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Dispersive Estimate for the Wave Equation with Short-Range Potential

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Summary. - *In this paper we consider a potential type perturbation of the three dimensional wave equation:*

$$\begin{cases} \Box u + V(x)u = 0 \\ u(x,0) = 0, \partial_t u(x,0) = f, \end{cases}$$

where the potential $V \geq 0$ satisfies the following decay assumption:

$$|V(x)| \leq \frac{C}{(1+|x|)^{2+\epsilon_0}},$$

for some $C, \epsilon_0 > 0$. We establish some dispersive estimates for the associated propagator.

1. Introduction

The semilinear wave equation:

$$(\partial_t^2 - \Delta + V)u = F(u), \tag{1}$$

can be considered as a perturbation of the classical semilinear wave equation

$$(\partial_t^2 - \Delta)u(t,x) = F(u). \tag{2}$$

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To study the local and global existence of solution to the corresponding Cauchy problem one needs Strichartz type estimates for the linear Cauchy problem:

$$\begin{cases} (\partial_t^2 - \Delta)u(t,x) + V(x)u(t,x) = 0, & x \in \mathbb{R}^3 \\ u(0,x) = 0, \partial_t u(0,x) = f \ . \end{cases} \tag{3}$$

If the potential $V(x) \geq 0$ is such that the operator $-\Delta + V$ is self-adjoint [13], then one can prove that the solution $u(t,x)$ can be represented as follows:

$$u(t,x) := \mathcal{U}_V(t)f = \frac{\sin\left(t\sqrt{-\Delta+V}\right)}{\sqrt{-\Delta+V}} f.$$

One of the classical Strichartz estimates (valid for 3-dimensional space and for $V=0$) is the following one:

$$\|\mathcal{U}_0(t)f\|_{L^4(\mathbb{R}^3)} \leq \frac{C}{\sqrt{t}} \|f\|_{L^{4/3}(\mathbb{R}^3)}, \tag{4}$$

where $f \in C_0^\infty(\mathbb{R}^3)$ and $\mathcal{U}_0(t)$ is the free propagator

$$\mathcal{U}_0(t) := \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}.$$

The Strichartz estimate for the case of potential type perturbation is established in [2] and [6]. The assumptions in these works require that the potential decays very rapidly at infinity. For instance, the assumption $V(x) = O(|x|^{-3-\epsilon_0}), \varepsilon_0 > 0$, is assumed in [6]. Here we shall relax the assumptions on the potential to the following one, $V(x) = O(|x|^{-2-\epsilon_0})$ as $|x| \to \infty$. We shall consider potentials that are not necessarily radial. The case $V(x) = a/|x|^2$ with radially symmetric data f is studied in [11]. The same potential is considered in [5], where the case of the non-radial initial data is treated.

In our case we shall assume that the potential $V(x) \geq 0$ is a measurable function and for some $\varepsilon_0 > 0$ and $C > 0$ the following inequalities is satisfied:

$$|V(x)| \leq \frac{C}{(1+|x|)^{2+\epsilon_0}} \tag{5}$$

The above conditions in particular guarantee that the operator $-\Delta + V(x)$ is a self - adjoint operator in $L^2(\mathbb{R}^3)$. This property implies that the resolvent operator

$$R_V(z) := (z + \Delta - V)^{-1}$$

is a well - defined bounded operator in $L^2(\mathbb{R}^3)$, if $z \in \mathbb{C} \setminus \mathbb{R}$. Using suitable L^2 weighted estimates of $R_V(\lambda^2 \pm i\varepsilon)$ for $\varepsilon \in (0,1]$ and $\lambda \in \mathbb{R}$ it is possible to prove the existence of a natural limit operator $R_V(\lambda^2 \pm i0)$ defined as follows,

$$R_V(\lambda^2 \pm i0)f = \lim_{\varepsilon \to 0} R_V(\lambda^2 \pm i\varepsilon)f.$$

(see theorem 4.2, page 166 in [1]).

In the sequel we will use the following notation :

$$R_0(\lambda^2 \pm i0)f := \lim_{\varepsilon \to 0}(\lambda^2 \pm i\varepsilon + \Delta)^{-1}f,$$

for $f \in C_0^\infty(\mathbb{R}^3)$.

Here and below for $p \geq 1$ and $\sigma \in \mathbb{R}$ we denote by $L^{p,\sigma}$ the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the following norm:

$$\|f\|^p_{L^{p,\sigma}} = \int_{\mathbb{R}^3} |f|^p \langle x \rangle^\sigma dx.$$

where $\langle x \rangle$ denotes the following quantity:

$$\langle x \rangle := \sqrt{1 + |x|^2}.$$

In [7] the following result is proved.

THEOREM 1.1. *Assume that the potential $V \geq 0$ satisfies the decay assumption* (5) *then the following properties are satisfied:*

1. *for any $\delta > 0$ small there exists a real constant $C = C(\delta) > 0$ such that*

$$\|R_0(\lambda^2 \pm i0)VU\|_{L^{2,-3-\delta}} \leq C\|U\|_{L^{2,-3-\delta}}, \ \forall U \in C_0^\infty(\mathbb{R}^3)$$

2. *the following equation $R_0(\lambda^2 \pm i0)VU = U$ has no solution $U \in L^{2,-3-\delta} \setminus \{0\}$ for $\delta > 0$ small and for any $\lambda \in \mathbb{R}$.*

The main dispersive estimate obtained in [7] is the following:

$$\|\mathcal{U}_V(t)f\|_{L^4(\mathbb{R}^3)} \leq \frac{C}{\sqrt{t}}\|f\|_{L^{2,\frac{3}{2}+\epsilon}}, \tag{6}$$

where $C = C(\epsilon) > 0$ is a real constant.

We are now ready to state the first result of this paper. This is a local (in time) dispersive estimate.

THEOREM 1.2. *Assume that the potential satisfies the hypothesis* (5), *then for* $0 < t < 1$ *there exists a real constant* $C > 0$ *such that the following estimate holds:*

$$\|\mathcal{U}_V(t)f\|_{L^4} \leq \frac{C}{\sqrt{t}}\|f\|_{L^{4/3}}. \tag{7}$$

For large values of t we shall obtain the following decay estimate.

THEOREM 1.3. *Assume that the potential* $V \geq 0$ *satisfies the assumption* (5). *Then for any* $\epsilon > 0, \delta > 0$ *and* $2 < p < \infty$ *such that* $p\delta > 6$, $\epsilon p < 2$ *and*

$$-\epsilon^2 p^2 + 2\epsilon p + 2p - 4 - 2\epsilon p^2 - 2\delta p - 2\delta\epsilon p + 4\delta \geq 0,$$

there exists a real constant $C = C(p, \delta, \epsilon) > 0$ *such that for any* $t > 1$ *the following estimate is satisfied:*

$$\|\mathcal{U}_V(t)f\|_{L^{p_\epsilon}} \leq \frac{C}{\sqrt{t}}\|f\|_{L^{q_\epsilon,\rho(p,\epsilon,\delta)}}$$

where:

$$p_\epsilon = \frac{4p + 2\epsilon p - 8}{p + \epsilon p - 2}, \frac{1}{p_\epsilon} + \frac{1}{q_\epsilon} = 1,$$

$$\rho(p, \epsilon, \delta) = \max\{\frac{8 - 2p_\epsilon}{p_\epsilon}, \beta(p, \delta)(1 - \theta_\epsilon)q_\epsilon + \delta\},$$

and

$$\theta_\epsilon = \frac{p - 4 + \epsilon p}{2p - 4 + \epsilon p}, \qquad \beta(p, \delta) = \frac{6}{p} + \frac{2\delta}{p}.$$

For the proof of theorem 1.3 the key point is the following representation of the perturbed propagator:

$$\mathcal{U}_V(t)f := \int_0^\infty \sin\lambda t \left[R_V(\lambda^2+i0) - R_V(\lambda^2-i0)\right] f d\lambda, \qquad (8)$$

that we will use in combination with some L^p weighted estimates for the limit resolvent operator $R_V(\lambda^2 \pm i0)$.

The idea to use the representation (8) for the proof of Strichartz type estimate has been used in [16], where an alternative proof of the classical Strichartz estimate is obtained in the case $V = 0$. The same representation formula was used in [7] to obtain the estimate (6). The type of estimate for the resolvent limit that we need in the sequel have the general form

$$\|R_V(\lambda^2 \pm i0)f\|_{L^{p,\sigma}} \leq \frac{C}{\lambda^A}\|f\|_{L^{2,s}} \qquad (9)$$

with $C = C(p,s,\sigma) > 0, A = A(p,s,\sigma) \geq 0$ suitably chosen.

In this paper the estimates of the type (9) will be useful for the estimate of the following truncated operators:

$$\mathcal{U}_{V,j}(t)f = \int_{\mathbb{R}} \phi_j(\lambda)\sin\lambda t \left[R_V(\lambda^2+i0) - R_V(\lambda^2-i0)\right] f d\lambda, \quad (10)$$

for $j < 0$ and

$$\mathcal{U}_{V,high}(t)f = \int_{\mathbb{R}} \psi(\lambda)\sin\lambda t \left[R_V(\lambda^2+i0) - R_V(\lambda^2-i0)\right] f d\lambda, \ (11)$$

where $\psi(\lambda)$ and $\phi_j(\lambda)$ are smooth functions such that $\sum_{j<0}\phi_j(\lambda) + \psi(\lambda) = 1$ and $\phi_j(\lambda) = \phi(\frac{\lambda}{2^j})$, where $\phi \in C_0^\infty(\mathbb{R})$ is such that supp$\phi \subset [\frac{1}{2}, 1]$).
Note that the following identity is fulfilled trivially:

$$\mathcal{U}_V(t)f = \sum_{j<0}\mathcal{U}_{V,j}(t)f + \mathcal{U}_{V,high}(t)f.$$

The plan of the work is the following. In Section 2 we give various estimate of the free resolvent and its square. To evaluate $R_V(\lambda^2 \pm i0)$ uniformly in λ we will use the Fredholm theory. These estimates are discussed in Sections 3 and 4. In Section 5 we prove theorems 1.2, and we prove a weak form of theorem 1.3. In section 6 we prove theorem 1.3 using the finite propagation speed property and the weaker form of the theorem obtained in the previous section.

2. Resolvent estimates for the free Laplacian

In this section we give some a priori weighted estimates for the resolvent of the free Laplacian $R_0(\lambda^2 \pm i0)$ and for its square $R_0^2(\lambda^2 \pm i0)$. They will be usefull to prove some a priori estimates for the perturbed resolvent $R_V(\lambda^2 \pm i0)$ and for its square $R_V^2(\lambda^2 \pm i0)$.

Here and in next sections we will use the following notations:

$$\alpha(p, \delta) = \frac{2}{p} + \frac{2\delta}{p}, \quad \beta(p, \delta) = \frac{6}{p} + \frac{2\delta}{p},$$

where $p \in [1, \infty]$ and $\delta > 0$.

Lemma 2.1. *The following estimates are satisfied:*

1. *For any $\delta > 0$ there exists a real constant $C = C(\delta) > 0$ such that for any $\lambda \in \mathbb{R}$ we have the following estimates:*

$$\|R_0(\lambda^2 \pm i0)f\|_\infty \leq C\|f\|_{L^{2,1+\delta}}; \tag{12}$$

$$\|R_0(\lambda^2 + i0)f - R_0(\lambda^2 - i0)f\|_{L^\infty} \leq C\lambda\|f\|_{L^{2,3+\delta}}; \tag{13}$$

2. *for any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta, \delta') > 0$ such that for any $\lambda \in \mathbb{R}$ we have the following estimates:*

$$\|R_0(\lambda^2 \pm i0)f\|_{L^{2,-1-\delta}} \leq C\|f\|_{L^{2,3+\delta'}}; \tag{14}$$

$$\|R_0(\lambda^2 \pm i0)f\|_{L^{2,-1-\delta}} \leq \frac{C}{\lambda}\|f\|_{L^{2,1+\delta'}}; \tag{15}$$

$$\|R_0(\lambda^2 + i0)f - R_0(\lambda^2 - i0)f\|_{L^{2,-3-\delta}} \leq C\|f\|_{L^{2,1+\delta'}}; \tag{16}$$

3. *for any $\delta, \delta' > 0$ and for any $2 < p < \infty$ there exists a real constant $C = C(p, \delta, \delta') > 0$ such that for any $\lambda \in \mathbb{R}$ we have the following estimates:*

$$\|R_0(\lambda^2 \pm i0)f\|_{L^{p,-\alpha(p,\delta)}} \leq \frac{C}{\lambda^{\frac{2}{p}}}\|f\|_{L^{2,1+\delta'}} \tag{17}$$

$$\|R_0(\lambda^2 \pm i0)f\|_{L^{p,-\alpha(p,\delta)}} \leq C\|f\|_{L^{2,1+\delta'+\frac{4}{p}}} \tag{18}$$

$$\|R_0(\lambda^2 + i0)f - R_0(\lambda^2 - i0)f\|_{L^{p,-\beta(p,\delta)}} \leq C\lambda^{1-\frac{2}{p}}\|f\|_{L^{2,3-\frac{4}{p}+\delta'}} \tag{19}$$

Proof of (12). Using the following representation formula:

$$R_0(\lambda^2 \pm i0)f(x) = \int_{\mathbb{R}^3} \frac{e^{\pm i\lambda|x-y|}}{|x-y|} f(y)dy, \tag{20}$$

we deduce the following estimate:

$$|R_0(\lambda^2 \pm i0)f(x)| \leq \int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|} dy \leq$$

$$\leq \left(\int_{\mathbb{R}^3} \frac{1}{|x-y|^2 \langle y \rangle^{1+\delta}} dy \right)^{\frac{1}{2}} \|f\|_{L^{2,1+\delta}}.$$

In the proof of lemma 2.1 in [7] the following estimate is proved:

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|^2 \langle y \rangle^{1+\delta}} dy \leq \frac{C}{\langle x \rangle^{\delta}}, \tag{21}$$

where $C > 0$. Then we obtain the desired estimate. □

Proof of (13). Using the previous representation formula for the fundamental solution we deduce the following identity:

$$R_0(\lambda^2 + i0)f(x) - R_0(\lambda^2 - i0)f(x) = \int_{\mathbb{R}^3} \frac{\sin \lambda|x-y|}{|x-y|} f(y)dy. \tag{22}$$

This identity and the boundedness of the real function $\frac{\sin t}{t}$ imply the following estimate:

$$|R_0(\lambda^2 + i0)f(x) - R_0(\lambda^2 - i0)f(x)| \leq$$
$$\leq |\lambda| \int_{\mathbb{R}^3} |\frac{\sin \lambda|x-y|}{\lambda|x-y|}||f(y)|dy \leq |\lambda| \|f\|_{L^1}$$

Using the Hölder inequality we obtain $\|f\|_{L^1} \leq C(\delta)\|f\|_{L^{2,3+\delta}}$ that combined with the previous estimate give (13). □

Proof of (14). Using the representation formula (20) we have:

$$\|R_0(\lambda^2 \pm i0)\|^2_{L^{2,-1-\delta}} \leq \int_{\mathbb{R}^3} \langle x \rangle^{-1-\delta} \left(\int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|} dy \right)^2 dx$$

then for the Cauchy inequality we have:

$$\|R_0(\lambda^2 \pm i0)f\|^2_{L^{2,-1-\delta}} \leq \|f\|^2_{L^{2,1+\delta'}} \left(\int_{\mathbb{R}^6} \frac{1}{\langle x\rangle^{1+\delta}|x-y|^2\langle y\rangle^{1+\delta'}} dxdy \right).$$

Since the integral $\int_{\mathbb{R}^6} \frac{1}{\langle x\rangle^{1+\delta}|x-y|^2\langle y\rangle^{1+\delta'}} dxdy$ is bounded we have the result. □

- *Proof of* (15). A complete proof of this estimate can be found in [3]. The key point of the estimate is theorem 14.2.2 in [9].
- *Proof of* (16). It is the adjoint estimate of (14) for $\delta = \delta'$.
- *Proof of* (17). It is an interpolation between (12) and (15).
- *Proof of* (18). It is obtained interpolating between (12) and (14).
- *Proof of* (19). It is obtained interpolating between (13) and (16).

In next lemma we prove some estimates for the square of the free resolvent that will be useful to perform an integration by parts in the representation formula (8) to obtain a decay estimate in time for the perturbed propagator $\mathcal{U}_V(t)$.

Lemma 2.2. *The following estimates are satisfied:*

1. *For any $\delta > 0$ there exists a real constant $C = C(\delta) > 0$ such that for any $\lambda > 0$ the following estimate is satisfied*

$$\|R_0^2(\lambda^2 \pm i0)f\|_{L^\infty} \leq \frac{C}{\lambda}\|f\|_{L^{2,3+\delta}}; \tag{23}$$

2. *for any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta, \delta') > 0$ such that for any $\lambda > 0$ the following estimate is satisfied*

$$\|R_0^2(\lambda^2 \pm i0)f\|_{L^{2,-3-\delta}} \leq \frac{C}{\lambda}\|f\|_{L^{2,3+\delta'}}; \tag{24}$$

$$\|R_0^2(\lambda^2 \pm i0)f\|_{L^{2,-3-\delta}} \leq \frac{C}{\lambda^2}\|f\|_{L^{2,3+\delta'}} \tag{25}$$

3. *for any* $2 < p < \infty$ *and for any* $\delta, \delta' > 0$ *there exists a real constant* $C = C(p, \delta, \delta') > 0$ *such that for any* $\lambda \in \mathbb{R}$ *the following estimate is satisfied*

$$\|R_0^2(\lambda^2 \pm i0)f\|_{L^{p,-\beta(p,\delta)}} \le \frac{C}{\lambda^{1+\frac{2}{p}}}\|f\|_{L^{2,3+\delta'}}; \tag{26}$$

$$\|R_0^2(\lambda^2 \pm i0)f\|_{L^{p,-\beta(p,\delta)}} \le \frac{C}{\lambda}\|f\|_{L^{2,3+\delta'}}. \tag{27}$$

Proof of (23). We have the following identity:

$$2\lambda R_0^2(\lambda^2 \pm i0)f = \frac{d}{d\lambda}R_0(\lambda^2 \pm i0)f = ie^{\pm i\lambda|x|} * f,$$

then

$$R_0^2(\lambda^2 \pm i0)f(x) = \frac{C}{\lambda}\left(\int_{\mathbb{R}^3} e^{\pm i\lambda|x-y|}f(y)dy\right). \tag{28}$$

Using the boundednes of the function e^{it} we obtain the following estimate:

$$|R_0^2(\lambda^2 \pm i0)f(x)| \le \frac{C}{\lambda}\|f\|_{L^1}$$

that combined with the inequality,

$$\|f\|_{L^1} \le C(\delta)\|f\|_{L^{2,3+\delta}}$$

give (23). □

Proof of (24). Using (28) we have the following estimate:

$$\|R_0^2(\lambda^2 \pm i0)\|^2_{L^{2,-3-\delta}} \le \frac{C}{\lambda}\int_{\mathbb{R}^3}\langle x\rangle^{-3-\delta}\left(\int_{\mathbb{R}^3}|f(y)|dy\right)^2 dx$$

then for the Hölder inequality we have:

$$\|R_0^2(\lambda^2 \pm i0)\|^2_{L^{2,-3-\delta}} \le \frac{C}{\lambda}\|f\|^2_{L^{2,3+\delta'}}\left(\int_{\mathbb{R}^6}\frac{1}{\langle x\rangle^{3+\delta}\langle y\rangle^{3+\delta'}}dxdy\right).$$

Since the integral $\int_{\mathbb{R}^6}\frac{1}{\langle x\rangle^{3+\delta}\langle y\rangle^{3+\delta'}}dxdy$ is bounded we have the desired estimate. □

- *Proof of* (25). This estimate is contained in Theorem 1.1 of [10].
- *Proof of* (26). It is an interpolation between (23) and (25).
- *Proof of* (27). It is an interpolation between (23) and (24).

3. Fredholm theory

In this section we use the Fredholm theory to prove the invertiblility of a one-parameter family of linear operators in some usefull Banach spaces. The key point of the proof will be a compactness result.

The main result of the section is the following.

THEOREM 3.1. *Assume that the potential* $V(x) \geq 0$ *satisfies hypothesis* (5), *then there exists a* $\bar{\delta} > 0$ *such that given any* $\bar{\delta} > \delta > 0$ *and given any* $p > 2$ *such that* $p\delta > 6$, *the family of operators* $\left[I - R_0(\lambda^2 \pm i0)V\right]$ *are invertible in* $\mathcal{L}(L^{p,-\alpha(p,\delta)}, L^{p,-\alpha(p,\delta)}$.

Moreover there exists a real constant $C = C(p, \delta) > 0$ *such that the following estimate is satisfied for any* $\lambda \in \mathbb{R}$:

$$\|[I - R_0(\lambda^2 \pm i0)V]f\|_{L^{p,-\alpha(p,\delta)}} > C\|f\|_{L^{p,-\alpha(p,\delta)}}.$$

The proof is a consequence of the Fredholm theory and of next two lemmas.

LEMMA 3.2. *Assume that the potential satisfies hypothesis* (5). *Then there exists a* $\bar{\delta} > 0$ *such that given any* $\bar{\delta} > \delta > 0$, $p > 2$ *and* $p\delta > 6$ *then the one - parameter family of operators* $R_0(\lambda^2 \pm i0)V$ *are compact endomorphism of the space Banach space* $L^{p,-\alpha(p,\delta)}$.

Proof. First we prove the continuity of the operators $R_0(\lambda^2 \pm i0)V$. Using the Hölder inequality and the decay assumption for $V(x)$ we can prove that there exists a real constant $C = C(p, \delta) > 0$ such that

$$\left(\int_{\mathbb{R}^3} V^2 u^2 \langle x\rangle^{1+\delta} dx\right)^{\frac{1}{2}} \leq C\left(\int_{\mathbb{R}^3} |u|^p \langle x\rangle^{-\alpha(p,\delta)} dx\right)^{\frac{1}{p}}. \qquad (29)$$

provided that $\frac{3p+2\epsilon_0 p-\delta p-2\alpha(p,\delta)}{p-2} > 3$ or equivalently $p(\delta - 2\epsilon_0) + \frac{4}{p} + \frac{4\delta}{p} < 6$. It is easy to prove that this inequality is satisfied if $\delta < min\{2\epsilon_0, 2\}$ and $p > 2$.
A combination of estimates (29) and (17) imply that the operators $R_0(\lambda^2 \pm i0)V$ belong to the space $\mathcal{L}(L^{p-\alpha(p,\delta)}, L^{p-\alpha(p,\delta)})$.

We now prove the compactness of the operators.
Let be given a sequence $\{u_n\}$ bounded in $L^{p,-\alpha(p,\delta)}$ and let us consider the following sequences $f_n = R_0(\lambda^2 \pm i0)Vu_n$ and $v_n = Vu_n$. We split the proof of the compactness of f_n in two parts.

Compactness on bounded set.

Using elliptic regularity, see for example [8], we deduce that for any $R > 0$ there exists a constant $C = C(R) > 0$ such that

$$\|f_n\|_{H^2(B(0,R))} \leq C(R)\|u_n\|_{L^{p,-\alpha(p,\delta)}} \tag{30}$$

where $B(R)$ denotes the boule centered in the origin and of radius R. Given a cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ with $supp\chi \subset B(0,1)$ we consider the following truncated functions $f_{n,R} := \chi(\frac{x}{R})f_n$, for (30) this sequence is bounded in $H^2(B(0,2R)$ then for the compactness of the Sobolev embedding on bounded sets up to a subsequence $\{f_{n,R}\}$ converges strongly in $L^p(\mathbb{R}^3)$ for any fixed $R > 0$.

Compactness at infinity

Using the explicit representation for $R_0(\lambda^2 \pm i0)$ and the Hölder inequality we have we have the following pointwise estimate:

$$|R_0(\lambda^2 \pm i0)f(x)| \leq (\int_{\mathbb{R}^3} \frac{1}{|x-y|^2\langle y\rangle^{1+\delta}}dy)^{\frac{1}{2}}\|f\|_{1+\delta}.$$

From this estimate and from the definiton of f_n and v_n we deduce:

$$\int_{|x|>R} |f_n|^p\langle x\rangle^{-\alpha(p,\delta)}dx \leq \int_{|x|>R} |R_0(\lambda^2 \pm i0)v_n|^p dx$$

$$\leq \|v_n\|_{1+\delta}^p \left(\int_{|x|>R}\left(\int_{\mathbb{R}^3} \frac{1}{|x-y|^2\langle y\rangle^{1+\delta}}dy\right)^{\frac{p}{2}} dx\right),$$

Combining this estimate and 29 we deduce:

$$\int_{|x|>R} |f_n|^p\langle x\rangle^{-\alpha(p,\delta)}dx \leq \|u_n\|_{L^{p,-\alpha(p,\delta)}}^p C(R)$$

where we denote by $C(R)$ the following quantity.
Using estimate (21) we have the following inequality:

$$C(R) := \int_{|x|>R} |\int \frac{1}{|x-y|^2\langle y\rangle^{1+\delta}}dy|^{\frac{p}{2}}dx \leq \int_{|x|>R} \frac{C}{\langle x\rangle^{\frac{\delta p}{2}}}.$$

Then if $p\delta > 6$ we have that $lim_{R\to\infty}C(R) = 0$ and from this estimate we deduce the compactness at infinity of the sequence $\{f_n\}$. $\square$

Lemma 3.3. *The functional equation* $U = R_0(\lambda^2 \pm i0)VU$ *has no nontrivial solutions in* $L^{p,-\alpha(p,\delta)}(\mathbb{R}^3)$.

Proof. Using 1.1 it is sufficient to prove that the space $L^{p,-\alpha(p,\delta)}(\mathbb{R}^3)$ is included in the space $L^{2,-3-\delta}(\mathbb{R}^3)$, or equivalently that there exists a real constant $C > 0$ such that the following inequality is satisfied:

$$\int_{\mathbb{R}^3} |f|^2 \langle x \rangle^{-3-\delta} dx \leq C \left(\int_{\mathbb{R}^3} |f|^p \langle x \rangle^{\frac{2}{p} + \frac{2\delta}{p}} \right)^{\frac{2}{p}}.$$

It is easy to check using the Hölder inequality and the decay assumption (5) that this inequality is satisfied. □

Proof of Theorem 3.1. Using lemmas 3.2, 3.3 and the abstract Fredholm theory we deduce that the operators $[I - R_0(\lambda^2 \pm i0)V]$ are invertible in the space $\mathcal{L}(L^{p,-\alpha(p,\delta)}, L^{p,-\alpha(p,\delta)})$ provided that p and δ satisfy the hypothesis.
To prove the uniform boundedness of the one parameter family of the inverses of these operators we remark that for $\lambda \leq 1$ it is a consequence of the compactness of the intervall $[0, 1]$. For large λ we remark that for 17 we have $\|R_0(\lambda^2 + i0)\| \to 0$ if $\lambda \to 0$ in $\mathcal{L}(L^{2,1+\delta}, L^{p,-\alpha(p,\delta)})$, then the operators $[I - R_0(\lambda^2 + i0)V]$ are close to the identity at infinity and then also their inverse are uniformly bounded. □

In the same way we can prove the following result.

Theorem 3.4. *Assume that the potential* $V(x) \geq 0$ *satisfies hypothesis* (5)*, then there exists a* $\bar{\delta} > 0$ *such that for any* $\bar{\delta} > \delta > 0$ *and for any* $p > 2$ *such that* $p\delta > 6$ *the family of operators* $[I - R_0(\lambda^2 \pm i0)V]$ *are invertible in* $\in \mathcal{L}(L^{p,-\beta(p,\delta)}(\mathbb{R}^3), L^{p,-\beta(p,\delta)}(\mathbb{R}^3))$*. Moreover there exists a real constant* $C = C(p, \delta) > 0$ *such that*

$$\|[I - R_0(\lambda^2 \pm i0)V]f\|_{L^{p,-\beta(p,\delta)}(\mathbb{R}^3)} > C\|f\|_{L^{p,-\beta(p,\delta)}(\mathbb{R}^3)}$$

for any $\lambda \in \mathbb{R}$.

The proof of the following theorem can be found in [7].

THEOREM 3.5. *Assume that the potential $V(x) \geq 0$ satisfies hypothesis* (5)*, then given $\delta > 0$, $1 < a < 3$ there exists a real constant $C = C(\delta, a) > 0$ such that*

$$\|[I - R_0(\lambda^2 \pm i0)V]f\|_{L^{2,-a-\delta}} > C\|f\|_{L^{2,-a-\delta}}$$

for any $\lambda \in \mathbb{R}$.

4. Estimates for the perturbed resolvent

In this section we give some usefull estimate for the perturbed Laplacian resolvent $R_V(\lambda^2 \pm i0)$ and for its square $R_V^2(\lambda^2 \pm i0)$, using the resolvent identity and the results of the previous section.

LEMMA 4.1. *Assume that the potential $V(x) \geq 0$ satisfies hypothesis* (5) *and* (1.1)*, then the following estimates are satisfied:*

1. *there exists a $\bar{\delta} > 0$ such that for any $\bar{\delta} > delta, \delta' > 0$ there exists a real constant $C = C(\delta, \delta') > 0$ such that we have the following estimate*

$$\|R_V(\lambda^2 \pm i0)f\|_{L^{2,-1-\delta}} \leq C\|f\|_{L^{2,3+\delta'}}; \tag{31}$$

   for any $\lambda > 0$;

2. *there exists a $\bar{\delta} > 0$ such that for any $\bar{\delta} > \delta, \delta' > 0$ there exists a real number such that for any $p > 2$ such that $p\delta > 6$, there exists a constant $C = C(p, \delta, \delta') > 0$ such that the following estimate are satisfied:*

$$\|R_V(\lambda^2 \pm i0)f\|_{L^{p,-\alpha(p,\delta)}} \leq \frac{C}{\lambda^{\frac{2}{p}}}\|f\|_{L^{2,1+\delta'}} \tag{32}$$

$$\|R_V(\lambda^2 \pm i0)f\|_{L^{p,-\alpha(p,\delta)}} \leq C\|f\|_{L^{2,1+\delta'+\frac{4}{p}}} \tag{33}$$

$$\|R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)\|_{L^{p,-\beta(p,\delta)}} \leq C\lambda^{1-\frac{2}{p}}\|f\|_{L^{2,3-\frac{4}{p}+\delta'}} \tag{34}$$

We use the resolvent identity to obtain the following identity of operators

$$R_V(\lambda^2 \pm i0) = [I - R_0(\lambda^2 \pm i0)V]^{-1}R_0(\lambda^2 \pm i0). \tag{35}$$

- *Proof of* (31). It is a direct consequence of the identity (35), of the estimate (14) and of theorem 3.5.

- *Proof of* (32). It is a direct consequence of the identity (35), of theorem 3.1 and of the estimate (17)

- *Proof of* (33). It is a direct consequence of the identity (35), of theorem 3.1 and of the estimate (18).

Proof of (34). Using the resolvent identity we have the following equality of operators

$$[R_V(\lambda^2+i0)-R_V(\lambda^2-i0)] = \\ [I-R_0(\lambda^2-i0)V]^{-1}[R_0(\lambda^2+i0)-R_0(\lambda^2-i0)][I+VR_V(\lambda^2+i0)]. \tag{36}$$

First we prove that the family of operators

$$[I+VR_V(\lambda^2+i0)]$$

is uniformly bounded in $\mathcal{L}(L^{2,3-\frac{4}{p}+\delta}, L^{2,3-\frac{4}{p}+\delta})$. Using the decay assumption of the potential (5) we obtain the following estimate:

$$\|VR_V(\lambda^2+i0)f\|_{L^{2,3-\frac{4}{p}+\delta}} \leq C\|R_V(\lambda^2+i0)f\|_{L^{2,-1-2\epsilon_0-\frac{4}{p}+\delta}},$$

that combined with (31) give that there exists a real constant $C=C(\delta)>0$ such that:

$$\|VR_V(\lambda^2+i0)f\|^2_{L^{2,3-\frac{4}{p}+\delta}} \leq C\|f\|_{L^{2,3-\frac{4}{p}+\delta}}, \tag{37}$$

if $\delta>0$ is small. The estimate (34) is then a consequence of the identity (36), of the uniform boundednes of the operators $VR_V(\lambda^2\pm i0)$ proved above, of theorem 3.4 and of estimate (19).

□

In next lemma we generalize to the square of the perturbed resolvent $R_V^2(\lambda^2\pm i0)$ the estimates proved for the square of the free resolvent $R_0^2(\lambda^2\pm i0)$.

Lemma 4.2. *Assume that the potential $V(x) \geq 0$ satisfies hypothesis (5) then there exists a $\bar{\delta} > 0$ such that for any $\bar{\delta} > \delta, \delta' > 0$ and for any $p > 2$ such that $p\delta > 6$, there exists a constant $C = C(p, \delta, \delta') > 0$ such that the following estimates are satisfied:*

$$\|R_V^2(\lambda^2 \pm i0)f\|_{L^{p,-\beta(p,\delta)}} \leq \frac{C}{\lambda^{1+\frac{2}{p}}}\|f\|_{L^{2,3+\delta'}} \tag{38}$$

$$\|R_V^2(\lambda^2 \pm i0)f\|_{L^{p,-\beta(p,\delta)}} \leq \frac{C}{\lambda}\|f\|_{L^{2,3+\delta'}}. \tag{39}$$

Proof. Using the resolvent identity we obtain the following identity of operators:

$$\begin{aligned} R_V^2(\lambda^2 \pm i0) &= \left[R_0(\lambda^2 \pm i0) + R_0(\lambda^2 \pm i0)VR_V(\lambda^2 \pm i0)\right] R_V(\lambda^2 \pm i0) \\ &= R_0(\lambda^2 \pm i0)\left[R_0(\lambda^2 \pm i0) + R_0(\lambda^2 \pm i0)VR_V(\lambda^2 \pm i0)\right] \\ &\quad + R_0(\lambda^2 \pm i0)VR_V^2(\lambda^2 \pm i0) \end{aligned}$$

then

$$\begin{aligned} R_V^2(\lambda^2 \pm i0) &= \left[I - R_0(\lambda^2 \pm i0)V\right]^{-1} R_0^2(\lambda^2 \pm i0) \\ &\quad + \left[I - R_0(\lambda^2 \pm i0)V\right]^{-1} R_0^2(\lambda^2 \pm i0)VR_V(\lambda^2 \pm i0). \end{aligned} \tag{40}$$

As above using the decay assumption (5) and the estimate (31) we can prove the following estimate for a constant $C = C(\delta) > 0$:

$$\|VR_V(\lambda^2 \pm i0)f\|_{L^{2,3+\delta}} \leq C\|f\|_{L^{2,3+\delta}}, \tag{41}$$

if $\delta > 0$ is small. □

- *Proof of* (38). It is a consequence of the identity (40), of (41), of theorem 3.4 and of the estimate (26)
- *Proof of* (39). It is a consequence of the identity (40), of (41) , of theorem 3.4 and of the estimate (27).

5. Dispersion for the propagator $\mathcal{U}_V(t)$

In this section we prove theorem 1.2 stated in the introduction.
We will also prove a dispersive estimate that is different from the

estimate stated in theorem 1.3. In last section we will see that this estimate can be improved using the finite propagation speed principle and we will obtain finally the proof of theorem 1.3.

Proof of Theorem 1.2. Using the Duhamel formula we obtain the following identity:

$$\mathcal{U}_V(t)f = \mathcal{U}_0(t)f + \int_0^t \mathcal{U}_0(t-s)V\mathcal{U}_V(s)f ds.$$

For estimate (4) and for the Minkowsky inequality we have:

$$\|\mathcal{U}_V(t)f\|_{L^4} \leq \frac{C}{\sqrt{t}}\|f\|_{L^{\frac{4}{3}}} + \int_0^t \frac{C}{\sqrt{t-s}}\|V\mathcal{U}_V(s)f\|_{L^{\frac{4}{3}}} ds. \tag{42}$$

For the decay assumption on the potential (5) and the Hölder inequality we have:

$$\|Vg\|_{L^{\frac{4}{3}}} \leq C(\delta)\|g\|_{L^{2,-\frac{5}{2}-2\epsilon_0+\delta}} \tag{43}$$

Moreover if $\delta > 0$ is small, using the adjoint of the estimate (6) we obtain the following estimate:

$$\|\mathcal{U}_V(t)f\|_{L^{2,-\frac{5}{2}-2\epsilon_0+\delta}} \leq \|\mathcal{U}_V(t)f\|_{L^{2,-\frac{3}{2}-\delta}} \leq \frac{C}{\sqrt{t}}\|f\|_{L^{\frac{4}{3}}}, \forall t > 0. \tag{44}$$

Putting together (43) and (44) we have:

$$\|V\mathcal{U}_V(t)f\|_{L^{\frac{4}{3}}} \leq \frac{C}{\sqrt{t}}\|f\|_{L^{\frac{4}{3}}}, \forall t > 0.$$

Using this estimate and (42) we have

$$\|\mathcal{U}_V(t)f\|_{L^4} \leq \frac{C}{\sqrt{t}}\|f\|_{L^{\frac{4}{3}}} + \left(\int_0^t \frac{C}{\sqrt{t-s}\sqrt{s}} ds\right)\|f\|_{L^{\frac{4}{3}}}, \forall t > 0.$$

It is easy to prove that there exists a real constant $C > 0$ such that

$$\int_0^t \frac{C}{\sqrt{t-s}\sqrt{s}} ds \leq \frac{C}{\sqrt{t}}$$

for $0 < t < 1$ and then we can conclude the proof.

$\square$

In next theorem we will prove as stated at the beginning of the section a type of dispersive estimate. More exactly we have the following result.

THEOREM 5.1. *Given any $\epsilon > 0$, $\delta > 0$ and $2 < p < \infty$ such that, $\epsilon p < 2$ and $p\delta > 6$, there exists a real constant $C = C(p, \epsilon, \delta) > 0$ such that the following estimate is satisfied:*

$$\|[\mathcal{U}_V(t)f - \mathcal{U}_0(t)f]\|_{L^{p_\epsilon,-(1-\theta_\epsilon)\beta(p,\delta)}} \leq \frac{C}{t^{1-\theta_\epsilon}}\|f\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)}}, \quad (45)$$

where p_ϵ and θ_ϵ are defined as follows:

$$p_\epsilon = \frac{4p + 2\epsilon p - 8}{p + \epsilon p - 2}, \theta_\epsilon = \frac{p - 4 + \epsilon p}{2p - 4 + \epsilon p},$$

$$\beta(p, \delta) = \frac{6}{p} + \frac{2\delta}{p}.$$

The proof will be a consequence of the following few lemmas. The operators $\mathcal{U}_{V,j}(t)$ and $\mathcal{U}_{V,high}(t)$ that appear in next lemmas are defined in the introduction in (10) and (11), while the operators $\mathcal{U}_{0,j}(t)$ and $\mathcal{U}_{0,high}(t)$ are defined in the same way assuming that $V = 0$.

LEMMA 5.2. *The following estimates are satisfied:*

1. *there exists a constant $C > 0$ such that for any $j < 0$ we have the following estimate*

$$\|\mathcal{U}_{0,j}(t)f\|_{L^2} \leq \frac{C}{2^j}\|f\|_{L^2};$$

2. *there exists a real constant $C > 0$ such that for any $j < 0$ we have the following estimate*

$$\|\mathcal{U}_{V,j}(t)f\|_{L^2} \leq \frac{C}{2^j}\|f\|_{L^2};$$

3. *for any $\sigma \in \mathbb{R}$ and $\psi \in C^\infty(\mathbb{R})$ with supp$\psi(\lambda) \subset (1,\infty)$ there exists a real constant $C > 0$ such that we have the following estimate*
$$\|\sqrt{-\Delta}^{1+i\sigma}\mathcal{U}_{0,high}(t)\|_{L^2} \leq C\|f\|_{L^2};$$

4. *there exists a real constant $C > 0$ such that for any $\sigma \in \mathbb{R}$ and $\psi \in C^\infty(\mathbb{R})$ with supp$\psi(\lambda) \subset (1,\infty)$ we have the following estimate*
$$\|\sqrt{-\Delta_V}^{1+i\sigma}\mathcal{U}_{V,high}(t)\|_{L^2} \leq C\|f\|_{L^2}.$$

Proof. The estimates 1 and 3 follow from the representation on the operators in the Fourier variable and from the Plancherel formula. The estimates 2 and 4 follow from the representation on the operators in the generalized Fourier variable and from the generalized Plancherel formula. □

LEMMA 5.3. *For any $\delta > 0$ and for any $2 < p < \infty$ such that $p\delta > 6$ there exists a real constant $C = C(p,\delta) > 0$ such that for any $j < 0$ the following estimate is satisfied*

$$\|\mathcal{U}_{V,j}(t) - \mathcal{U}_{0,j}(t)\|_{L^{p,-\beta(p,\delta)}} \leq \frac{C2^{j(1-\frac{2}{p})}}{t}\|f\|_{L^{q,\beta(p,\delta)}} \tag{46}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta(p,\delta) = \frac{6}{p} + \frac{2\delta}{p}$.

Proof. We start from the following identity that can be deduced from the resolvent identity:

$$\begin{aligned}&[\mathcal{U}_{V,j}(t) - \mathcal{U}_{0,j}(t)]f = \\ &= \int_0^\infty \phi_j(\lambda)\sin\lambda t[R_0(\lambda^2 + i0)VR_V(\lambda^2 + i0) - \\ &- R_0(\lambda^2 - i0)VR_V(\lambda^2 - i0)]fd\lambda.\end{aligned} \tag{47}$$

Using integration by parts in the variable λ we have the following identity:

$$[\mathcal{U}_{V,j}(t) - \mathcal{U}_{0,j}(t)]f = \frac{1}{t}(I^+ - I^- + II^+ - II^- + III)$$

where:

$$I^\pm = \int \lambda \cos \lambda t \phi_j R_0(\lambda^2 \pm i0) V R_V^2(\lambda^2 \pm i0) f; \tag{48}$$

$$II^\pm = \int \lambda \cos \lambda t \phi_j [R_0^2(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0) f; \tag{49}$$

$$III = \int \phi_j'(\lambda) \cos \lambda t [R_0(\lambda^2 + i0) V R_V(\lambda^2 + i0) - \tag{50}$$
$$- R_0(\lambda^2 - i0) V R_V(\lambda^2 - i0)] f d\lambda.$$

We now estimate separately the integrals $I^\pm, II^\pm, III$.

Estimate for $I^\pm$. Using estimate (18) we have:

$$\|I^\pm\|_{L^{p,-\alpha(p,\delta)}} \leq \int_{\mathrm{supp}\phi_j} \lambda \|R_0(\lambda^2 \pm i0) V R_V^2(\lambda^2 \pm i0) f\|_{L^{p,-\alpha(p)}}$$
$$\leq \int_{\mathrm{supp}\phi_j} \lambda \|V R_V^2(\lambda^2 \pm i0) f\|_{L^{2,1+\delta+\frac{4}{p}}} d\lambda$$

Using the decay assumption of the potential at infinity (5) we have:

$$\|I^\pm\|_{L^{p,-\alpha(p,\delta)}} \leq \int_{\mathrm{supp}\phi_j} \lambda \|R_V^2(\lambda^2 \pm i0) f\|_{L^{2,-3-2\epsilon_0+\delta+\frac{4}{p}}}$$

and for the dual estimate of (39) we have finally:

$$\|I^\pm\|_{L^{p,-\alpha(p,\delta)}} \left(\int_{\mathrm{supp}\phi_j} C d\lambda \right) \|f\|_{L^{q,\beta(p,\delta)}} \leq C 2^j \|f\|_{L^{q,\beta(p,\delta)}}. \tag{51}$$

Estimate for $II^\pm$. Using (26) and the dual estimate of (33) we obtain exactly as in the estimate of $I^\pm$ the following estimate:

$$\|II^\pm\|_{L^{p,-\beta(p,\delta)}} \leq C 2^j \|f\|_{L^{q,\alpha(p,\delta)}}.$$

Estimate for III. From the resolvent identity we obtain the following identity of operators:

$$R_0(\lambda^2 + i0) V R_V(\lambda^2 + i0) - R_0(\lambda^2 - i0) V R_V(\lambda^2 - i0) =$$
$$[R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)] V R_V(\lambda^2 + i0)$$
$$+ R_0(\lambda^2 - i0) V [R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)].$$

Using this identity we have:

$$\|III\|_{L^{p,-\beta(p,\delta)}} \leq \int_{\mathrm{supp}\phi_j} sup|\phi_j'|\|[(R_0(\lambda^2+i0)$$
$$- R_0(\lambda^2-i0)]VR_V(\lambda^2+i0)f\|_{L^{p,-\bar{\alpha}(p)}}d\lambda+$$
$$+\int_{\mathrm{supp}\phi_j} sup|\phi_j'|\|R_0(\lambda^2-i0)V[R_V(\lambda^2+i0)$$
$$- R_V(\lambda^2-i0)]f\|_{L^{p,-\bar{\alpha}(p)}}d\lambda.$$

Using (18), (19) we obtain the following estimate:

$$\|III\|_{L^{p,-\beta(p,\delta)}} \leq C2^{-j}\int_{\mathrm{supp}\phi_j}\lambda^{1-\frac{2}{p}}\|VR_V(\lambda^2+i0)f\|_{L^{2,3-\frac{4}{p}+\delta}}d\lambda$$
$$+C2^{-j}\int_{\mathrm{supp}\phi_j}\|V[R_V(\lambda^2+i0)-R_V(\lambda^2-i0)f]\|_{L^{2,1+\delta+\frac{4}{p}}}d\lambda$$

Using now the decay assumption on the potential (5) we have:

$$\|III\|_{L^{p,-\beta(p,\delta)}} \leq C2^{-j}\int_{\mathrm{supp}\phi_j}\lambda^{1-\frac{2}{p}}\|R_V(\lambda^2+i0)f\|_{L^{2,-1-\frac{4}{p}-2\epsilon_0+\delta}}d\lambda$$
$$+C2^{-j}\int_{\mathrm{supp}\phi_j}\|[R_V(\lambda^2+i0)-R_V(\lambda^2-i0)]f\|_{L^{2,-3-2\epsilon_0+\delta}}d\lambda,$$

and finally for (33) and (34) we have:

$$\|III\|_{L^{p,-\beta(p,\delta)}} \leq C2^{-j}\left(\int_{\mathrm{supp}\phi_j}\lambda^{1-\frac{2}{p}}d\lambda\right)\|f\|_{L^{q,\alpha(p,\delta)}}$$
$$+C2^{-j}\left(\int_{\mathrm{supp}\phi_j}\lambda^{1-\frac{2}{p}}d\lambda\right)\|f\|_{L^{q,\beta(p,\delta)}}$$
$$\leq C2^{j(1-\frac{2}{p})}\|f\|_{L^{q,\beta(p,\delta)}}.$$

□

Using interpolation between lemmas 5.2 and 5.3 we obtain the following result.

LEMMA 5.4. *For any $\delta > 0$, for any $2 < p < \infty$ such that $p\delta > 6$ and for any $\epsilon > 0$ there exists a real constant $C = C(p, \delta, \epsilon) > 0$ such that for any $j < 0$ we have the following estimate,*

$$\|[\mathcal{U}_{V,j}(t)f - \mathcal{U}_{0,j}(t)f]\|_{L^{p_\epsilon, -(1-\theta_\epsilon)\beta(p,\delta)}} \leq$$
$$\leq \frac{C}{t^{1-\theta_\epsilon}} 2^{j\frac{(p-2)+2\theta_\epsilon(1-p)}{p}} \|f\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)}},$$

where

$$p_\epsilon = \frac{4p + 2\epsilon p - 8}{p + \epsilon p - 2}, \theta_\epsilon = \frac{p - 4 + \epsilon p}{2p - 4 + \epsilon p},$$
$$\beta(p, \delta) = \frac{6}{p} + \frac{2\delta}{p}, \frac{1}{p_\epsilon} + \frac{1}{q_\epsilon} = 1.$$

We now give an estimate of the high frequency part of the propagator.

LEMMA 5.5. *For any $\epsilon > 0$, $\delta > 0$ and $2 < p < \infty$, such that $p\delta > 6$, there exists a real constant $C = C(p, \delta, \epsilon) > 0$ such that we have the following estimate:*

$$\|(\sqrt{-\Delta_V})^{-1+\frac{4}{p}-\epsilon+i\sigma}\mathcal{U}_{V,high}(t)f -$$
$$-(\sqrt{-\Delta})^{-1+\frac{4}{p}-\epsilon+i\sigma}\mathcal{U}_{0,high}(t)f\|_{L^{p,-\beta(p,\delta)}} \leq \frac{C}{t}\|f\|_{L^{q,\beta(p,\delta)}}.$$

Proof. Using the resolvent identity we deduce the following identity of operators:

$$(\sqrt{-\Delta_V})^{-1-\epsilon+\frac{4}{p}+i\sigma}\mathcal{U}_{V,high}(t)f - (\sqrt{-\Delta})^{-1-\epsilon+\frac{4}{p}+i\sigma}\mathcal{U}_{0,high}(t)f =$$
$$\int_0^\infty \psi(\lambda)\frac{\sin\lambda t}{\lambda^{1+\epsilon-\frac{4}{p}+i\sigma}}[R_0(\lambda^2 + i0)VR_V(\lambda^2 + i0) -$$
$$-R_0(\lambda^2 - i0)VR_V(\lambda^2 - i0)]f d\lambda.$$

After integration by parts we have:

$$(\sqrt{-\Delta_V})^{-1-\epsilon+\frac{4}{p}+i\sigma}\mathcal{U}_{V,high}(t)f - (\sqrt{-\Delta})^{-1-\epsilon+\frac{4}{p}+i\sigma}\mathcal{U}_{0,high}(t)f$$
$$=\frac{1}{t}(I^+ - I^- + II^+ - II^- + III^+ - III^- + IV^+ - IV^-),$$

where:

$$I^{\pm} = \int \lambda \cos \lambda t \frac{\psi(\lambda)}{\lambda^{1+\epsilon-\frac{4}{p}+i\sigma}} R_0(\lambda^2 \pm i0) V R_V^2(\lambda^2 \pm i0) f; \tag{52}$$

$$II^{\pm} = \int \lambda \cos \lambda t \frac{\psi(\lambda)}{\lambda^{1+\epsilon-\frac{4}{p}+i\sigma}} R_0^2(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0) f; \tag{53}$$

$$III^{\pm} = c \int \psi(\lambda) \frac{\cos \lambda t}{\lambda^{2+\epsilon-\frac{4}{p}+i\sigma}} R_0(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0) f d\lambda, \tag{54}$$

where $c := c(p, \epsilon, \sigma) = -1 - \epsilon + \frac{4}{p} - i\sigma$;

$$IV^{\pm} = \int \psi'(\lambda) \frac{\cos \lambda t}{\lambda^{1+\epsilon-\frac{4}{p}+i\sigma}} R_0(\lambda^2 \pm i0) V R_V(\lambda^2 \pm i0) f d\lambda. \tag{55}$$

We estimates $I^{\pm}, II^{\pm}, III^{\pm}, IV^{\pm}$ separately.

Estimate for $I^{\pm}$. Using (17) and the decay assumption (5) we have:

$$\|I^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq C \int_{\mathrm{supp}\psi} \frac{1}{\lambda^{\epsilon-\frac{2}{p}}} \|R_V^2(\lambda^2 \pm i0) f\|_{L^{2,-3-2\epsilon_0+\delta}} d\lambda.$$

We can now conclude the estimate using the dual estimate of (38) as follows:

$$\|I^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq C \left(\int_{supp\psi} \frac{1}{\lambda^{1+\epsilon}} d\lambda \right) \|f\|_{L^{q,\beta(p,\delta)}} \leq C \|f\|_{L^{q,\beta(p,\delta)}}.$$

Estimate for $II^{\pm}$. We use (26) and we obtain:

$$\|II^{\pm}\|_{L^{p,-\bar{\alpha}(p,\delta)}} \leq C \int_{\mathrm{supp}\psi} \frac{1}{\lambda^{1+\epsilon-\frac{2}{p}}} \|V R_V(\lambda^2 \pm i0) f\|_{L^{2,3+\delta}} d\lambda.$$

For the assumption (5) we arrive at the following estimate:

$$\|II^{\pm}\|_{L^{p,-\bar{\alpha}(p,\delta)}} \leq \int_{supp\psi} \frac{C}{\lambda^{1+\epsilon-\frac{2}{p}}} \|R_V(\lambda^2 \pm i0) f\|_{L^{2,-1+\delta-2\epsilon_0}} d\lambda.$$

Using now (32) we arrive finally at the following inequality:

$$\|II^{\pm}\|_{L^{p,-\bar{\alpha}(p,\delta)}} \leq C \left(\int_{supp\psi} \frac{1}{\lambda^{1+\epsilon}} d\lambda \right) \|f\|_{L^{q,\alpha(p,\delta)}}$$

Estimate for $III^{\pm}$. We use (17), then:

$$\|III^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq C\int_{\mathrm{supp}\psi} \frac{C}{\lambda^{2+\epsilon-\frac{2}{p}}}\|VR_V(\lambda^2 \pm i0)f\|_{L^{2,1+\delta}}d\lambda.$$

Using now the assumption (5) we arrive at the following estimate:

$$\|III^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq C\int_{\mathrm{supp}\psi} \frac{C}{\lambda^{2+\epsilon-\frac{2}{p}}}\|R_V(\lambda^2 \pm i0)f\|_{L^{2,-3+\delta-2\epsilon_0}}d\lambda$$

and finally for (32):

$$\|III^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq \left(\int_{\mathrm{supp}\psi} \frac{C}{\lambda^{2+\epsilon}}d\lambda\right)\|f\|_{L^{q,\alpha(p,\delta)}}$$

Estimate for $IV^{\pm}$. Using (17) we obtain the following inequalities

$$\|IV^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq \int_{\mathrm{supp}\psi'} \frac{C}{\lambda^{1+\epsilon-\frac{2}{p}}}\|VR_V(\lambda^2 \pm i0)f\|_{L^{2,1+\delta}}d\lambda.$$

Using (5) we have:

$$\|IV^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq \int_{\mathrm{supp}\psi'} \frac{C}{\lambda^{1+\epsilon-\frac{2}{p}}}\|R_V(\lambda^2 \pm i0)f\|_{L^{2,-3-2\epsilon_0+\delta}}d\lambda$$

and finally for (32) we have:

$$\|IV^{\pm}\|_{L^{p,-\alpha(p,\delta)}} \leq \left(\int_{\mathrm{supp}\psi'} \frac{C}{\lambda^{1+\epsilon}}d\lambda\right)\|f\|_{L^{q,\alpha(p,\delta)}} \leq C\|f\|_{L^{q,\alpha(p,\delta)}}$$

□

Using the complex interpolation theorem of Stein between the estimates given in lemmas 5.2 and 5.5, we find the following result.

Lemma 5.6. *For any $\delta > 0$, for any $2 < p < \infty$ such that $p\delta > 6$ and for any $\epsilon > 0$ there exists a real constant $C = C(p,\delta,\epsilon) > 0$ such that we have the following estimate is satisfied:*

$$\|[\mathcal{U}_{V,high}(t)f - \mathcal{U}_{0,high}(t)f]\|_{L^{p_\epsilon,-\beta(1-\theta_\epsilon)}} \leq \frac{C}{t^{1-\theta_\epsilon}}\|f\|_{L^{q_\epsilon,\beta(p,\delta)(1-\theta_\epsilon)}},$$

where
$$p_\epsilon = \frac{4p+2\epsilon p-8}{p+\epsilon p-2}, \theta_\epsilon = \frac{p-4+\epsilon p}{2p-4+\epsilon p},$$
$$\beta(p,\delta) = \frac{6}{p} + \frac{2\delta}{p}, \frac{1}{p_\epsilon} + \frac{1}{q_\epsilon} = 1.$$

Proof of theorem 5.1. With a simple computation we can prove that the condition:
$$\frac{(p-2)+2\theta_\epsilon(1-p)}{p} > 0,$$
is equivalent to the condition $\epsilon p < 2$.
We use the following decomposition
$$\mathcal{U}_V(t)f - \mathcal{U}_0(t)f =$$
$$= [\sum_{j\leq 0} \mathcal{U}_{V,j}(t)f - \mathcal{U}_{0,j}(t)f] + \mathcal{U}_{V,high}(t)f - \mathcal{U}_{0,high}(t)f.$$
Using the previous decomposition and the Minkowsky inequality we have:
$$\begin{aligned}&\|\mathcal{U}_V(t)f - \mathcal{U}_0(t)f\|_{L^{p_\epsilon,-(1-\theta_\epsilon)\beta(p,\delta)}} \leq \\ &\leq \sum_j \|\mathcal{U}_{V,j}(t)f - \mathcal{U}_{0,j}(t)f\|_{L^{p_\epsilon,-(1-\theta_\epsilon)\beta(p,\delta)}} \leq \\ &+ \|\mathcal{U}_{V,high}(t)f - \mathcal{U}_{0,high}(t)f\|_{L^{p_\epsilon,-(1-\theta_\epsilon)\beta(p,\delta)}}\end{aligned} \tag{56}$$
then for lemmas 5.4 and 5.6 we have:
$$\begin{aligned}&\|\mathcal{U}_V(t)f - \mathcal{U}_0(t)f\|_{L^{p_\epsilon,-(1-\theta_\epsilon)\beta(p,\delta)}} \leq \\ &\leq \frac{C}{t^{1-\theta_\epsilon}} \left(\sum_{j<0} 2^{j\frac{(p-2)+2\theta_\epsilon(1-p)}{p}} + C\right) \|f\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)}},\end{aligned}$$

If $\epsilon p < 2$, for the remark done at the beginning of the proof, we have the following property:
$$\sum_{j<0} 2^{j\frac{(p-2)+2\theta_\epsilon(1-p)}{p}} < \infty.$$

Then finally we have:
$$\|\mathcal{U}_V(t)f - \mathcal{U}_0(t)f\|_{L^{p_\epsilon,-(1-\theta_\epsilon)\beta(p,\delta)}} \leq \frac{C}{t^{1-\theta_\epsilon}} \|f\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)}}.$$
□

6. Finite propagation speed and dispersive estimate for $\mathcal{U}_V(t)$

This section is devoted to the proof of theorem 1.3.
Given a Paley - Littlewood partition of unity $\varphi_j(x)$ for $j \in \mathbb{N}$, where $supp\varphi_0(x) \subset |x|\{\leq 2\}$ and $\mathrm{supp}\varphi_j(x) \subset \{2^{j-1} \leq |x| \leq 2^{j+1}\}$ for $j \in \mathbb{N}$, and given a function $f(x)$ we will use the following notation

$$f_j(x) := \varphi_j(x)f(x).$$

In the proof of theorem 1.3 we need a lemma whose proof will be a consequence of the finite propagation speed principle that we state in the following proposition.

PROPOSITION 6.1. *Let be given two real positive numbers $R_1 < R_2$ and a function $f(x)$ such that,*

$$supp_x f(x) \subseteq \{R_1 \leq |x| \leq R_2\}$$

then

$$supp_x \mathcal{U}_0(t)f \subseteq \{R_1 - t \leq |x| \leq R_2 + t\}$$

and

$$supp_x \mathcal{U}_V(t)f \subseteq \{R_1 - t \leq |x| \leq R_2 + t\}.$$

We can now prove the following lemma.

LEMMA 6.2. *For any $\delta > 0$, for any $2 < p < \infty$ such that $p\delta > 6$, $\epsilon p < 2$ and*

$$-\epsilon^2 p^2 + 2\epsilon p + 2p - 4 - 2\epsilon p^2 - 2\delta p - 2\delta\epsilon p + 4\delta \geq 0,$$

there exists a real constant $C = C(p, \delta, \epsilon) > 0$ such that for any $j \in \mathbb{Z}$ the following estimate is satisfied:

$$\|\mathcal{U}_V(t)f_j - \mathcal{U}_0(t)f_j\|_{L^{p_\epsilon}} \leq \frac{C}{\sqrt{t}} 2^{j(1-\theta_\epsilon)\beta(p,\delta)} \|f_j\|_{L^{q_\epsilon}}. \tag{57}$$

where:

$$p_\epsilon = \frac{4p + 2\epsilon p - 8}{p + \epsilon p - 2}, \theta_\epsilon = \frac{p - 4 + \epsilon p}{2p - 4 + \epsilon p},$$

$$\beta(p, \delta) = \frac{6}{p} + \frac{2\delta}{p}, \frac{1}{p_\epsilon} + \frac{1}{q_\epsilon} = 1.$$

Proof. It is sufficient to prove the estimate for the case $t > 1$, since for $t < 1$ it can be proved combining estimates (4) and (7).
We split the proof for $t > 1$ in two cases.

Case A : $2^j \leq 4t$. In this case proposition 6.1 implies:

$$\mathrm{supp}_x \mathcal{U}_V(t) f_j \cup \mathrm{supp}_x \mathcal{U}_0(t) f_j \subseteq \{|x| \leq 9t\}.$$

Moreover the estimate (45) implies:

$$\begin{aligned} &\frac{1}{t^{\frac{(1-\theta_\epsilon)\beta(p,\delta)}{p_\epsilon}}} \|\mathcal{U}_V(t) f_j - \mathcal{U}_0(t) f_j\|_{L^{p_\epsilon}} \leq \\ &\leq C\|\mathcal{U}_V(t) f_j - \mathcal{U}_0(t) f_j\|_{L^{p_\epsilon, -(1-\theta_\epsilon)\beta(p,\delta)}} \leq \\ &\leq \frac{C}{t^{1-\theta_\epsilon}} \|f_j\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)}} \end{aligned} \tag{58}$$

then if:

$$(1-\theta_\epsilon) - \frac{(1-\theta_\epsilon)\beta(p,\delta)}{p_\epsilon} \geq \frac{1}{2},$$

we obtain, since $t > 1$, the following estimate:

$$\|\mathcal{U}_V(t) f_j - \mathcal{U}_0(t) f_j\|_{L^{p_\epsilon}} \leq \frac{C}{\sqrt{t}} 2^{j\frac{(1-\theta_\epsilon)\beta}{q_\epsilon}} \|f_j\|_{L^{q_\epsilon}}.$$

Case B : $2^j \geq 4t$. In this case for proposition 6.1 we have:

$$\mathrm{supp}_x \mathcal{U}_V(t) f_j \cup \mathrm{supp}_x \mathcal{U}_0(t) f_j \subseteq \{2^{j-2} \leq |x| \leq 2^{j+2}\}.$$

Using (45) and the property $t > 1$ we obtain:

$$\begin{aligned} &2^{-j\frac{\beta(p,\delta)(1-\theta_\epsilon)}{p_\epsilon}} \|\mathcal{U}_V(t) f_j - \mathcal{U}_0(t) f_j\|_{L^{p_\epsilon}} \leq \\ &\leq C\|\mathcal{U}_V(t) f_j - \mathcal{U}_0(t) f_j\|_{L^{p_\epsilon, -(1-\theta_\epsilon)\beta(p,\delta)}} \leq \frac{C}{t^{1-\theta_\epsilon}} \|f_j\|_{L^{q_\epsilon}} 2^{j\frac{(1-\theta_\epsilon)\beta(p,\delta)}{q_\epsilon}} \end{aligned} \tag{59}$$

so we get

$$\|\mathcal{U}_V(t) f_j - \mathcal{U}_0(t) f_j\|_{L^{p_\epsilon}} \leq \frac{C}{t^{1-\theta_\epsilon}} 2^{j(1-\theta_\epsilon)\beta(p,\delta)} \|f_j\|_{L^{q_\epsilon}}. \tag{60}$$

If moreover $1 - \theta_\epsilon \geq \frac{1}{2}$ we have:

$$\|\mathcal{U}_V(t) f_j - \mathcal{U}_0(t) f_j\|_{L^{p_\epsilon}} \leq \frac{C}{\sqrt{t}} 2^{j(1-\theta_\epsilon)\beta(p,\delta)} \|f_j\|_{L^{q_\epsilon}}.$$

Then (57) is established in both cases if the following conditions are satisfied:

$$(1-\theta_\epsilon) - \frac{(1-\theta_\epsilon)\beta(p,\delta)}{p_\epsilon} \geq \frac{1}{2},$$

$$1-\theta_\epsilon \geq \frac{1}{2}.$$

Since the second inequality is weaker than the first we concentrate to study the first one. Considering the explicit representation of θ_ϵ and p_ϵ and studying the inequality we arrive at the following condition for ϵ, p, δ:

$$-\epsilon^2 p^2 + 2\epsilon p + 2p - 4 - 2\epsilon p^2 - 2\delta p - 2\delta\epsilon p + 4\delta \geq 0$$

□

Next lemma will be very important to prove theorem 1.3.

LEMMA 6.3. *We have the following estimate for the free propagator:*

$$\|\mathcal{U}_0(t)f\|_{L^{p_\epsilon}} \leq \frac{C}{t^{\gamma(\epsilon,p)}}\|f\|_{L^{q_\epsilon,\sigma(\epsilon,p)}}$$

where $\gamma(\epsilon,p) = \frac{p_\epsilon - 2}{p_\epsilon}$, $\sigma(\epsilon,p) = \frac{8-2p_\epsilon}{p_\epsilon}$ *and* $\frac{1}{p_\epsilon} + \frac{1}{q_\epsilon} = 1$.

Proof. The following Hardy inequality:

$$\|\frac{1}{\sqrt{-\Delta}}f\|_{L^2} \leq C\|f\|_{L^{2,2}}$$

implies easily the following estimate for the free propagator:

$$\|\mathcal{U}_0(t)f\|_{L^2} \leq C\|f\|_{L^{2,2}}. \tag{61}$$

Making interpolation between this estimate and estimate (4) we have the result. □

Proof of Theorem 1.3. We remark that the following inequality is satisfied for any $\beta, \mu > 0$ and $1 < r < \infty$:

$$\sum_{j=0}^{\infty} \|f_j\|_{L^{r,\beta}} \leq C\|f\|_{L^{r,\beta+\mu}}. \tag{62}$$

Using triangular inequality and lemma 6.2 we obtain the following chain of inequalities

$$\|\mathcal{U}_V(t)f - \mathcal{U}_0(t)f\|_{L^{p_\epsilon}} \leq \sum_j \|\mathcal{U}_V(t)f_j - \mathcal{U}_0(t)f_j\|_{L^{p_\epsilon}} \leq$$

$$\leq \frac{C}{\sqrt{t}} \sum_j \|f_j\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)q_\epsilon}}.$$

Then for (62) we have:

$$\|\mathcal{U}_V(t)f - \mathcal{U}_0(t)f\|_{L^{p_\epsilon}} \leq \frac{C}{\sqrt{t}} \|f\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)q_\epsilon+\delta}}$$

From this estimate and lemma 6.3 we deduce

$$\|\mathcal{U}_V(t)f\|_{L^{p_\epsilon}} \leq \|\mathcal{U}_V(t)f - \mathcal{U}_0(t)f\|_{L^{p_\epsilon}} + \|\mathcal{U}_0(t)f\|_{L^{p_\epsilon}}$$

$$\leq \frac{C}{t^{\gamma(\epsilon,p)}} \|f\|_{L^{q_\epsilon,\sigma(\epsilon,p)}} + \frac{C}{\sqrt{t}} \|f\|_{L^{q_\epsilon,(1-\theta_\epsilon)\beta(p,\delta)q_\epsilon+\delta}}.$$

Since $\gamma(\epsilon, p) < \frac{1}{2}$ we have the result.

□

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