

# Isomorphism of Commutative Group Algebras over all Fields

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SUMMARY. - *It is argued that the commutative group algebra over each field determines up to an isomorphism its group basis for any of the following group classes:*

- *Direct sums of cocyclic groups*
- *Splitting countable modulo torsion groups whose torsion parts are direct sums of cyclics;*
- *Splitting groups whose torsion parts are separable countable*
- *Groups whose torsion parts are algebraically compact*
- *Algebraically compact groups*

*These give a partial positive answer to the R.Brauer's classical problem.*

## 1. Introduction

Let  $K$  be a field and let  $G$  be an arbitrary multiplicative group. In the theory of the group algebras the following major problem due to R. Brauer (see, for instance, [3]) is well-known: Whether the group algebra  $K[G]$  over every field  $K$  determines  $G$ , i.e. is then  $G$  isomorphic to any group  $H$  provided  $K[G]$  is isomorphic to  $K[H]$  as  $K$ -algebras over each field  $K$ ? This problem has a negative answer

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AMS Subject Classification (1995): 20C07, 16S34, 20K21

in general. Actually, there exist two counterexamples: first, E. Dade (see, for instance, [20]) showed that there are two nonisomorphic metabelian (noncommutative) groups  $G$  and  $H$  such that  $K[G]$  and  $K[H]$  are isomorphic over all  $K$ ; secondly, W. May in [18, 14] has proved such parallel assertion for special countables abelian  $G$  and  $H$ .

In this paper we treat the Brauer's problem in its commutative variant so that new commutative group classes are obtained, for which the above question holds (in this aspect the reader can see also [5, 6, 8]). These classes of abelian groups are the following: direct sums of cocyclic groups, splitting countable modulo torsion groups with a torsion part that is a direct sum of cyclics; splitting groups with a countable separable torsion part; groups with an algebraic compact torsion part; algebraically compact groups. We also discuss the question for the so-called Rotman's and Warfield's groups.

This work is organized as follows. In the second paragraph, we set the notation and recall the formulation of certain facts and results, given by us in [4]. In the third paragraph, we state the main results, distributed to five sections. As a final, the fourth paragraph contains some left-open problems and conjectures.

## 2. Notation and conventions

Throughout this research paper  $K$  denotes a field, and all groups are commutative and written multiplicatively. For  $G$  an abelian group and  $p$  a prime natural,  $G_p$  denotes the  $p$ -primary component of the torsion subgroup  $T(G)$  in  $G$ . Besides,  $V_p K[G]$  will denote the  $p$ -torsion subgroup of the group  $VK[G]$  of normed units (i.e. of units with augmentation 1), and  $I(K[G]; A)$  will denote the relative fundamental (augmentation) ideal of a ring  $K[G]$  with respect to the subgroup  $A$  of  $G$ .

First we recognize that  $G$  splits as a mixed group if  $T(G)$  is its direct factor. We follow for the most part the notations and terminology of [10, 11] and we refer the reader to [15] for a more information, too.

In the theory of the group rings the isomorphism problem posed by R. Brauer (see [3]) for an isomorphism of group algebras over all

fields plays a central role. The formulation of this problem is the following: Suppose  $K[G] \cong K[H]$  as  $K$ -algebras over all  $K$  and for some groups  $G$  and  $H$ . Then whether or not  $G \cong H$ ? Although this principal question has a negative solution in the general case, the discovery of large sorts of groups, for which the Brauer's conjecture is true, is of some significance. Here, in this investigation, we will touch on the commutative case.

In that direction we also examine the same conception in question but provided that  $K[G] = K[H]$  as  $K$ -algebras for every field  $K$ . This condition, however, provides us with much more information than  $K[G] \cong K[H]$  as  $K$ -algebras over all fields  $K$ .

In [5] the following was documented.

**THEOREM 2.1 (ISOMORPHISM THEOREMS [5]).** *Let  $G$  be a direct sum of cyclic groups or be a divisible group or be a simply presented torsion group. Then for any group  $H$ ,  $K[H] \cong K[G]$  as  $K$ -algebras over all fields  $K$  if and only if  $H \cong G$ .*

The above theorem illustrates that a complete system of invariants for the  $K$ -algebra  $K[G]$  over all  $K$  consists of  $G$ . The explanation is that  $G$  is a splitting group, and that  $K[G]$  determines the splitting of  $G$  up to an isomorphism by a reason of the specific character of the group classes. In some else cases this situation is impossible. As a matter of fact W. May ([18], page 17, Example 2) shows that there exist two nonisomorphic mixed countable of torsion-free rank one  $G$  and  $H$  (as  $H$  splits) such that for all choices of the field  $K$ , the algebras  $K[G]$  and  $K[H]$  are  $K$ -isomorphic.

*Commentary on the May's example and detailed analysis.* Suppose  $G$  is the special selected nonsplitting countable group ([18], p.17) and

$$H = T(G) \times G/T(G) \cong T(H) \times H/T(H),$$

for which

$$H/T(G) \cong G/T(G) \cong Q$$

(the field of all rationals regarded as an additive group). Therefore  $G \not\cong H$ , but it was established that  $K[G] \cong K[H]$  over all  $K$  (see

again the cited paper [18]). The cause of this phenomenon is that for any field  $K$ , the group ring  $K[G]$  (as  $K$ -algebra) does not in general provide us with enough information to decide whether or not  $G$  splits as a mixed group.

In addition, in this example  $G$  and  $H$  are not direct sums of cyclics and are not algebraically compact. By the same token,  $G$  is not a direct sum of a divisible group and of a direct sum of cyclic groups;  $T(H) = T(G)$  is not  $p$ -torsion and finally  $H_p = G_p$  is not separable for any prime number  $p$ , i.e. is not a direct sum of cyclics. Thus  $T(G)$  is not separable, is not a direct sum of cyclics and is not algebraic compact. Moreover, in the May's counterexample (see again [18])  $G$  and  $H$  are not Rotman's groups, but are simply presented whence Warfield groups.

The present study is a sequel to [4]. Here the Brauer's question will be solved in the affirmative for the following group classes which are splitting, namely:

1. Splitting  $G$  whose  $T(G)$  is a direct sum of cyclics and  $G/T(G)$  is countable.
2. Splitting  $G$  whose  $T(G)$  is countable separable.
3.  $T(G)$  is algebraic compact.
4.  $G$  is a direct sum of cyclic and quasicyclic groups.

We also settle the question for other no splitting classes of mixed abelian groups such as the algebraically compact groups.

In the proofs of these affirmations, we shall base again on the property of the group algebra to determine the splitting of the group and also to determine some special group functions which are cardinal numbers. On that tactic, we will specially utilize to the fullest extent the following result due to Berman and Mollov [2], slightly modified in a more convenient for us form.

**THEOREM 2.2.** *Let  $G$  be a group so that  $T(G)$  is separable and either  $T(G)$  or  $G/T(G)$  is countable, and let  $E$  be a finite extension of the field of rational numbers. Then  $E[G]$  splits if and only if  $G$  splits (see [2]).*

Now we can state the important results in the next paragraph entitled.

### 3. Isomorphic commutative group algebras over all fields

The following results are announced in [4] only.

#### 3.1. Direct sums of cyclic and quasicyclic groups

**PROPOSITION 3.1.** *Let  $K$  be a field of  $\text{char}K = p \neq 0$ . Then  $V_p K[G]$  is a direct sum of cyclics if and only if  $G_p$  is a direct sum of cyclics [5, 7].*

Now we are in a position to prove

**THEOREM 3.2.** *Let  $G$  be a splitting group whose torsion part  $T(G)$  is a direct sum of cyclics and such that  $G/T(G)$  is countable. Then  $K[H] \cong K[G]$  as  $K$ -algebras over every field  $K$  and for any group  $H$  if and only if  $H \cong G$ .*

*Proof.* Write

$$G \cong T(G) \times G/T(G).$$

But  $T(G)$  is a direct sum of cyclics, i.e.  $G_p$  is the same for each prime  $p$ . The  $K_p$ -isomorphism

$$K_p[G] \cong K_p[H]$$

implies

$$V_p K_p[G] \cong V_p K_p[H],$$

where  $K_p$  is a field of  $\text{char}K_p = p > 0$ . Thus, using the proposition, we have that  $H_p \subseteq V_p K_p[H]$  is a direct sum of cyclics. Moreover, the Ulm  $p$ -invariants of  $G$ , i.e. the Ulm-Kaplansky functions of  $G_p$ , are determined by  $K_p[G]$  (see [17] or, [13, 14], [1] and [3]). That is why  $G_p \cong H_p$  for all primes  $p$ , i.e.  $T(G) \cong T(H)$ .

Suppose now  $E$  is a finite extension of the field of rationals. Under the existing circumstances  $G$  splits and consequently  $E[G] \cong E[H]$  splits. From the result of Berman-Mollov,  $H$  splits as well, because a principal known assertion due to May [17] (see [9] as well) implies

$$G/T(G) \cong H/T(H),$$

hence  $H/T(H)$  is countable. Finally

$$H \cong T(H) \times H/T(H)$$

and  $G \cong H$ . The theorem is proved.  $\square$

As a direct consequence we deduce

**COROLLARY 3.3.** *Let  $G$  be a direct sum of cyclic and quasicyclic groups such that  $T(G)$  is separable and  $G/T(G)$  is countable. Then  $K[H] \cong K[G]$  as  $K$ -algebras over every field  $K$  and some group  $H$  if and only if  $H \cong G$ .*

*Proof.* We see elementarily that  $G$  splits as a direct sum of two splitting groups. Hence we can apply the theorem to obtain  $G \cong H$ .  $\square$

The first corollary may be strengthened by dropping the restrictive conditions on  $T(G)$  and  $G/T(G)$ . But before made this, we need a few preliminaries, starting with

**DEFINITION 3.4.** *We shall say that the subgroup  $N$  of the abelian group  $A$  is nice if it is  $p$ -nice for each prime natural  $p$ , i.e. in other words*

$$(A/N)^{p^\tau} = A^{p^\tau}N/N$$

for all ordinals  $\tau$  and primes  $p$ , or, equivalently,

$$\bigcap_{\alpha < \tau} (A^{p^\alpha}N) = A^{p^\tau}N$$

for all limit ordinals  $\tau$  and primes  $p$ .

The following is useful.

**LEMMA 3.5.**  *$T(A)$  is nice in  $A$  if and only if  $A_p$  is nice in  $A$  for every prime number  $p$ .*

*Proof.* The necessity is trivial since  $A_p$  is a direct factor of  $T(A)$  for each prime number  $p$ .

For the reverse, using the above formulated definition, we detect

$$\begin{aligned} \bigcap_{\alpha < \tau} (A^{p^\alpha}T(A)) &= \bigcap_{\alpha < \tau} (A^{p^\alpha}A_p) \\ &= A^{p^\tau}A_p \\ &= A^{p^\tau}T(A), \end{aligned}$$

for arbitrary, but a fixed, prime  $p$  and every limit ordinal  $\tau$ , because

$$A_q^p = A_q$$

for each prime  $q \neq p$ , i.e. because all  $A_q$  are  $p$ -divisible.  $\square$

**THEOREM 3.6.** *Suppose  $G$  is a direct sum of cyclic and quasicyclic groups. Then for some arbitrary group  $H$  the  $K$ -equality of algebras  $K[H] = K[G]$  for every field  $K$  implies  $H \cong G$ .*

*Proof.* Write

$$G \cong T(G) \times G/T(G)$$

and

$$G = d(G) \times r(G),$$

hence

$$T(G) = T(d(G)) \times T(r(G))$$

and

$$G/T(G) \cong [d(G)/T(d(G))] \times [r(G)/T(r(G))].$$

By similar conclusions,

$$H/T(H) \cong [d(H)/T(d(H))] \times [r(H)/T(r(H))].$$

But  $T(G)$  as a direct factor of the splitting  $G$  is its nice subgroup. The main purpose that we pursue is to argue that  $T(H)$  is nice in  $H$ . In fact, the Lemma means that it is sufficient to establish that  $H_p$  is a nice subgroup of  $H$  for every prime  $p$ . Next, the niceness of  $G_p$  in  $G$  guarantees that

$$V_p K_p[G] = 1 + I(K_p[G]; G_p) = V_p K_p[H]$$

is nice in  $V K_p[G] = V K_p[H]$  for some perfect field  $K_p$  of  $\text{char} K_p = p \neq 0$  (for instance see a lemma due to W. May stated and proved in [14]). But  $H_p$  is balanced in  $V_p K_p[H]$ , thus in  $V K_p[H]$ , so  $H_p$  is balanced in  $H$ , as required. So,  $T(H)$  is indeed a nice subgroup of  $H$ , whence  $r(H)/T(r(H))$  is reduced.

Further, observing via [17] that  $G/T(G) \cong H/T(H)$ , we extract that  $r(H)/T(r(H))$  is a direct sum of cyclics whence  $T(r(H))$  is a direct factor of  $r(H)$ . Therefore  $H$  splits, i.e. in other words,

$$H \cong T(H) \times H/T(H);$$

but  $T(G) \cong T(H)$ ; indeed, it is easy to see that  $T(G)$  is a direct sum of cyclic and quasicyclic groups, i.e.  $G_p$  is so for all prime integers  $p$ . Consulting with [8],  $G_p \cong H_p$  for each so  $p$ , hence the desired isomorphism of the torsion subgroups holds valid.

Finally, we derive  $G \cong H$ , as stated. The proof is completed.  $\square$

We shall proceed by proving the following key assertion.

**PROPOSITION 3.7.** *Suppose  $K[G] = K[H]$  as  $K$ -algebras for every field  $K$ . Then  $d(G) \cong d(H)$ .*

*Proof.* Complying with [10], we write down

$$d(G) \cong \left[ \prod_p d(G_p) \right] \times \prod_{r_0(d(G))} \mathbb{Q}$$

and similarly

$$d(H) \cong \left[ \prod_p d(H_p) \right] \times \prod_{r_0(d(H))} \mathbb{Q}.$$

Exploiting [17],

$$\prod_p d(G_p) \cong \prod_p d(H_p).$$

Besides, conforming with [10],

$$\begin{aligned} d(G) &= G^{\sigma+1} \\ &= G^\sigma \\ &= \bigcap_p G^{p^{\omega\sigma}}, \end{aligned}$$

where  $\sigma$  is the smallest ordinal that satisfies this property. By the reason of symmetry

$$d(H) = \bigcap_p H^{p^{\omega\sigma}}.$$

Given  $F_p$  is a field in  $\text{char} F_p = p \neq 0$  and set

$$R = \times_p \times_\omega F_p.$$



Then  $R \times F_p = R$  for each prime  $p$ . Apparently,  $\text{char} R = 0$  and  $\text{inv}(R)$ , the set of all primes  $q$  for which  $q \cdot 1_R$  is a unit in  $R$ , is empty. Since

$$F_p[G^{p^{\omega\sigma}}] = F_p \cdot (F_p[G])^{p^{\omega\sigma}}$$

and similarly

$$F_p[H^{p^{\omega\sigma}}] = F_p \cdot (F_p[H])^{p^{\omega\sigma}},$$

for every prime number  $p$  we obtain

$$R[G^{p^{\omega\sigma}}] = R[H^{p^{\omega\sigma}}]$$

because of the formal equalities

$$\begin{aligned} R[G^{p^{\omega\sigma}}] &= (R \times F_p)[G^{p^{\omega\sigma}}] \\ &= R \times (F_p[G^{p^{\omega\sigma}}]) \end{aligned}$$

and

$$\begin{aligned} R[H^{p^{\omega\sigma}}] &= (R \times F_p)[H^{p^{\omega\sigma}}] \\ &= R \times (F_p[H^{p^{\omega\sigma}}]). \end{aligned}$$

That is why

$$\begin{aligned} R[d(G)] &= R \left[ \bigcap_p G^{p^{\omega\sigma}} \right] = \bigcap_p R[G^{p^{\omega\sigma}}] \\ &= \bigcap_p R[H^{p^{\omega\sigma}}] \\ &= R \left[ \bigcap_p H^{p^{\omega\sigma}} \right] \\ &= R[d(H)]. \end{aligned}$$

Since  $p \cdot 1_R$  is not in  $\text{inv}(R)$ , there exists a maximal ideal  $M$  of  $R$  with  $p \cdot 1_R \in M$ . Putting  $F = R/M$ , a field of characteristic  $p$ , we establish  $F[d(G)] \cong F[d(H)]$ , hence in view of [17],

$$r_0(d(G)) = r_0(d(H)).$$

Finally, we infer that  $d(G) \cong d(H)$ , as claimed. The proof is finished.  $\square$

The next, formulated more generally, is a direct consequence of the above attainment when the group bases are both divisible.

**COROLLARY 3.8.** *Assume  $G$  is divisible and  $K[H] \cong K[G]$  as  $K$ -algebras for any group  $H$  and over all fields  $K$ . Then  $H \cong G$  [4, 5].*

### 3.2. Countable groups

**THEOREM 3.9.** *Let  $G$  be a splitting group for which  $T(G)$  is countable separable and let  $H$  be any group. Then  $K[H] \cong K[G]$  as  $K$ -algebras over each field  $K$  if and only if  $H \cong G$ .*

*Proof.* It is obvious that by a criterion due to Prüfer [15, 10],  $T(G)$  is a direct sum of cyclics. Therefore, as in the proof of the theorem from the above section,  $T(G) \cong T(H)$ . Further, utilizing step-by-step the above preceding schema for a proof, we may conclude that  $G \cong H$ . The proof is fulfilled.  $\square$

*Remark.* Assuming that  $T(G)$  is not separable, we detect that there is  $G$  so that it can not be retrieved from  $K[G]$  (of course as  $K$ -algebra) over all  $K$  [18].

**COROLLARY 3.10.** *Let  $G$  be a splitting countable group whose torsion part  $T(G)$  is separable and let  $H$  be a group. Then  $K[H] \cong K[G]$  as  $K$ -algebras over each  $K$  if and only if  $H \cong G$ .*

The proof follows immediately from the theorem.

*Remark.* This corollary may be obtained also from the Theorem (3.2) established in the above section.

### 3.3. Algebraically compact and cotorsion groups

**THEOREM 3.11.** *Let  $G$  be a group whose torsion part  $T(G)$  is algebraic compact. Then  $K[H] \cong K[G]$  as  $K$ -algebras for every field  $K$  and any group  $H$  if and only if  $H \cong G$ .*

*Proof.* It is well-known that (see [10])  $G$  splits, i.e.

$$G \cong T(G) \times G/T(G).$$

Suppose again  $K_p$  is a field of char  $K_p = p$ . For each prime  $p$ , we will prove that  $G_p \cong H_p$  and so  $T(G) \cong T(H)$ .

Really,  $G_p$  is algebraic compact since  $T(G)$  is. Therefore  $G_p$  has a bounded reduced part  $r(G_p)$  [10]. But this is equivalent to  $G_p^{p^i}$  is divisible for some natural  $i$ .

Select now  $A = G^{p^i}$ , so  $A_p = G_p^{p^i}$  is divisible; and select  $B = H^{p^i}$ , so  $B_p = H_p^{p^i}$  for this  $i$ . We have  $K_p[H] = K_p[G]$ , where  $H$  is a normalized group basis ( $H \subseteq VK_p[G]$ ), and besides without loss of generality we may assume that  $K_p$  is perfect. Hence  $K_p[A] = K_p[B]$  along with [5] imply

$$\begin{aligned} 1 + I(K_p[A]; A_p) &= V_p K_p[A] \\ &= V_p K_p[B] \\ &= 1 + I(K_p[B]; B_p), \end{aligned}$$

and

$$\begin{aligned} 1 + I^p(K_p[A]; A_p) &= 1 + I(K_p[A^p]; A_p) \\ &= 1 + I(K_p[B^p]; B_p^p) \\ &= 1 + I^p(K_p[B]; B_p). \end{aligned}$$

Furthermore

$$I(K_p[A]; A_p) = I(K_p[B]; B_p^p),$$

because  $K_p[A] = K_p[B]$ . Finally,

$$I(K_p[B]; B_p^p) = I(K_p[B]; B_p),$$

i.e. clearly  $B_p^p = B_p$  and thus  $B_p$  is divisible. Consequently  $H_p$  has a bounded reduced part  $r(H_p)$ , i.e.  $H_p$  is algebraic compact by virtue of [10]. But as we see in [17], the Ulm-Kaplansky invariants of  $r(G_p)$  and  $r(H_p)$  are equal and thus immediately  $r(G_p) \cong r(H_p)$  [11]. Using the classical result of May [17] (see also [13, 14]),  $K_p[G] = K_p[H]$  implies  $d(G_p) \cong d(H_p)$ , where  $d(G_p)$  and  $d(H_p)$  are the maximal divisible parts, respectively in  $G_p$  and in  $H_p$ . Thus,  $G_p \cong H_p$  for all primes  $p$ . Further

$$H \cong T(H) \times H/T(H).$$

But we employ again a statement of May in [17] (see [1] too) to obtain,

$$G/T(G) \cong H/T(H).$$

So,  $G \cong H$  and we are done.  $\square$

**COROLLARY 3.12.** *Let  $G$  be a splitting algebraic compact group and  $H$  is a group. Then  $K[H] \cong K[G]$  as  $K$ -algebras for every  $K$  if and only if  $H \cong G$ .*

*Proof.* By a supposition,  $T(G)$  is a direct factor of  $G$ . Hence  $T(G)$  is algebraic compact invoking to [10] and we have need only apply the theorem.  $\square$

**COROLLARY 3.13.** *Let  $G$  be a direct sum of a divisible group and of a bounded group and  $H$  is a group. Then  $K[H] \cong K[G]$  as  $K$ -algebras for every  $K$  if and only if  $H \cong G$ .*

*Proof.* Trivially,  $G$  is a splitting algebraic compact group. Consequently the last corollary is applicable.  $\square$

**COROLLARY 3.14.** *Let  $G$  be countable algebraic compact or torsion algebraic compact and  $H$  is a group. Then  $K[H] \cong K[G]$  as  $K$ -algebras for all  $K$  if and only if  $H \cong G$ .*

*Proof.* The facts in [10] yield that  $G$  is a direct sum of a divisible group and a bounded group. Furthermore the last corollary verifies the claim.  $\square$

*Remark.* If  $G$  is torsion or torsion-free, then via [10]  $G$  is algebraic compact if and only if  $G$  is cotorsion. Therefore if  $G$  splits, then the same conclusion is valid. Besides if  $G$  is countable, then the same observation is again true (see [10]). A well-known connection between these two group classes is the important result due to K. Rangaswamy that any abelian cotorsion group is algebraically compact if and only if its periodical (= torsion) part is torsion-complete.

We end the current section with

**THEOREM 3.15.** *Suppose  $G$  and  $H$  are both algebraic compact abelian groups and  $K[G] = K[H]$  as  $K$ -algebras over each field  $K$ . Then  $G \cong H$ .*

*Proof.* Employing Proposition (3.7), we have  $d(G) \cong d(H)$ . Moreover, by making use of [17], the  $p$ -Ulm-Kaplansky invariants of  $G$  and  $H$  are equal for every prime number  $p$ . After this, we consider the cardinal number

$$\dim_{F_p}(G/(G^p T(G))) = \dim_{F_p}(G/(G^p G_p))$$

taken over the simple field of  $p$ -elements  $F_p$  for some fixed but arbitrary prime integer  $p$ ; it is an invariant for the group  $G$ . Evidently the groups  $G$  and  $G/G_p$  have equal such functions, and besides  $K[G] \cong K[H]$  implies

$$K[G/G_p] \cong K[H/H_p]$$

(see for example [17] or [9]), so we will assume further in the sequel that all group algebras considered are semisimple. But the application of [13, 14] leads us to this that the studied cardinal is equal to

$$\dim_F(I(F[G]; G)/[I(F[G]; G^p)I(F[G]; G) + I(F[G]; T(G))])$$

for the field  $F$  which is in characteristic  $p$ . It is a simple matter to see that  $I(F[G]; G)$  and

$$\begin{aligned} F[G]I^p(F[G]; G) &= F[G].I(F^p[G^p]; G^p) \\ &= I(F[G]; G^p) \end{aligned}$$

are both deduced from  $F[G]$ . On the other hand  $I(F[G]; T(G))$  or  $I(F[G]; G_p)$  is a structural invariant for  $F[G]$  consulting with [17] or [9]. Finally, the application of [16] ensures that  $G \cong H$ , as desired. The proof is concluded.  $\square$

**COROLLARY 3.16.** *Suppose  $G$  and  $H$  are both reduced algebraic compact abelian groups and  $K[G] \cong K[H]$  are  $K$ -isomorphic over every field  $K$ . Then  $G \cong H$ .*

### 3.4. Rotman's groups

Following [21], a mixed countable abelian group of torsion-free rank one is said to be a Rotman's group (a KM-group in the terms of [21]) if for any prime number  $p$  its element has an infinite  $p$ -height if and only if this element has a finite order relatively prime to  $p$ . These groups are known to be strongly presented, hence Warfield.

Foremost, we mention that in the May's counterexample (see again [18])  $G$  and  $H$  are not groups of Rotman.

The next attainment is useful.

PROPOSITION 3.17. *Suppose  $G$  is a Rotman group and  $K[H] \cong K[G]$  as  $K$ -algebras for some arbitrary but a fixed group  $H$  and over each field  $K$ . Then  $H$  is a Rotman group.*

*Proof.* Clearly  $H$  is abelian. Let now we fix an arbitrary prime  $p$  and choose an arbitrary element  $x \in H$  such that  $p - \text{height}_H(x)$  is infinite. On the other hand we may write  $K_p[H] = K_p[G]$  for some perfect field  $K_p$  in characteristic  $p > 0$ . Write

$$x = \alpha_1 g_1 + \cdots + \alpha_n g_n,$$

where  $\alpha_i \in K_p$  with  $\alpha_1 + \cdots + \alpha_n = 1$ ,  $g_i \in G$ , and  $1 \leq i \leq n \in \mathbb{N}$ . Since

$$p - \text{height}_H(x) = \min_{1 \leq i \leq n \in \mathbb{N}} \{p - \text{height}_G(g_i)\},$$

we can deduce

$$p - \text{height}(g_1), \cdots, p - \text{height}(g_n)$$

are infinite. Thus by the definition of a Rotman group,

$$\text{order}(g_1), \cdots, \text{order}(g_n)$$

are finite and moreover

$$(\text{order}(g_1), p) = 1, \cdots, (\text{order}(g_n), p) = 1.$$

That is why  $\text{order}(x)$  is finite and  $(\text{order}(x), p) = 1$ . Really, the first is evident because of the formula  $T(VP[C]) = VP[T(C)]$ , valid for each semisimple group algebra  $P[C]$  (see, for example, [9]). For the second, if  $p$  divides  $\text{order}(x)$ , we can write

$$\text{order}(x) = p^s r,$$

where  $s$  is natural and  $(p, r) = 1$ . Therefore,

$$\begin{aligned} 1 &= x^{p^s r} \\ &= (\alpha_1 g_1 + \cdots + \alpha_n g_n)^{p^s r} \\ &= (\alpha_1^{p^s} g_1^{p^s} + \cdots + \alpha_n^{p^s} g_n^{p^s})^r, \end{aligned}$$

and so we may deduce that there are indices

$$1 \leq i \neq j \neq \cdots \neq k \leq n$$

with the property

$$(g_i^{\epsilon_i} g_j^{\epsilon_j} \cdots g_k^{\epsilon_k})^{p^s} = 1,$$

where  $\epsilon_i, \epsilon_j, \cdots, \epsilon_k$  are integers that satisfy

$$\epsilon_i + \epsilon_j + \cdots + \epsilon_k = r.$$

But this is a contradiction since it is easy to verify that

$$(\text{order}(g_i^{\epsilon_i} g_j^{\epsilon_j} \cdots g_k^{\epsilon_k}), p) = 1$$

conforming with the conditions

$$(\text{order}(g_i), p) = (\text{order}(g_j), p) = \cdots = (\text{order}(g_k), p) = 1.$$

This gives our claim.

Further,  $K[G] \cong K[H]$  implies  $|G| = |H|$ , so  $H$  is countable. Moreover, bearing in mind [17], it implies  $G/T(G) \cong H/T(H)$ , so  $H$  is with torsion-free rank equal to this of  $G$ .

Finally, inspired by the definition, we can conclude  $H$  is a Rotman's group, thus completing the proof.  $\square$

### 3.5. Warfield groups

The definition of global Warfield abelian groups was given in [12]. It is proved there that the Warfield  $p$ -invariants defined as above in the case of algebraically compact groups, namely

$$\text{rank}(G^{p^\alpha}/(T(G^{p^\alpha})G^{p^{\alpha+1}})) = \text{rank}(G^{p^\alpha}/(G_p^{p^\alpha}G^{p^{\alpha+1}})),$$

where  $p$  is a prime and  $\alpha$  is an ordinal, along with the classical Ulm-Kaplansky  $p$ -invariants (see, for instance, [11]) taken for any prime number  $p$  serve up to an isomorphism the  $p$ -mixed objects form this group class. Nevertheless, this is not the case for arbitrary global Warfield groups even order all primes.

So, we come to

PROPOSITION 3.18. *Suppose  $G$  and  $H$  are both commutative Warfield groups and for every field  $K$  the  $K$ -algebras equality  $K[G] = K[H]$  holds valid. Then  $d(G) \cong d(H)$  and  $T(G) \cong T(H)$ .*

*Proof.* The first half follows from Proposition (3.7). The second one follows from the fact that the torsion subgroups are  $S$ -groups (see [11, 20]), in virtue of the invariance of the Warfield  $p$ -functions just demonstrated by us above, of the invariance of Ulm-Kaplansky  $p$ -functions by [17] and together with the previous discussion.  $\square$

*Remark.* As we have above observed, when  $G$  and  $H$  are both reduced, the condition on  $K$ -equality of algebras may be replaced by the more weak condition on  $K$ -isomorphism of algebras.

We close the article with

#### 4. Open questions and problems

As we verified, if

$$G \cong T(G) \times G/T(G),$$

where  $T(G)$  is a direct sum of cyclics and  $G/T(G)$  is countable, then  $G$  may be gotten from  $K[G]$  over every field  $K$ . The first query is that if  $G$  is uncountable modulo torsion, does the above property hold? Moreover, whether the property algebraic compactness of  $G$  can be invariantly recaptured from the algebras  $K[G]$  over every field  $K$ .

Now, we will pose some questions for nonsplitting in general mixed groups.

First, suppose  $G$  a mixed group of torsion-free rank one so that  $T(G)$  is closed or a direct sum of cyclics (see [19]). Is then  $G$  retrieved from  $K[G]$  for all fields  $K$ ? The answer is probably no.

Secondly, following the paper [21], a mixed countable abelian group of torsion-free rank one is said to be a Rotman's group (a KM-group) if for any prime  $p$  its element has an infinite  $p$ -height if and only if this element has a finite order relatively prime to  $p$ . So, if  $G$  is such a group, is then it recovered from  $K[G]$  for all fields  $K$ ? The solution is probably no as well.

It is appeared in [18] that  $K[G] \cong K[H]$  as  $K$ -algebras over every field  $K$  is not enough to imply that the Ulm-height matrices  $U(G)$



and  $U(H)$  are equivalent. But whether or not this remains valid under the more stronger conditions  $K[G] = K[H]$  as  $K$ -algebras for each field  $K$ , is unknown yet.

As a final question, we raise the following:

**General Conjecture.** *Suppose  $F[G] = F[H]$  as  $F$ -algebras for each field  $F$ . Then  $G \cong H$  holds.*

In particular, does it follow that  $G$  splitting yields  $H$  is splitting?

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Received April 9, 2003.