the equation (24), we shall extend Theorem1 to the following problem.

$$\begin{cases} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r} \right) u(r, t) = H_{pq}(u(r, t), u_t(r, t)) & \text{in } [0, \infty)^2, \\ u(r, 0) = u_0(r), \quad u_t(r, 0) = u_1(r). \end{cases} \quad (27)$$

$H_{pq} \in C^1(\mathbf{R} \times \mathbf{R})$  satisfies

$$H_{pq}(u, u_t) \geq B|u|^p|u_t|^q \quad (28)$$

with  $B > 0$ , where  $p = 0$  or  $p > 1$ , and  $q > 1$ .

Then we have the following result.

**THEOREM 1.9.** *Assume that*

$$u_0(r) \equiv 0, \quad u_1(r) \geq \frac{\phi(r)}{(1+r)^{1+\kappa_2}}, \quad (29)$$

where  $\phi$  is a positive and monotonously increasing function in  $[0, \infty)$  and

$$\kappa_2 = \frac{2-q}{p+q-1}. \quad (30)$$

Then (27) with (28) admits no global  $C^2$ -solution if  $\lim_{r \rightarrow \infty} \phi(r) = \infty$ .

We also have the following result on the local in time existence similar to Corollary1.3



COROLLARY 1.10. *Suppose that the same assumptions as in Theorem3 are fulfilled. Assume that there exists a function  $\phi_0$  such that*

$$\phi(r) = (1+r)^{1+\kappa_2}\phi_0(r), \quad \lim_{r \rightarrow \infty} \phi_0(r) = \infty. \quad (31)$$

*Then (8) with (9) admits no  $C^2$ -solution till any positive time.*

This paper is organized as follows. In the next section we prove the iteration frame for (8) including the comparison argument. Theorem1 and its corollary are proved in Section3. Theorem3 and its corollary are also proved in Section4. The proofs of Theorem2 and its corollary are given in Section5 without the one of a priori estimate. The last section is devoted to a priori estimate in three space dimensions.

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## 2. Integral inequality with a comparison argument

In order to prove Theorem1, we need Lemma2.6 and Lemma2.9 in [23] with a revised argument at the initial time.

LEMMA 2.1. *Let  $n = 2m + 1$  or  $n = 2m$ ,  $m \in \mathbf{N}$  and  $u$  be a classical solution to (8) with  $u_0(r) \equiv 0$ ,  $u_1(r) > 0$ . Assume that (9). Then  $u > 0$  in  $\Sigma$  where*

$$\Sigma = \left\{ (r, t) \in (0, \infty)^2 \mid r - t \geq \frac{2}{\delta_m}t > 0 \right\} \quad (32)$$

*Here  $\delta_m$  is a positive constant to ensure a positivity of the kernel in the integral representation of a solution. More precisely,  $\delta_m$  verifies that*

$$P_{m-1}(s), T_{m-1}(s) \geq \frac{1}{2} \quad \text{for} \quad 1 \geq s \geq \frac{1}{1 + \delta_m}, \quad (33)$$

where  $P_{m-1}, T_{m-1}$  denotes respectively Legendre, Tschebyscheff polynomials of degree  $m - 1$ .

Moreover,  $u$  satisfies the following inequality.

$$u(r, t) \geq \frac{1}{8r^m} \int_{r-t}^{r+t} \lambda^m u_1(\lambda) d\lambda + \frac{A}{8r^m} \int \int_{\Gamma(r,t)} \lambda^m u(\lambda, \tau)^p d\lambda d\tau \quad \text{in } \Sigma, \tag{34}$$

where  $\Gamma(r, t)$  is a backward cone with a vertex  $(r, t)$  ;

$$\Gamma(r, t) = \{(\lambda, \tau) \in (0, \infty)^2 \mid |r - \lambda| \leq t - \tau\}. \tag{35}$$

*Proof.* First we note that

$$\Gamma(r_0, t_0) \subset \Sigma \quad \text{for an arbitrarily fixed point } (r_0, t_0) \in \Sigma. \tag{36}$$

Setting

$$t_1 = \inf\{t > 0 \mid u(r, t) = 0 \text{ where } (r, t) \in \Gamma(r_0, t_0)\}, \tag{37}$$

we have that  $t_1 > 0$ . Because  $u_t$  is positive till a small time in  $\Gamma(r_0, t_0)$  due to  $u_t(r, 0) = u_1(r) > 0$  and its continuity together with compactness of the closure of  $\Gamma(r_0, t_0)$ . Hence so is  $u$  in  $\Gamma(r_0, t_0)$ .

Suppose that there exists  $r_1 > 0$  such that  $u(r_1, t_1) = 0$  and  $(r_1, t_1) \in \Gamma(r_0, t_0)$ . First we consider the odd dimensional case,  $n = 2m + 1$ . Then it follows from Lemma 2.2 in [23] and Duhamel's principle that

$$\begin{aligned} u(r_1, t_1) = & \frac{1}{2r_1^m} \int_{r_1-t_1}^{r_1+t_1} \lambda^m u_1(\lambda) P_{m-1} \left( \frac{\lambda^2 + r_1^2 - t_1^2}{2r_1\lambda} \right) d\lambda \\ & + \frac{1}{2r_1^m} \int \int_{\Gamma(r_1, t_1)} \lambda^m F_p(u(\lambda, \tau)) P_{m-1} \left( \frac{\lambda^2 + r_1^2 - (t_1 - \tau)^2}{2r_1\lambda} \right) d\lambda d\tau. \end{aligned} \tag{38}$$

By definition of  $t_1$  we have that

$$u > 0 \quad \text{in } \Gamma(r_1, t_1) \setminus \{(r_1, t_1)\}. \tag{39}$$

Hence the second term in (38) is estimated from below by

$$\frac{A}{4r_1^m} \int \int_{\Gamma(r_1, t_1)} \lambda^m u^p(\lambda, \tau) d\lambda d\tau > 0 \tag{40}$$

because  $(r_1, t_1) \in \Sigma$  implies that

$$\begin{aligned} \frac{\lambda^2 + r_1^2 - (t_1 - \tau)^2}{2r_1\lambda} &\geq \frac{(r_1 - t_1 + \tau)^2 + r_1^2 - (t_1 - \tau)^2}{2r_1(r_1 + t_1 - \tau)} \\ &= \frac{r_1 - t_1 + \tau}{r_1 + t_1 - \tau} \geq \frac{r_1 - t_1}{r_1 + t_1} \geq \frac{1}{1 + \delta_m}. \end{aligned} \tag{41}$$

Similarly to this, the first term in (38) is bounded from below by

$$\frac{1}{4r_1^m} \int_{r_1-t_1}^{r_1+t_1} \lambda^m u_1(\lambda) d\lambda > 0. \tag{42}$$

Therefore the same inequality as (34) is valid in which  $(r, t)$  is replaced by  $(r_1, t_1)$ . Such an inequality implies that  $u(r_1, t_1) > 0$ . But this contradicts the definition of  $t_1$  which means  $u(r_1, t_1) = 0$ . Consequently we have that  $u > 0$  in  $\Gamma(r_0, t_0)$ .  $(r_0, t_0)$  is arbitrarily fixed in  $\Sigma$ . Therefore we can conclude that  $u > 0$  in  $\Sigma$ . The same procedure as estimating  $u(r_1, t_1)$  above immediately gives us (34).

Next we consider the even dimensional case,  $n = 2m$ . Instead of (38), it follows from Lemma 2.3 in [23] and Duhamel's principle that

$$u(r_1, t_1) = \frac{2}{\pi r_1^{m-1}} (I_1(r_1, t_1) + I_2(r_1, t_1)), \tag{43}$$

where

$$\begin{aligned} I_1(r, t) &= \int_0^t \frac{\rho d\rho}{\sqrt{t^2 - \rho^2}} \int_{r-\rho}^{r+\rho} \frac{\lambda^m u_1(\lambda)}{\sqrt{\lambda^2 - (r - \rho)^2} \sqrt{(r + \rho)^2 - \lambda^2}} T_{m-1} \times \\ &\times \left( \frac{\lambda^2 + r^2 - \rho^2}{2r\lambda} \right) d\lambda \end{aligned} \tag{44}$$

and

$$\begin{aligned} I_2(r, t) &= \int_0^t d\tau \int_0^{t-\tau} \frac{\rho d\rho}{\sqrt{(t-\tau)^2 - \rho^2}} \times \\ &\times \int_{r-\rho}^{r+\rho} \frac{\lambda^m F_p(u(\lambda, \tau))}{\sqrt{\lambda^2 - (r - \rho)^2} \sqrt{(r + \rho)^2 - \lambda^2}} T_{m-1} \times \\ &\times \left( \frac{\lambda^2 + r^2 - \rho^2}{2r\lambda} \right) d\lambda. \end{aligned} \tag{45}$$

In  $I_2(r_1, t_1)$  we find that

$$\begin{aligned} \frac{\lambda^2 + r_1^2 - \rho^2}{2r_1\lambda} &\geq \frac{(r_1 - \rho)^2 + r_1^2 - \rho^2}{2r_1(r_1 + \rho)} = \frac{r_1 - \rho}{r_1 + \rho} \\ &\geq \frac{r_1 - t_1 + \tau}{r_1 + t_1 - \tau} \geq \frac{r_1 - t_1}{r_1 + t_1} \geq \frac{1}{1 + \delta_m}. \end{aligned} \tag{46}$$

Then the positivity of  $u$  for  $0 < t < t_1$  again yields that

$$\begin{aligned} I_2(r_1, t_1) &\geq \frac{A}{2} \int_0^{t_1} d\tau \int_0^{t_1 - \tau} \frac{\rho d\rho}{\sqrt{(t_1 - \tau)^2 - \rho^2}} \times \\ &\quad \times \int_{r_1 - \rho}^{r_1 + \rho} \frac{\lambda^m u^p(\lambda, \tau) d\lambda}{\sqrt{\lambda^2 - (r_1 - \rho)^2} \sqrt{(r_1 + \rho)^2 - \lambda^2}} > 0. \end{aligned} \tag{47}$$

Similarly to this, we also have that

$$\begin{aligned} I_1(r_1, t_1) &\geq \frac{1}{2} \int_0^{t_1} \frac{\rho d\rho}{\sqrt{t_1^2 - \rho^2}} \int_{r_1 - \rho}^{r_1 + \rho} \frac{\lambda^m u_1(\lambda) d\lambda}{\sqrt{\lambda^2 - (r_1 - \rho)^2} \sqrt{(r_1 + \rho)^2 - \lambda^2}} > 0. \end{aligned} \tag{48}$$

Therefore the desired contradiction  $u(r_1, t_1) > 0$  is established also in the even dimensional case. (34) now follows from the completely same proof as Lemma2.6 in [23]. The proof is ended.  $\square$

### 3. Proof of Theorem1

By virtue of Lemma2.1, one can prove the theorem by iteration argument which was originally introduced in [11]. The proof is almost the same to the one of Theorem1.1 in [23] in which the blow-up result for sub-critical decay was proved.

Let  $u(r, t)$  be a global solution of the problem, (8), and  $\Sigma$  be the one in Lemma2.1. We note that

$$\Gamma(r, t) \subset \Sigma \quad \text{if } (r, t) \in \Sigma. \tag{49}$$

Taking the second term away from (34) and substituting  $u_1$  by lower bound in the assumption on  $u_1$ , we have that

$$u(r, t) \geq \frac{1}{8r^m} \int_r^{r+t} \frac{\lambda^m \phi(\lambda) d\lambda}{(1 + \lambda)^{\kappa_0 + 1}} \quad \text{in } \Sigma. \tag{50}$$

Here we cut the domain of the integration by positivity of  $\phi$ . Hence the monotonicity of  $\phi$  yields the first step of the iteration of the estimate for  $u$  in  $\Sigma$ , namely,

$$u(r, t) \geq \frac{\phi(r)t}{8(1+r+t)^{\kappa_0+1}} \quad \text{in } \Sigma. \quad (51)$$

Now we assume the  $j$ -th step ( $j \in \mathbf{N}$ ) of the form

$$u(r, t) \geq \frac{c_j \phi(r)^{p^{j-1}} t^{a_j}}{(1+r+t)^{b_j}} \quad \text{in } \Sigma, \quad (52)$$

where  $a_j, b_j, c_j$  are positive constants. Then taking the first term away from (34) and substituting  $u$  by quantity of the right-hand side of the  $j$ -step, we have that

$$u(r, t) \geq \frac{A}{8r^m} \int_0^t d\tau \int_r^{r+t-\tau} \lambda^m \left( \frac{c_j \phi(\lambda)^{p^{j-1}} \tau^{a_j}}{(1+\lambda+\tau)^{b_j}} \right)^p d\lambda \quad \text{in } \Sigma. \quad (53)$$

Here we cut the domain of  $\lambda$ -integration by replacing  $r-t+\tau$  by  $r$ . Hence the monotonicity of  $\phi$  again yields that

$$u(r, t) \geq \frac{Ac_j^p \phi(r)^{p^j}}{8(1+r+t)^{pb_j}} \int_0^t (t-\tau) \tau^{pa_j} d\tau \quad \text{in } \Sigma. \quad (54)$$

which shows that the  $(j+1)$ -step should start with

$$u(r, t) \geq \frac{Ac_j^p \phi(r)^{p^j} t^{pa_j+2}}{8(pa_j+2)^2(1+r+t)^{pb_j}} \quad \text{in } \Sigma \quad (55)$$

by making use of the integration by parts in  $\tau$ -integration.

In order to investigate infinitely many times of this procedure, we define  $\{a_j\}, \{b_j\}$  by

$$\begin{aligned} a_{j+1} &= pa_j + 2, & a_1 &= 1, \\ b_{j+1} &= pb_j, & b_1 &= \kappa_0 + 1. \end{aligned} \quad (56)$$

One can readily solve them and reach to expressions

$$a_j = (\kappa_0 + 1)p^{j-1} - \kappa_0, \quad b_j = (\kappa_0 + 1)p^{j-1} \quad (57)$$

because  $\kappa_0 = 2/(p-1)$ . Noticing that  $p\kappa_0 - 2 = \kappa_0$ , we know that  $c_{j+1}$  should be defined as follows to keep the iteration.

$$c_{j+1} \geq \frac{Ac_j^p}{8(\kappa_0 + 1)^2 p^{2j}} \quad (58)$$

which inductively implies that

$$\begin{aligned} & \log c_{j+1} \\ & \geq p \log c_j - j \log p^2 + \log \frac{A}{8(\kappa_0 + 1)^2} \\ & \geq p^j \log c_1 - \sum_{k=1}^j kp^{j-k} \log p^2 + \sum_{k=1}^j p^{k-1} \log \frac{A}{8(\kappa_0 + 1)^2} \\ & \geq p^j \left( -\log 8 - \sum_{k=1}^j kp^{-k} \log p^2 + \frac{1-1/p^j}{p-1} \log \frac{A}{8(\kappa_0 + 1)^2} \right). \end{aligned} \quad (59)$$

The sum in the last line converges as  $j$  tends to infinity. Hence there exists a positive constant  $c_{p,A}$  depending only on  $p, A$  such that

$$c_j \geq \exp(-c_{p,A}p^{j-1}) \quad \text{for all } j \in \mathbf{N}. \quad (60)$$

Therefore we obtain for all  $j \in \mathbf{N}$  that

$$u(r, t) \geq \frac{1}{t^{\kappa_0}} \exp(U(r, t)p^{j-1}) \quad \text{in } \Sigma, \quad (61)$$

where we set

$$U(r, t) = \log \phi(r) + (\kappa_0 + 1) \log \frac{t}{1+r+t} - c_{p,A}. \quad (62)$$

Now we restrict ourselves on the half line;

$$\{r = d_m^{-1}t \mid t \geq 1\} \subset \Sigma, \quad \text{where } d_m = \frac{\delta_m}{2 + \delta_m} > 0. \quad (63)$$

On this line we have that

$$U(r, d_m r) \geq \log \phi(r) + (\kappa_0 + 1) \log \frac{d_m}{3} - c_{p,A} \quad (64)$$

because of  $d_m < 1$ . Letting  $r$  be large, we can find a point  $(r_0, t_0) \in \Sigma$  such that

$$U(r_0, t_0) > 0, \quad \text{where } t_0 = d_m r_0 \quad (65)$$

by assumption of  $\lim_{r \rightarrow \infty} \phi(r) = \infty$ . Therefore it follows from (61) with  $j \rightarrow \infty$  that  $u(r_0, t_0) = \infty$  which contradicts the assumption that  $u$  is a global solution. The proof is now completed.

**Proof of Corollary 1.3.**

First we fix a time as  $t = t_1$ . Suppose that (8) with (9) admits a local  $C^2$ -solution  $u(r, t)$  for  $0 \leq t \leq t_1$ . Then we have the same inequality (61). In this case the assumption on  $\phi$  and (62) yield that

$$U(r, t_1) = \log \phi_0(r) + (\kappa_0 + 1) \log \frac{(1+r)t_1}{1+r+t_1} - c_{p,A} \quad \text{for } (r, t_1) \in \Sigma. \tag{66}$$

Hence we can find a  $r_1$  such that  $U(r_1, t_1) > 0$  which implies again the desired contradiction  $u(r_1, t_1) = \infty$ .

**4. Proof of Theorem 3**

Before giving a proof of Theorem 2, we shall prove Theorem 3 because its proof is very similar to the one of Theorem 1.

LEMMA 4.1. *Let  $n = 2m + 1$  or  $n = 2m$ ,  $m \in \mathbf{N}$  and  $u$  be a classical solution to (27) with  $u_0(r) \equiv 0$ ,  $u_1(r) > 0$ . Assume that (28). Then  $u$  satisfies the following inequality in  $\Gamma_0$ , where  $\Gamma_0$  is the one in Lemma 2.1.*

$$u(r, t) \geq \frac{1}{8} \int_r^{r+t} u_1(\lambda) d\lambda + \frac{B}{8} \left(1 + \frac{p}{q}\right)^{-q} \int_r^{r+t} (r+t-\lambda)^{1-q} |u(\lambda, r+t-\lambda)|^{p+q} d\lambda. \tag{67}$$

*Proof.* It follows from Lemma 2.6 in [23] and the positivity on the nonlinear term that

$$u(r, t) \geq \frac{1}{8r^m} \int_{r-t}^{r+t} \lambda^m u_1(\lambda) d\lambda + \frac{B}{8r^m} \int \int_{\Gamma(r,t)} \lambda^m |u(\lambda, \tau)|^p |u_t(\lambda, \tau)|^q d\lambda d\tau \tag{68}$$

in  $\Gamma_0$ . Then the second term of this inequality is estimated from below by

$$\begin{aligned} & \frac{B}{8r^m} \left( \int_{r-t}^r d\lambda \int_0^{\lambda-(r-t)} d\tau + \int_r^{r+t} d\lambda \int_0^{r+t-\lambda} d\tau \right) \times \\ & \quad \times \lambda^m |u(\lambda, \tau)|^p |u_t(\lambda, \tau)|^q \\ & \geq \frac{B}{8r^m} \int_r^{r+t} \lambda^m d\lambda \int_0^{r+t-\lambda} |u(\lambda, \tau)|^p |u_t(\lambda, \tau)|^q d\tau. \end{aligned} \quad (69)$$

Hölder's inequality yields that

$$\begin{aligned} & \left| \int_0^{r+t-\lambda} u^{p/q}(\lambda, \tau) u_t(\lambda, \tau) d\tau \right|^q \\ & \leq (r+t-\lambda)^{q-1} \int_0^{r+t-\lambda} |u(\lambda, \tau)|^p |u_t(\lambda, \tau)|^q d\tau. \end{aligned} \quad (70)$$

Therefore noticing that  $u(r, 0) = u_0(r) \equiv 0$ , we have (67) with a trivial cancellation of  $r^m$  and  $\lambda^m$ .  $\square$

**Proof of Theorem 3.** Similarly to the proof of Theorem 1, let  $u(r, t)$  be a global solution of the problem (27). Introducing the same domain  $\Sigma$ , we have the first step of the iteration

$$u(r, t) \geq \frac{\phi(r)t}{8(1+r+t)^{\kappa_2+1}} \quad \text{in } \Sigma \quad (71)$$

by assumption on  $u_1$ . Here we pick up the first term on the righthand side of (67).

Again we assume the  $j$ -th step ( $j \in \mathbf{N}$ ) of the form

$$u(r, t) \geq \frac{c_j \phi(r)^{(p+q)^{j-1}} t^{a_j}}{(1+r+t)^{b_j}} \quad \text{in } \Sigma, \quad (72)$$

where  $a_j, b_j, c_j$  are constants, especially  $a_j \geq 1$ . Taking the first term away from (67) and substituting  $u$  by quantity of the right-hand side



of the  $j$ -step, we have that

$$\begin{aligned}
 u(r, t) &\geq \frac{B}{8} \left(1 + \frac{p}{q}\right)^{-q} \times \\
 &\quad \times \int_r^{r+t} (r+t-\lambda)^{1-q} \left( \frac{c_j \phi(\lambda)^{(p+q)^{j-1}} (r+t-\lambda)^{a_j}}{(1+r+t)^{b_j}} \right)^{p+q} d\lambda \\
 &\text{in } \Sigma.
 \end{aligned} \tag{73}$$

Here we use the fact that

$$\text{segment } \{(\lambda, r+t-\lambda) \mid \lambda \in [r, r+t]\} \subset \Sigma \quad \text{if } (r, t) \in \Sigma. \tag{74}$$

Hence the monotonicity of  $\phi$  yields that

$$\begin{aligned}
 u(r, t) &\geq \frac{B}{8} \left(1 + \frac{p}{q}\right)^{-q} \times \\
 &\quad \times \frac{c_j^{p+q} \phi(r)^{(p+q)^j}}{(1+r+t)^{(p+q)b_j}} \int_r^{r+t} (r+t-\lambda)^{1-q+(p+q)a_j} d\lambda \quad \text{in } \Sigma.
 \end{aligned} \tag{75}$$

Therefore we obtain at the  $(j+1)$ -step, that

$$u(r, t) \geq \frac{B}{8} \left(1 + \frac{p}{q}\right)^{-q} \frac{c_j^{p+q} \phi(r)^{(p+q)^j} t^{2-q+(p+q)a_j}}{(2-q+(p+q)a_j)(1+r+t)^{(p+q)b_j}} \quad \text{in } \Sigma. \tag{76}$$

Here we use the fact that  $a_j \geq 1$ .

In order to investigate infinitely many times of this procedure, we define  $\{a_j\}, \{b_j\}$  by

$$\begin{aligned}
 a_{j+1} &= (p+q)a_j + 2 - q, & a_1 &= 1, \\
 b_{j+1} &= (p+q)b_j, & b_1 &= \kappa_2 + 1.
 \end{aligned} \tag{77}$$

One can readily solve them and reach to expressions

$$a_j = (\kappa_2 + 1)(p+q)^{j-1} - \kappa_2 \geq 1, \quad b_j = (\kappa_2 + 1)p^{j-1} \tag{78}$$

because  $\kappa_2 = (2-q)/(p+q-1)$ . When  $q \geq 2$  i.e.  $\kappa_2 \leq 0$ , we have

$$2-q+(p+q)a_j = 2-q+(\kappa_2+1)(p+q)^j - \kappa_2(p+q) \leq (p+q)^j. \tag{79}$$

When  $1 < q < 2$  i.e.  $0 < \kappa_2 < 1$ , we have

$$2-q+(p+q)a_j \leq 1+2(p+q)^j. \tag{80}$$

So  $c_{j+1}$  should be defined as follows to keep the iteration.

$$c_{j+1} \geq \frac{B}{8} \left(1 + \frac{p}{q}\right)^{-q} \frac{c_j^{p+q}}{3(p+q)^j}. \quad (81)$$

In the same manner as in the proof of Theorem1, there exists a positive constant  $c_{p,q,B}$  depending only on  $p, q, B$  such that

$$c_j \geq \exp(-c_{p,q,B}(p+q)^{j-1}) \quad \text{for all } j \in \mathbf{N}. \quad (82)$$

Hence we obtain for all  $j \in \mathbf{N}$  that

$$u(r, t) \geq \frac{1}{t^{\kappa_2}} \exp\left(V(r, t)(p+q)^{j-1}\right) \quad \text{in } \Sigma, \quad (83)$$

where we set

$$V(r, t) = \log \phi(r) + (\kappa_2 + 1) \log \frac{t}{1+r+t} - c_{p,q,B}. \quad (84)$$

Therefore the proof follows from the same argument as the related part of Theorem1, in which  $U, \kappa_0, c_{p,A}$  should be replaced by  $V, \kappa_2, c_{p,q,B}$  respectively.

**Proof of Corollary1.10.** The proof immediately follows from the one of Corollary1.3 with replaced  $U$  by  $V$ .

## 5. Lifespan in three space dimensions

Following [11] and [2], we shall prove Theorem2 in this section. First we note that the solution  $u$  of (14) has to satisfy

$$u(x, t) = u^0(x, t) + L(G_p(u))(x, t), \quad (85)$$

where  $u^0$  is the solution of  $\square u^0 = 0$  with the initial data  $u^0(x, 0) = \varepsilon f(x), u_t^0 = \varepsilon g(x)$  and

$$L(G_p(u))(x, t) = \frac{1}{4\pi} \int_0^t (t-\tau) d\tau \int_{|\omega|=1} G_p(u(x+(t-\tau)\omega, \tau)) d\omega \quad (86)$$

is the solution of  $\square u = G_p(u)$  with zero data.

Now we start with a pointwise estimate of  $u^0$ .

PROPOSITION 5.1. *Assume (15). Then there exists a positive constant  $C_p$  depending only on  $p$  such that*

$$\sum_{|\alpha| \leq 2} |\nabla_x^\alpha u^0(x, t)| \leq \frac{C_p \varepsilon \psi(t+r)}{1+t+r} \times \begin{cases} \frac{1}{(1+|t-r|)^{\kappa_0-1}} & \text{if } \kappa_0 > 1, \\ 1 + \log \frac{1+t+r}{1+|t-r|} & \text{if } \kappa_0 = 1, \\ \frac{1}{(1+t+r)^{\kappa_0-1}} & \text{if } 0 < \kappa_0 < 1, \end{cases} \tag{87}$$

where  $r = |x|$ .

*Proof.* This is a direct consequence of the monotonicity of  $\psi$  and the proof of almost the same estimate in [2] and [26]. Actually, according to [11] and [2], the radial symmetricity of  $\psi$  yields that

$$\sum_{|\alpha| \leq 2} |\nabla_x^\alpha u^0(x, t)| \leq \frac{C_\varepsilon}{r} \int_{|t-r|}^{t+r} \frac{\lambda \psi(\lambda) d\lambda}{(1+\lambda)^{1+\kappa_0}}. \tag{88}$$

□

Therefore the proposition is now established by the following lemma, which was proved in [2] and [26].

LEMMA 5.2. *Let  $\kappa > 0$ . Then there exists a positive constant  $C_\kappa$  depending only on  $\kappa$  such that*

$$\frac{1}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{-\kappa} d\alpha \leq \frac{C_\kappa}{1+t+r} \times \begin{cases} \frac{1}{(1+|t-r|)^{\kappa-1}} & \text{if } \kappa > 1, \\ 1 + \log \frac{1+t+r}{1+|t-r|} & \text{if } \kappa = 1, \\ \frac{1}{(1+t+r)^{\kappa-1}} & \text{if } 0 < \kappa < 1 \end{cases} \tag{89}$$

for all  $t, r \geq 0$ .

Next we introduce a weight function

$$w(r, t) = \frac{1+t+r}{\psi(t+r)} \times \begin{cases} (1+|t-r|)^{\kappa_0-1} & \text{if } \kappa_0 > 1, \\ \left(1 + \log \frac{1+t+r}{1+|t-r|}\right)^{-1} & \text{if } \kappa_0 = 1, \\ (1+t+r)^{\kappa_0-1} & \text{if } 0 < \kappa_0 < 1. \end{cases} \quad (90)$$

Define a norm for functions  $u(x, t)$  which are continuous in  $\mathbf{R}^3 \times [0, T]$  by

$$\|u\| = \sup_{(x,t) \in \mathbf{R}^3 \times [0, T]} w(|x|, t) |u(x, t)|. \quad (91)$$

The following estimate guarantees the existence of local in time solution.

**PROPOSITION 5.3.** *Assume that  $p > 1 + \sqrt{2}$ . Then there exists a positive constant  $C_{p,\psi}$  depending on  $p, \psi$  such that the following inequality holds for  $T > 0$ .*

$$\|L|u|^p\| \leq C_{p,\psi} \psi(T)^{p-1} \|u\|^p. \quad (92)$$

We shall prove this in the next section. Here we give a remark on the relation between this proposition and the result of [26].

**REMARK 5.4.**  *$\psi \equiv \text{const.} > 0$  is admissible in the proof of Proposition 5.3. See the assumption on  $\psi$  in the key tool, Lemma 6.1 below. This means that our proof also shows the global existence for the critical decay in [26]. The key estimate in [26] for  $\kappa_0 = 1$  is Proposition 4.3 in [26]. It can be replaced by more simple estimate, Lemma 6.2 in the next section.*

**Proof of Theorem 3.** Let  $X$  be a linear space defined by

$$X = \{u \mid \nabla_x^\alpha u(x, t) \in C(\mathbf{R}^3 \times [0, T]), \|\nabla_x^\alpha u\| < \infty \text{ for } |\alpha| \leq 2\}. \quad (93)$$

One can readily check that  $X$  is complete with respect to the norm

$$\|u\|_X = \sum_{|\alpha| \leq 2} \|\nabla_x^\alpha u\|. \quad (94)$$

Define a sequence of functions  $\{u_n\}_{n \in \mathbb{N}}$  by

$$u_{n+1} = u^0 + L(G_p(u_n)), \quad u_1 = u^0. \tag{95}$$

Then Proposition 5.1 implies that

$$\|u^0\|_X \leq C_p \varepsilon. \tag{96}$$

Hence we have  $u^0 \in X$ .

We now take  $\varepsilon > 0$  to satisfy that

$$2^p p A_1 C_{p,\psi} \psi(T)^{p-1} (C_p \varepsilon)^{p-1} \leq 1, \quad C_p \varepsilon \leq \frac{1}{2} \tag{97}$$

which yields that

$$2^p p A_1 C_{p,\psi} \psi(T)^{p-1} \|u^0\|^{p-1} \leq 1, \quad \|u^0\| \leq \frac{1}{2}. \tag{98}$$

According to [11] or [2], (98) and Proposition 5.3 guarantee that there exists a unique solution of (14) in time interval  $[0, T]$  which is obtained as a limit in  $X$  of the sequence  $\{u_n\}$ . The upper bound of  $T$  is determined by (97). This completes the proof of Theorem 2.

**Proof of Corollary 1.5.** The proof immediately follows from the one of Theorem 1. The uniqueness of the solution shows us that it is enough to consider the radially symmetric solution because of the assumption  $g = g(|x|)$ . Let  $u(r, t)$  be a  $C^2$ -solution of (14) in the time interval  $[0, T]$ , where  $r = |x|$ . Setting  $\phi(r) = \varepsilon g_0 \psi(r)$  in (62) in the proof of Theorem 1, we know that the nonexistence of  $u$  follows from the inequality

$$\log \left[ \varepsilon g_0 \psi(d_1^{-1} t) \right] + (\kappa_0 + 1) \log \frac{d_1}{3} - c_{p,A} > 0 \quad \text{for } t \geq 1. \tag{99}$$

We note that  $d_1 > 0$  is a numerical constant. This means that  $u$  cannot exist as far as  $T$  satisfies

$$T \geq \max \left\{ \psi^{-1} \left[ \varepsilon^{-1} \frac{e^{c_{p,A}}}{g_0} \left( \frac{d_1}{3} \right)^{-(\kappa_0+1)} \right], 1 \right\}. \tag{100}$$

Making  $\varepsilon$  to be small, max has to choose the quantity of left-hand side. Therefore there exist  $\varepsilon_0 = \varepsilon_0(p, g_0, A) > 0$  such that the lifespan  $T(\varepsilon)$  of  $C^2$ -solution of (14) satisfies

$$T(\varepsilon) \leq \psi^{-1}(C\varepsilon^{-1}) \quad \text{for all } 0 < \varepsilon \leq \varepsilon_0, \tag{101}$$

where  $C = C(p, g_0, A) > 0$ . The lower bound of  $T(\varepsilon)$  is already obtained in Theorem2 in which  $A_1$  is replaced by  $A$ . Corollary is now established.

**6. A priori estimate in three space dimensions**

In this section we shall prove Proposition5.3 in the previous section. According to [11] or [2] again, it is enough to show that there exists  $C_{p,\psi} > 0$  such that

$$P_w(r, t) \leq C_{p,\psi} \psi(T)^{p-1} w(r, t)^{-1}, \tag{102}$$

where  $r = |x|$  and  $P_w$  is defined by

$$P_w(r, t) = \frac{1}{2r} \int_0^t d\tau \int_{|r-t+\tau|}^{r+t-\tau} \lambda w(\lambda, \tau)^{-p} d\lambda. \tag{103}$$

Because we have

$$\|L(|u|^p)\| \leq \|u\|^p \|P_w\|. \tag{104}$$

See LemmaII in [11], or p.1470 in [2].

Now we shall devide the proof into three cases up the value of  $\kappa_0$ . In each case, we shall use the decomposition  $\mathbf{R}^3 \times [0, T] = \cup_{j=1}^3 D_j$ , where

$$\begin{aligned} D_1 &= \{2t \leq r, r \leq 1\}, \\ D_2 &= \{2t \leq r, r \geq 1\}, \\ D_3 &= \{2t \geq r\}. \end{aligned} \tag{105}$$

We note that

$$w(r, t) \leq C \quad \text{in } D_1 \tag{106}$$

and

$$r \geq \frac{1+r+2t}{3} \geq \frac{1+r+t}{3} \quad \text{in } D_2. \tag{107}$$

The definition of  $\kappa_0$  yields that

$$1 - p(\kappa_0 - 1) = -\kappa_0 + p - 1 > 0 \quad \text{when } p > 1 + \sqrt{2}. \tag{108}$$

Hereafter constant  $C$  may change from line to line, and we shall omit the dependence on  $p, \psi$ .

**Case1 ;  $\kappa_0 > 1$ , i.e.  $1 + \sqrt{2} < p < 3$ .** It follows (90) that

$$\begin{aligned}
 & P_w(r, t) \\
 & \leq \frac{1}{2r} \int_0^t d\tau \int_{|r-t+\tau|}^{r+t-\tau} (1 + |\tau - \lambda|)^{-p(\kappa_0-1)} (1 + \lambda + \tau)^{1-p} \psi(\lambda + \tau)^p d\lambda.
 \end{aligned}
 \tag{109}$$

In  $D_1$  and  $D_2$ , we have

$$P_w(r, t) \leq \frac{1}{2r} \int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} (1 + \lambda - \tau)^{-p(\kappa_0-1)} (1 + \lambda + \tau)^{1-p} \psi(\lambda + \tau)^p d\lambda
 \tag{110}$$

since  $\lambda - \tau \geq r - t \geq t \geq 0$ .

First we consider  $P_w$  in  $D_1$ . Then we obtain

$$P_w(r, t) \leq \frac{(1 + r - t)^{-p(\kappa_0-1)+1-p} \psi(r + t)^p}{r} \int_0^t (t - \tau) d\tau \leq C. \tag{111}$$

Therefore (102) follows from (106).

Next we consider  $P_w$  in  $D_2$ . It follows that

$$P_w(r, t) \leq \frac{(1 + r - t)^{-p(\kappa_0-1)}}{2r} \int_0^t d\tau \int_{t-\tau}^{r+t-\tau} (1 + \lambda + \tau)^{1-p} \psi(\lambda + \tau)^p d\lambda.
 \tag{112}$$

We employ the following lemma in order to avoid to have  $\psi(r + t)^{p-1}$  in this case.

**LEMMA 6.1.** *Let  $a > 1$  and  $b, d, p \geq 0$  with  $b \leq d$ . Assume that  $\psi \in C^1([0, \infty))$  satisfies*

$$0 \leq (1 + r)\psi'(r) \leq K\psi(r)^{1-\delta}, \quad \psi(r) > 0, \tag{113}$$

where  $K, \delta > 0$  are constants. Then there exists positive constant  $C$  depending on  $a, \psi$  such that

$$\int_b^d (1 + \alpha)^{-a} \psi(\alpha)^p d\alpha \leq C(1 + b)^{1-a} \psi(b)^p. \tag{114}$$

*Proof.* The integration by parts and the assumption on  $\psi$  yield that

$$I_0 \leq \frac{(1+b)^{1-a}\psi(b)^p}{a-1} + \frac{pK}{a-1}I_1, \tag{115}$$

where we set

$$I_j = \int_b^d (1+\alpha)^{-a}\psi(\alpha)^{p-j\delta}d\alpha \tag{116}$$

for  $j \in \mathbf{N} \cup \{0\}$ . Again the itegration by parts yields that

$$I_1 \leq \frac{(1+b)^{1-a}\psi(b)^{p-\delta}}{a-1} + \frac{p-\delta}{a-1} \int_b^d (1+\alpha)^{1-a}\psi(\alpha)^{p-\delta-1}\psi'(\alpha)d\alpha. \tag{117}$$

If  $p-\delta \leq 0$  the proof is ended with  $C = 1/(a-1) + pK\psi(0)^{-\delta}/(a-1)^2 > 0$ . If  $p-\delta > 0$ , we continue to estimate the integration with the assumption on  $\psi$  as follows.

$$\int_b^d (1+\alpha)^{1-a}\psi(\alpha)^{p-\delta-1}\psi'(\alpha)d\alpha \leq KI_2. \tag{118}$$

In this way, we inductively have that

$$I_0 \leq \frac{(1+b)^{1-a}\psi(b)^p}{a-1} \left( 1 + \sum_{j=1}^{J-1} \frac{K^j \prod_{k=1}^j [p - (k-1)\delta]}{(a-1)^j \psi(0)^{j\delta}} \right) + \frac{K^J \prod_{k=1}^J [p - (k-1)\delta]}{(a-1)^J} I_J, \tag{119}$$

where  $J = \min\{j \mid p - j\delta < 0\} \geq 2$ . Therefore the integration by parts completes the proof of this lemma with

$$C = \frac{1}{a-1} \left( 1 + \sum_{j=1}^J \frac{K^j \prod_{k=1}^j [p - (k-1)\delta]}{(a-1)^j \psi(0)^{j\delta}} \right) > 0. \tag{120}$$

□

Now we continue to prove (102) in  $D_2$ . Making use of Lemma6.1 with  $a = p - 1 > 1, \alpha = \lambda + \tau, b = t, d = r + t$ , we have

$$P_w(r, t) \leq Cr^{-1}(1+r-t)^{-p(\kappa_0-1)}t(1+t)^{2-p}\psi(t)^p \leq Cr^{-1}(1+r-t)^{-p(\kappa_0-1)+3-p}\psi(r+t)\psi(T)^{p-1}. \tag{121}$$



Therefore (102) follows from (108) and (107).

Finally we consider  $P_w$  in  $D_3$ . Changing variables by

$$\alpha = \tau + \lambda, \quad \beta = \tau - \lambda, \tag{122}$$

we have that

$$P_w(r, t) \leq \frac{C\psi(t+r)^p}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{1-p} d\alpha \int_{-\alpha}^{t-r} (1+|\beta|)^{-p(\kappa_0-1)} d\beta. \tag{123}$$

Since  $\beta$ -integral is dominated by  $C(1+\alpha)^{1-p(\kappa_0-1)}$ , (108) yields that

$$P_w(r, t) \leq \frac{C\psi(t+r)\psi(3T)^{p-1}}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{-\kappa_0} d\alpha. \tag{124}$$

Therefore (102) is now established by Lemma5.2.

**Case2 ;  $\kappa_0 = 1$ , i.e.  $p = 3$ .** It follows from (90) that

$$P_w(r, t) \leq \frac{1}{2r} \int_0^t d\tau \times \int_{|r-t+\tau|}^{r+t-\tau} (1+\lambda+\tau)^{-2} \psi(\lambda+\tau)^3 \left(1 + \log \frac{1+\tau+\lambda}{1+|\tau-\lambda|}\right)^3 d\lambda. \tag{125}$$

Similar to Case1, (102) in  $D_1$  is trivial by (106). So we consider  $P_w$  in  $D_2$ . Then it follows from (125) that

$$P_w(r, t) \leq \frac{1}{2r} \int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} (1+\lambda+\tau)^{-2} \psi(\lambda+\tau)^3 \left(2 \log 3 \frac{1+\lambda+\tau}{1+\lambda-\tau}\right)^3 d\lambda \tag{126}$$

because  $(1+\lambda+\tau)/(1+\lambda-\tau) \geq 1$ . In order to handle the logarithmic term, we shall employ the following lemma which can be verified by simple differentiation.

LEMMA 6.2. For any  $\eta > 0$ ,

$$\frac{X^\eta}{\eta} \geq \log X \quad \text{for } X \geq 1. \tag{127}$$

By making use of this lemma, we have that

$$\begin{aligned}
P_w(r, t) &\leq \frac{3^{3\eta}4}{\eta^{3r}} \int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} (1+\lambda+\tau)^{3\eta-2} \psi(\lambda+\tau)^3 (1+\lambda-\tau)^{-3\eta} d\lambda \\
&\leq \frac{3^{3\eta}4(1+r-t)^{-3\eta}}{\eta^{3r}} \int_0^t d\tau \int_{t-\tau}^{r+t-\tau} (1+\lambda+\tau)^{3\eta-2} \psi(\lambda+\tau)^3 d\lambda.
\end{aligned} \tag{128}$$

We now fix  $\eta$  to satisfy  $3\eta - 2 < -1$ , for example  $\eta = 1/4$ . Then, by virtue of Lemma6.1 with  $\alpha = \lambda + \tau$ ,  $a = 2 - 3\eta > 1$ ,  $b = t$ ,  $d = r + t$ , we obtain

$$\begin{aligned}
P_w(r, t) &\leq Cr^{-1}(1+r-t)^{-3/4}(1+t)^{3/4-1}t\psi(t)^3 \\
&\leq Cr^{-1}\psi(r+t)\psi(T)^2.
\end{aligned} \tag{129}$$

Therefore (102) follows (107) and  $(1+r+t)/(1+r-t) \geq 1$ .

Next we consider  $P_w$  in  $D_3$ . Changing variables by (122), we have that

$$P_w(r, t) \leq \frac{C\psi(t+r)^3}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{-2} d\alpha \int_{-\alpha}^{t-r} \log^3 3 \frac{1+\alpha}{1+|\beta|} d\beta \tag{130}$$

because  $(1+\alpha)/(1+|\beta|) \geq 1$ . Hence again Lemma6.2 with  $\eta = 1/4$  yields that

$$P_w(r, t) \leq \frac{C\psi(t+r)\psi(3t)^2}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{-5/4} d\alpha \int_{-\alpha}^{t-r} (1+|\beta|)^{-3/4} d\beta. \tag{131}$$

Since the  $\beta$ -integral is dominated by  $C(1+\alpha)^{1/4}$ , we obtain

$$P_w(r, t) \leq \frac{C\psi(t+r)\psi(3T)^2}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{-1} d\alpha. \tag{132}$$

Therefore (102) follows from Lemma5.2.

**Case3** ;  $0 < \kappa_0 < 1$ , **i.e.**  $p > 3$ . It follows (90) and (108) that

$$P_w(r, t) \leq \frac{1}{2r} \int_0^t d\tau \int_{|r-t+\tau|}^{r+t-\tau} (1+\lambda+\tau)^{-\kappa_0-1} \psi(\lambda+\tau)^p d\lambda. \tag{133}$$

Again (102) in  $D_1$  is trivial by (106). First we consider  $P_w$  in  $D_2$ . Since

$$P_w(r, t) \leq \frac{1}{2r} \int_0^t d\tau \int_{t-\tau}^{r+t-\tau} (1 + \lambda + \tau)^{-\kappa_0-1} \psi(\lambda + \tau)^p d\lambda, \quad (134)$$

we can apply Lemma 6.1 to  $\lambda$ -integral by setting  $\alpha = \lambda + \tau$ ,  $a = \kappa_0 + 1$ ,  $b = t$ ,  $d = r + t$ . Hence we obtain

$$\begin{aligned} P_w(r, t) &\leq Cr^{-1}(1+t)^{-\kappa_0} t \psi(t)^p \\ &\leq Cr^{-1}(1+r+t)^{1-\kappa_0} \psi(t+r) \psi(T)^{p-1}. \end{aligned} \quad (135)$$

Therefore (102) follows from (107).

Finally we consider  $P_w$  in  $D_3$ . Changing variables by (122), we have that

$$P_w(r, t) \leq \frac{C\psi(t+r)^p}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{-\kappa_0-1} d\alpha \int_{-\alpha}^{t-r} d\beta. \quad (136)$$

Hence we obtain

$$P_w(r, t) \leq \frac{C\psi(t+r)\psi(3T)^{p-1}}{r} \int_{|t-r|}^{t+r} (1+\alpha)^{-\kappa_0} d\alpha. \quad (137)$$

Therefore (102) follows from Lemma 5.2. This completes the proof of (102) for all cases.

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