

# Stress Boundary Value Problem in Linear Viscoelasticity

W. DESCH AND E. FAŠANGA (\*)

SUMMARY. - *We will solve the following boundary value problem in linear viscoelasticity: given the value of the stress on (a part of) the boundary of the domain find the stress in the whole body at all positive times. We are especially interested in the regularity of the stress. We use a constitutive relation giving rise to a partial integrodifferential equation.*

## 1. Introduction

Consider a linearly viscoelastic body subjected to forces on one part of its boundary, while another part of the boundary is fixed. The stress field inside the body is described by a Volterra equation involving an elliptic operator and a scalar convolution kernel which is singular at time 0. Given sufficient smoothness of the boundary forces, we show existence of the corresponding stress field, and relate its regularity to the regularity of the boundary forces.

Technically, the problem is characterized by two features. It is a time dependent problem with forcing by boundary conditions, on the other hand it involves an integral equation with a singular kernel.

---

(\*) Authors' addresses: W. Desch, Institut für Mathematik, Universität Graz, Heinrichstrasse 36, A-8010 Graz, Austria, e-mail: [georg.desch@kfunigraz.ac.at](mailto:georg.desch@kfunigraz.ac.at)  
E. Fašanga, Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 75 Prague, Czech Republic, e-mail: [fasanga@karlin.mff.cuni.cz](mailto:fasanga@karlin.mff.cuni.cz)  
Keywords: Volterra equation, linear viscoelasticity.

AMS Subject Classification: Primary: 45K05; Secondary: 73F15.

This work was partially supported by the grants AKTION Österreich–Tschechische Republik 23p8, 29p8, Spezialforschungsbereich Optimierung und Kontrolle F-003, MSM 113200007, GAUK 166/1999, GAČR 201/98/1450, and GAČR 201/01/D094.

Both types of problems have been treated in the literature. The inhomogeneous boundary conditions can be converted to a distributed forcing term by solving an elliptic stationary boundary value problem. Using this approach, we reduce the problem to a parabolic inhomogeneous Volterra equation with a self-adjoint operator in a Hilbert space. For this equation known results on wellposedness and regularity are applicable (see [6]).

Frequently, dynamic problems of elasticity and viscoelasticity are set in terms of the displacement or velocity fields. In our paper we write the integral equation in terms of the stress field. The reason to do so is that we want to get regularity of the stress field in terms of regularity of the boundary forces, which can be obtained directly if we utilize the maximal regularity properties of the integral equation in the stress setting. Notice that Hölder regularity of stress with respect to time is not equivalent to Hölder regularity of displacement, but of displacement in convolution with the relaxation modulus. While this problem can be overcome in a displacement setting, the approach by a stress setting is more direct.

This paper is structured as follows: In Section 2 we set up the viscoelastic problem and state the main result. Subsequently we decompose the problem into an elliptic boundary value problem and a parabolic Volterra equation. In Section 3 we solve the elliptic problem describing a tensor valued stress field given normal stresses on the boundary. The method used is a standard Lax-Milgram argument, and the result is what one would naturally expect in linear elasticity (e.g., [1], [4]). However, to our knowledge this result is not found in literature explicitly and seems worthwhile to be stated by itself. In Section 4 we prove our main result. Section 5 briefly sketches how the abstract formulation fits in the framework of a synchronous linearly viscoelastic body in three dimensions.

The authors wish to thank an anonymous referee for helpful suggestions.

## 2. Mathematical setting

A viscoelastic body with constant density is occupying a space domain  $\Omega$ . We will use the following notation:

- $x \in \Omega \subset \mathbf{R}^n$
- $\partial\Omega = \Gamma_0 \cup \Gamma_1$  the boundary of  $\Omega$ ,  $\Gamma_1 \cap \Gamma_0 = \emptyset$
- $n(x)$  is the outer normal to  $\partial\Omega$  at  $x$
- $t \geq 0$  time
- $u(t, x) \in \mathbf{R}^n$  displacement
- $v(t, x) \in \mathbf{R}^n$  velocity,  $v = u_t$
- $\sigma(t, x) \in \mathbf{R}_{sym}^{n \times n}$  stress
- $\varepsilon(t, x) \in \mathbf{R}_{sym}^{n \times n}$  strain

We will use the following model:

- linear strain:  $\varepsilon(t, x) = \frac{1}{2}(\nabla u(t, x) + \nabla u^T(t, x))$
- conservation of momentum:  $v_t = \operatorname{div} \sigma(t, x)$
- constitutive relation:  $\sigma(t, x) = \int_0^t \mu(t-s)P(x)\varepsilon_t(s, x) ds$

Here,  $\mu$  is a scalar valued function of time, independent of  $x$ , while  $P$  is a bounded measurable function of space into symmetric and positive definite fourth order tensors. This setting corresponds to synchronous viscoelasticity, when all elastic moduli at each material point have the same relaxation behavior. Also, we have assumed that the body is in a stress-free reference configuration up to time  $t = 0$ .

Concerning the material properties, we make the following assumptions:

- (H1)** For each  $x \in \Omega$ , the tensor  $P(x)$ , considered as an operator mapping  $\mathbf{R}^{n \times n} \rightarrow \mathbf{R}^{n \times n}$ , is symmetric and positive definite. Moreover,  $P(x)$  is a measurable bounded function of space, and the inverse operator  $P^{-1}(x)$  is uniformly bounded.
- (H2)** For some  $\gamma > 0$ , the integral  $\int_0^\infty e^{-\gamma s} |\mu(s)| ds$  is finite.

- (H3) There exists a constant  $\Theta < \pi/2$  such that  $\widehat{\mu} : \{\Re\lambda > 0\} \rightarrow \{|\Arg \lambda| < \Theta\} \setminus \{0\}$ .
- (H4) There exists a constant  $M > 0$  such that  $|\lambda\widehat{\mu}'(\lambda)| \leq M|\widehat{\mu}(\lambda)|$  for all  $\Re\lambda > 0$ .

(Here,  $\widehat{f}$  denotes the Laplace transform of  $f$ .)

For example, certain viscoelastic materials can be modelled by a fractional derivative model (they are then called a power-law solid materials)

$$\sigma(t, x) = P(x) (D_t^\alpha \varepsilon(\cdot, x))(t),$$

$\alpha \in (0, 1)$ , where

$$(D_t^\alpha f)(t) := \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t s^{-\alpha} f(t-s) ds$$

(note that  $D_t^\alpha f = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \star \frac{d}{dt} f$  if  $f(0) = 0$ ). This can be achieved by choosing

$$\mu(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

For the equations of linear viscoelasticity we refer to [5], see also [6, Section 5].

We suppose that  $\Omega$  is a bounded domain with Lipschitz continuous boundary. Its boundary  $\partial\Omega$  is divided into two relatively open disjoint sets  $\Gamma_0$  and  $\Gamma_1$ . For  $t \geq 0$  let the movement of the body be fixed on  $\Gamma_0$ , a given boundary force  $g(t, x)$  acts at  $x \in \Gamma_1$ . ( $\Gamma_0$  may be empty; if  $\Gamma_1 = \emptyset$  then we are in the situation of a homogeneous boundary condition which is solved e.g. in [2]). Let the initial velocity be  $v_0$ . The set of equations to be studied is hence written as

$$\begin{aligned} v_t &= \operatorname{div} \sigma, & x \in \Omega, & t > 0 \\ \sigma &= \mu \star P \frac{1}{2} (\nabla v + \nabla v^T), & x \in \Omega, & t > 0 \\ v &= 0, & x \in \Gamma_0, & t > 0 \\ \sigma \cdot n &= g, & x \in \Gamma_1, & t > 0 \\ v &= v_0, & x \in \Omega, & t = 0. \end{aligned} \tag{1}$$

For shorthand we introduce the following spaces and operators:

- $Y := L^2(\Omega, \mathbf{R}^n)$ ;
- $X := L^2(\Omega, \mathbf{R}_{sym}^{n \times n})$ ;
- $\tilde{D} : \text{dom } \tilde{D} \subset X \rightarrow Y$ ,  $\text{dom } \tilde{D} = \{\sigma \in X, \text{div } \sigma \in L^2(\Omega, \mathbf{R}^n)\}$ ,  
 $\tilde{D}\sigma = -\text{div } \sigma$ ;
- $D : \text{dom } D \subset X \rightarrow Y$ ,  $\text{dom } D = \{\sigma \in X, \text{div } \sigma \in L^2(\Omega, \mathbf{R}^n),$   
 $\sigma \cdot n = 0 \text{ on } \Gamma_1\}$ ,  $D\sigma = -\text{div } \sigma$ ;
- $P : X \rightarrow X$ ,  $(P\sigma)(x) = P(x)\sigma(x)$ .

(the boundary condition is meant in the sense of traces, in  $H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$ ;  $(\text{div } \sigma)_i := \sum_{j=1}^n \frac{\partial \sigma_{ij}}{\partial x_j}$  is meant in the sense of distributions).

LEMMA 2.1 ([2, LEMMA 4.1]).

1.  $D \subset \tilde{D}$ .
2.  $D$  is a densely defined closed linear operator.
3. We define an operator  $D^*$  by  $\text{dom } D^* = \{v \in Y, v \in H^1(\Omega, \mathbf{R}^n),$   
 $v = 0 \text{ on } \Gamma_0\}$ ,

$$(D^*v)_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

(the boundary condition is meant in the sense of traces in  $H^{\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$ ). Then  $D^*$  is the adjoint of  $D$ .

4.  $P$  is a self-adjoint, positive definite bounded linear operator on  $X$ .

With this notation (1) can be rewritten as follows: find  $v : [0, \infty) \rightarrow \text{dom } D^* \subset Y$  and  $\sigma : [0, \infty) \rightarrow \text{dom } \tilde{D} \subset X$  such that

$$\begin{aligned} v_t &= -\tilde{D}\sigma, & t > 0 \\ \sigma &= \mu \star PD^*v, & t > 0 \\ \sigma \cdot n &= g, & x \in \Gamma_1, \quad t > 0 \\ v &= v_0 & t = 0; \end{aligned} \tag{2}$$

Eliminating  $v$  in (2) we obtain an equation for  $\sigma$ :

$$\begin{aligned}\sigma &= \mu \star PD^*(v_0 - 1 \star \tilde{D}\sigma), & t > 0 \\ \sigma.n &= g, & x \in \Gamma_1, \quad t > 0\end{aligned}\quad (3)$$

We are now in the position to state our main result:

**THEOREM 2.2.** *Let  $X, D, \tilde{D}, P$  be given as above. Suppose that the hypotheses (H1), (H2), (H3), (H4) hold. Then for any  $g \in \mathcal{W}_{loc}^{1,1}([0, \infty); H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n))$  and  $v_0 \in \text{dom } D^*$  there exists a function  $\sigma \in \mathcal{C}([0, \infty), X)$  solving (1) in the following sense:*

$$\begin{aligned}1 \star \mu \star \sigma(t) &\in \text{dom } D^* \tilde{D}, & t \geq 0, \\ \sigma(t) &= -PD^* \tilde{D}(1 \star \mu \star \sigma(t)) + 1 \star \mu(t)PD^*v_0, & t \geq 0, \\ (1 \star \mu \star \sigma)(t, \cdot).n &= (1 \star \mu \star g(t, \cdot))(t) & \text{in } H^{-\frac{1}{2}}(\Gamma_1), \quad t \geq 0.\end{aligned}$$

If, in addition for some  $\alpha \in (0, 1)$ ,

- (1)  $|\lambda^2 \hat{\mu}''(\lambda)| \leq M |\hat{\mu}(\lambda)|$  whenever  $\Re \lambda > 0$ ,
- (2)  $v_0 = 0$  or  $\mu \in L_{loc}^{1/(1-\alpha)}([0, \infty))$ ,

then for each  $g \in \mathcal{C}_0^\alpha([0, \infty); H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n))$  there exists  $\sigma \in \mathcal{C}_0^\alpha([0, \infty); X)$  solving (1) in the sense above.

The rest of the paper is devoted to the proof of Theorem 2.2. For this purpose, we split (3) into two auxiliary problems:

- (I) for a fixed  $t \geq 0$  find  $r(t) \in \text{dom } D^* \tilde{D} \subset X$  such that  $r(t, x).n(x) = g(t, x)$  for  $x \in \Gamma_1$  (in the sense of traces, i.e. in  $H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$ )
- (II) find  $\varrho : [0, \infty) \rightarrow \text{dom } D^*D \subset X$  solving

$$\varrho = -1 \star A \star D^*D\varrho + 1 \star AD^*v_0 - 1 \star A \star D^* \tilde{D}r - r$$

In fact, if  $r$  and  $\varrho$  are given by (I) and (II), then formally we obtain a solution of (3) by putting

$$\sigma := r + \varrho.$$

### 3. Problem (I)

Before we give the technical details, we outline the content of this section. Let  $D, \tilde{D}$  be the operators defined in the previous section. We consider the following auxiliary problem: given  $g \in H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$  find  $w \in \text{dom } \tilde{D}D^*$  such that

$$\begin{aligned} \tilde{D}D^*w + w &= 0 & \text{in } \Omega \\ D^*w.n &= g & \text{on } \Gamma_1. \end{aligned} \quad (4)$$

( $H^{-\frac{1}{2}}(\Gamma_1)$  is the dual space of  $H^{\frac{1}{2}}(\Gamma_1)$  with respect to the  $L^2(\Gamma_1)$ -scalar product.) If  $w$  is a solution to (4) then  $r := D^*w$  will be a solution of problem (I), satisfying the equations

$$\begin{aligned} r &\in \text{dom}(D^*\tilde{D}), \\ D^*\tilde{D}r + r &= 0 & \text{in } \Omega, \\ r.n &= g & \text{on } \Gamma_1. \end{aligned} \quad (5)$$

We will denote  $r = T^{-1}g$ . The operator  $T^{-1}$  will therefore act as a generalized inverse for the trace operator  $Tr = r.n|_{\Gamma}$ .

The function  $r(t) = T^{-1}g(t)$  will then serve as a solution to Problem I on the way to prove our main result.

REMARK 3.1. *In linear elasticity the following problem is studied:*

$$\tilde{D}\sigma = f \text{ in } \Omega, \quad \sigma.n = g \text{ on } \Gamma_1 \quad w = 0 \text{ on } \Gamma_0$$

*with the constitutive relation  $\sigma = \lambda(\text{tr}D^*w)I + 2\mu D^*w$ , see [1, Section 6.6.3]; we employ the method used for this problem. Similar methods have been used to solve a boundary value problem in [4, Theorem I.3.5].*

We give now the details in form of a trace theorem:

THEOREM 3.2. *Let the open set  $\Omega \subset \mathbf{R}^n$  be a halfspace or a bounded set with Lipschitz continuous boundary. We set*

$$\begin{aligned} W &:= \{r \in L^2(\Omega, \mathbf{R}_{sym}^{n \times n}); \text{div } r \in L^2(\Omega, \mathbf{R}^n)\}, \\ \|r\|_W^2 &:= \|r\|_{L^2}^2 + \|\text{div } r\|_{L^2}^2, \end{aligned}$$

( $\operatorname{div} r$  is meant in the sense of distributions), a Banach space. Suppose that the boundary of  $\Omega$  is divided to two relatively open disjoint sets  $\partial\Omega = \Gamma_1 \cup \Gamma_0$ .

Then there exists a bounded linear operator

$$T : W \rightarrow H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$$

such that  $Tr = r.n|_{\Gamma_1}$  for smooth  $r$ .

Moreover,  $T$  is surjective and there exists a bounded linear operator

$$T^{-1} : H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n) \rightarrow W$$

such that  $T(T^{-1}g) = g$  for any  $g \in H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$ .

The operator  $T^{-1}$  can be constructed so that  $T^{-1}g \in \operatorname{dom}D^*\tilde{D}$  and  $D^*\tilde{D}Tg + Tg = 0$ .

*Proof.* We divide the proof into two steps.

**Step 1.** Here we prove the existence of  $T$ .

Suppose first that  $r \in H^1(\Omega, \mathbf{R}_{sym}^{n \times n}) \subset W$  and  $u \in H^{\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$ . By the trace theorem for scalar valued functions ([3]), for each  $u \in H^{\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$  there exists  $\tilde{u} \in H^1(\Omega, \mathbf{R}^n)$  satisfying  $u_1 := u$  on  $\Gamma_1$ , and  $u_1 := 0$  on  $\Gamma_0$ , such that  $\|\tilde{u}\|_{H^1(\Omega)} \leq \|u\|_{H^{\frac{1}{2}}(\Gamma_1)}$ . Then by Green's formula ([3]) we have

$$\int_{\Gamma_1} (r.n).udS(x) = \int_{\Omega} (\operatorname{div} r).\tilde{u}dx + \int_{\Omega} r.\nabla\tilde{u}dx.$$

For  $r \in W$  we define

$$\langle Tr, u \rangle_{H^{-\frac{1}{2}}(\Gamma_1), H^{\frac{1}{2}}(\Gamma_1)} := \int_{\Omega} \operatorname{div} r.\tilde{u}dx + \int_{\Omega} r.\nabla\tilde{u}dx.$$

Then

$$|\langle Tr, u \rangle| \leq \left( \|\operatorname{div} r\|_{L^2(\Omega)} + \|r\|_{L^2(\Omega)} \right) \|\tilde{u}\|_{H^1(\Omega)} \leq c\|r\|_W \|u\|_{H^{\frac{1}{2}}(\Gamma_1)},$$

and so by a density argument this is the unique extension of the restriction operator.

**Step 2.** Here we prove the surjectivity of  $T$  and the existence of  $T^{-1}$ . Let  $g \in H^{-\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$ . Consider the Banach space

$$V := \{u \in L^2(\Omega, \mathbf{R}^n); u \in H^1(\Omega, \mathbf{R}^n), u = 0 \text{ on } \Gamma_0\}, \quad \|\cdot\|_V := \|\cdot\|_{H^1}$$



(the boundary value is meant in the sense of traces, i.e. in  $H^{\frac{1}{2}}(\Gamma_0, \mathbf{R}^n)$ ), the bilinear form

$$a(u, w) := \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n \int_{\Omega} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) + \sum_{i=1}^n \int_{\Omega} u_i w_i$$

on  $V$  and the functional

$$F(u) := \sum_{i=1}^n \langle u_i, g_i \rangle_{H^{\frac{1}{2}}(\Gamma_1), H^{-\frac{1}{2}}(\Gamma_1)}$$

on  $V$  ( $u_i|_{\Gamma_1}$  is meant in the sense of traces). Then  $a$  is a bounded bilinear form on  $V$  which is coercive due to Korn's inequality ([3]),

$$a(u, u) \geq c \|u\|_{H^1(\Omega)}^2, \quad u \in V.$$

The functional  $F$  is bounded due to the trace theorem,

$$\begin{aligned} |F(u)| &\leq \|u\|_{H^{\frac{1}{2}}(\Gamma_1)} \|g\|_{H^{-\frac{1}{2}}(\Gamma_1)} = \|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \|g\|_{H^{-\frac{1}{2}}} \\ &\leq c \|u\|_{H^1(\Omega)} \|g\|_{H^{-\frac{1}{2}}}. \end{aligned}$$

So we can use the Lax-Milgram lemma to obtain a unique  $w \in V$ , depending continuously on  $g$ , satisfying

$$a(w, u) = F(u) \quad \text{for any } u \in V.$$

We see that  $w \in \text{dom } D^*$  and we will show that  $D^*w \in W$ . Then we will put  $T^{-1}g := D^*w$ .

In particular, for any  $\varphi \in \mathcal{D}(\Omega, \mathbf{R}^n) \subset V$  we have

$$\begin{aligned} 0 &= F(\varphi) = \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n \int \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) \left( \frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i} \right) + \sum_{i=1}^n \int w_i \varphi_i \\ &= -\frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n \langle \frac{\partial}{\partial x_j} \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right), \varphi_i \rangle_{\mathcal{D}', \mathcal{D}} - \\ &\quad -\frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n \langle \frac{\partial}{\partial x_i} \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right), \varphi_j \rangle_{\mathcal{D}', \mathcal{D}} + \sum_{i=1}^n \int_{\Omega} w_i \varphi_i dx = \\ &= -\sum_{i=1}^n \langle \text{div } (D^*w)_i, \varphi_i \rangle_{\mathcal{D}', \mathcal{D}} + \sum_{i=1}^n \int_{\Omega} w_i \varphi_i = \langle \tilde{D}D^*w + w, \varphi \rangle_{\mathcal{D}', \mathcal{D}}. \end{aligned}$$

Note that  $\tilde{D}D^*w \stackrel{\mathcal{D}'}{=} -w \in L^2(\Omega, \mathbf{R}^n)$  implies that  $D^*w \in \text{dom } \tilde{D} = W$ . We have

$$\begin{aligned} & \|D^*w\|_{L^2}^2 + \|\text{div } D^*w\|_{L^2}^2 = \\ & \|D^*w\|_{L^2}^2 + \|w\|_{L^2}^2 \leq c\|w\|_V^2 \leq c_1\|g\|_{H^{-\frac{1}{2}}(\Gamma_1)}^2, \end{aligned}$$

hence the continuity of  $T^{-1}$  follows.

Finally, we show that  $T(D^*w) = g$ . Since  $D^*w \in L^2$  and  $\text{div } D^*w \stackrel{\mathcal{D}'}{=} w \in L^2$ , by definition of  $T(D^*w)$  we have for any  $u \in H^{\frac{1}{2}}(\Gamma_1, \mathbf{R}^n)$  (we denote by  $\tilde{u} := T_1^{-1}u_1 \in H^1(\Omega, \mathbf{R}^n)$  the extension associated continuously by the trace theorem to  $u_1 := u$  on  $\Gamma_1$ ,  $u_1 := 0$  on  $\Gamma_0$ , see the Appendix):

$$\begin{aligned} & \langle T(D^*w), u \rangle_{H^{-\frac{1}{2}}(\Gamma_1), H^{\frac{1}{2}}(\Gamma_1)} := \int_{\Omega} \text{div } D^*w \cdot \tilde{u} + \\ & + \sum_{i,j=1}^n \int_{\Omega} (D^*w)_{ij} \frac{\partial \tilde{u}_i}{\partial x_j} = \int_{\Omega} w \cdot \tilde{u} + \sum_{i,j=1}^n \int_{\Omega} \frac{1}{2} \left( \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) \frac{\partial \tilde{u}_i}{\partial x_j} = \\ & = a(w, \tilde{u}) = F(\tilde{u}) = \langle g, u \rangle_{H^{-\frac{1}{2}}(\Gamma_1), H^{\frac{1}{2}}(\Gamma_1)}. \end{aligned}$$

□

REMARK 3.3. *The above proof does not use that  $D^*$  is the adjoint of  $D$ . In fact, the proof of Lemma 2.1 is based on Theorem 3.2 and Korn's inequality.*

#### 4. Proof of The main result

We write the equation in problem (II) as

$$\begin{aligned} P^{-\frac{1}{2}}\varrho(t) &= -1 \star \mu \star P^{-\frac{1}{2}}D^*DP^{\frac{1}{2}}P^{-\frac{1}{2}}\varrho(t) + 1 \star \mu(t)P^{\frac{1}{2}}D^*v_0 - \\ & - 1 \star \mu \star P^{\frac{1}{2}}D^*\tilde{D}r(t) - P^{-\frac{1}{2}}r(t) \end{aligned}$$

which is a Volterra equation for  $P^{-\frac{1}{2}}\varrho$  studied in [6].

THEOREM 4.1. ([6, Theorems 3.1, 3.3]) *Let  $A$  be a closed linear operator on a Banach space  $X$  with dense domain and  $T > 0$ . Let  $a \in L_{loc}^1([0, \infty))$  be of subexponential growth and such that*

- (1) for some constant  $M$  and any  $\Re\lambda > 0$  we have  $\widehat{a}(\lambda) \neq 0$ ,  $\frac{1}{\widehat{a}(\lambda)} \in \varrho(A)$ ,  $\|(I - \widehat{a}(\lambda)A)^{-1}\| \leq c$  (i.e. the equation is parabolic);
- (2)  $|\lambda\widehat{a}'(\lambda)| \leq c|\widehat{a}(\lambda)|$  for  $\Re\lambda > 0$  (i.e. the function  $a$  is 1-regular).

Then for all  $f \in \mathcal{W}^{1,1}([0, T]; X)$  there exists a unique  $u \in \mathcal{C}([0, T]; X)$  such that

$$a \star u(t) \in D(A), \quad u(t) = A(a \star u(t)) + f(t), \quad t \in [0, T].$$

If moreover

- (3)  $|\lambda^2\widehat{a}''(\lambda)| \leq c|\widehat{a}(\lambda)|$  for  $\Re\lambda > 0$  (i.e. the function  $a$  is 2-regular)

and  $\alpha \in (0, 1)$ , then for any  $f \in \mathcal{C}_0^\alpha([0, T]; X)$  the solution is in  $\mathcal{C}_0^\alpha([0, T]; X)$ .

With  $A := -P^{-\frac{1}{2}}D^*DP^{\frac{1}{2}}$ , a nonnegative self-adjoint operator on  $X$ ,  $a := 1 \star \mu$  and  $f := 1 \star \mu P^{\frac{1}{2}}D^*w_0 + 1 \star \mu \star P^{\frac{1}{2}}r - P^{-\frac{1}{2}}r$  we may apply this theorem to obtain  $P^{-\frac{1}{2}}\varrho(t)$ , and hence  $\varrho$ . We only have to check the regularity of  $f$  piece by piece. We note that the time regularity of  $g$  carries over to time regularity of  $r$  verbatim, and that a convolution of a locally integrable function ( $1 \star \mu$ ) with a locally  $\mathcal{W}^{1,1}$ - resp.  $\mathcal{C}_0^\alpha$ -function is again locally  $\mathcal{W}^{1,1}$  resp.  $\mathcal{C}_0^\alpha$ . Notice that if  $v_0 = 0$  then we do not need the regularity of  $1 \star \mu$  itself. The equation for  $\sigma := \varrho + r$  is straightforward, and the boundary condition for  $\sigma$  may be written only in a weak form, since only the convolution  $1 \star \mu \star \varrho$  is in the domain of  $D$ , and not  $\varrho$  itself.

## 5. Example

Let  $n = 3$  or  $2$  (for  $n = 1$  the problem (I) can be solved explicitly). We consider a linearly viscoelastic homogeneous isotropic medium in  $\mathbf{R}^n$ . The constitutive relation for this model has the form

$$\sigma_{ij}(t, x) = \sum_{k,l=1}^n [a_{ijkl}(\cdot) \star (\varepsilon_t)_{kl}(\cdot, x)](t)$$

with

$$a_{ijkl}(t) = \lambda(t)\delta_{ij}\delta_{kl} + \mu(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where  $\lambda$ ,  $\mu$  are the Lamé moduli ( $\mu$  is the shear modulus and  $\lambda+2\mu/3$  is the bulk or compression modulus). We suppose that the material is synchronous, i.e.

$$\lambda(t) + \frac{2}{3}\mu(t) = \beta\mu(t)$$

for some constant  $\beta > 0$ , i.e.  $\lambda = (\beta - \frac{2}{3})\mu$  (cf. [5, Chapter V] or [6, Section 5]). Then

$$a_{ijkl}(t) = \mu(t) \left[ \left(\beta - \frac{2}{3}\right)\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right] =: \mu(t)p_{ijkl}.$$

We denote by  $P$  the linear operator on  $\mathbf{R}_{sym}^{n \times n}$  defined by  $p_{ijkl}$ . Then  $P$  induces a symmetric and positive definite operator on  $X$ .

The constitutive relation is in the form

$$\sigma = \mu \star PD^*v$$

which fits the framework of our investigations, e.g. in the case of the fractional derivative model ( $\mu(t) = \frac{1}{\Gamma(1-\alpha)}t^{-\alpha}$ ,  $\alpha \in (0, 1)$ ,  $\hat{\mu}(\lambda) = \lambda^{\alpha-1}$ ).

#### REFERENCES

- [1] PH.G. CIARLET, *Mathematical elasticity. Volume I: Three-Dimensional elasticity*, North-Holland, Amsterdam, 1988.
- [2] W. DESCH, R. GRIMMER, *Singular relaxation moduli and smoothing in three dimensional viscoelasticity*, Trans. Amer. Math. Soc. (1989), no. 314, 381–404.
- [3] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.
- [4] O.A. OLEINIK, A.S. SHAMAEV, G.A. YOSIFIAN, *Mathematical problems in elasticity and homogenization*, North Holland, Amsterdam, 1992.
- [5] A. C. PIPKIN, *Lectures on viscoelasticity theory*, Springer, New York, 1972.
- [6] J. PRÜSS, *Evolutionary integral equations and applications*, Birkhäuser, Basel, 1993.

Received February 18, 2002.