Rend. Istit. Mat. Univ. Trieste
Vol. XXXV, 37–48 (2003)

From Hermite to Humbert Polynomials

G. Dattoli, S. Lorenzutta and C. Cesarano (*)

Summary. - *We use the multivariable Hermite polynomials to derive integral representations of Chebyshev and Gegenbauer polynomials. It is shown that most of the properties of these classes of polynomials can be deduced in a fairly straightforward way from this representation, which proves a unifying framework for a large body of polynomial families, including forms of the Humbert and Bessel type, which are a natural consequence of the point of view developed in this paper.*

1. Introduction

It is well known that Chebyshev polynomials of the second kind are defined by [1, 5]

$$U_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k)!(2x)^{n-2k}}{k!(n-2k)!} \tag{1}$$

(*) Authors' addresses: G. Dattoli, ENEA – Unità Tecnico Scientifica Tecnologie Fisiche Applicate, Centro Ricerche - Frascati - Via E. Fermi, 45, 00044 - Frascati, Roma, Italia, e-mail: dattoli@frascati.enea.it
S. Lorenzutta, ENEA – Dipartimento Innovazione, Centro Ricerche - Bologna, Via Don Fiammelli, 2, 40100 - Bologna, Italia, e-mail: lorenzutta@bologna.enea.it
C. Cesarano, University of Ulm, Germany and ENEA Guest - Frascati - Via E. Fermi, 45, 00044 - Frascati, Roma, Italia, e-mail: cesarano@frascati.enea.it
Keywords: Gegenbauer polynomials, Hermite polynomials, Humbert polynomials.
AMS Subject Classification: 33C45, 44A45, 33C50.

and the two variable Hermite polynomials reads [4]

$$H_n(x,y) = n! \sum_{k=0}^{[n/2]} \frac{y^k x^{n-2k}}{k!(n-2k)!}. \tag{2}$$

By recalling that

$$n! = \int_0^{+\infty} e^{-t} t^n dt, \tag{3}$$

we can state the following integral representation for the second kind Chebyshev polynomials, which is a slightly different version of the analogous representation proposed in [6]

$$U_n(x) = \frac{1}{n!} \int_0^{+\infty} e^{-t} t^n H_n\left(2x, -\frac{1}{t}\right) dt. \tag{4}$$

Along with (4) we introduce the further representations

$$T_n(x) = \frac{1}{2(n-1)!} \int_0^{+\infty} e^{-t} t^{n-1} H_n\left(2x, -\frac{1}{t}\right) dt, \tag{5}$$

$$W_n(x) = \frac{2}{(n+1)!} \int_0^{+\infty} e^{-t} t^{n+1} H_n\left(2x, -\frac{1}{t}\right) dt. \tag{6}$$

The polynomials $T_n(x)$ are first kind Chebyshev polynomials [1, 5], while the nature and the role of the $W_n(x)$, which for the moment will be indicated as *third kind Chebyshev polynomials*, will be specified in the following.

Just to give a preliminary idea of the usefulness of the previous representation, we note that the following well known property of the two variable Hermite polynomials [4]

$$\frac{\partial}{\partial x} H_n(x,y) = n H_{n-1}(x,y), \tag{7}$$

$$H_{n+1}(x,y) = \left(x + 2y \frac{\partial}{\partial x}\right) H_n(x,y), \tag{8}$$

allow to conclude that the $U_n(x)$ and $W_n(x)$ are linked by the relations [1, 5]

$$\frac{d}{dx} U_n(x) = n W_{n-1}(x), \tag{9}$$

$$\frac{d}{dx} T_n(x) = n U_{n-1}(x), \tag{10}$$

and

$$U_{n+1}(x) = xW_n(x) - \frac{n}{n+1}W_{n-1}(x), \qquad (11)$$

$$T_{n+1}(x) = xU_n(x) - U_{n-1}(x). \qquad (12)$$

The generating function of the above families of Chebyshev polynomials is also a direct consequence of that of the $H_n(x, y)$

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} H_n(x, y) = e^{x\xi + y\xi^2}. \qquad (13)$$

By multiplying indeed both sides of (4) by $\xi^n/n!$, by summing up over n, by exploiting (13) and then by integrating over t, we end up with the well known result [1, 5]

$$\sum_{n=0}^{\infty} \xi^n U_n(x) = \frac{1}{1 - 2\xi x + \xi^2}, \qquad -1 < x < 1, |\xi| < 1. \qquad (14)$$

By using the same procedure and the second of (7) we get [1, 5]

$$\sum_{n=0}^{\infty} \xi^n T_{n+1}(x) = \frac{x - \xi}{1 - 2\xi x + \xi^2}, \qquad (15)$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)\xi^n W_{n+1}(x) = \frac{8(x - \xi)}{(1 - 2\xi x + \xi^2)^3}. \qquad (16)$$

The use of the integral representations relating Chebyshev and Hermite polynomials is a fairly important tool of analysis allowing the derivation of a wealth of relations (others will be discussed in the concluding section) between first, second and third kind Chebyshev polynomials.

This paper is devoted to a first attempt towards this direction. We will show that Gegenbauer polynomials can be framed within such a framework, which offers interesting criteria of generalizations. It will be indeed shown that, by combining the wealth of Hermite polynomial forms and the flexibility of the proposed representations, we can develop a systematic procedure of generalization by including in a natural way the Humbert [8], Gould [7] and Bessel [9] polynomials.

2. Hermite and Gegenbauer polynomials

As already remarked, integral transforms relating Chebyshev and Hermite polynomials are not new, and in [3] it has been shown that:

$$U_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-s} H_n\left(2xs, -s\right) ds. \tag{17}$$

We can introduce a slightly modified form of $U_n(x)$ by defining the quantity (for its link with the ordinary case see below):

$$U_n(x, y; \alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha s} H_n\left(2xs, -ys\right) ds. \tag{18}$$

The use of the identity [4]

$$\frac{\partial}{\partial y} H_n(x, y) = n(n-1) H_{n-2}(x, y) \tag{19}$$

allows to conclude that

$$\frac{\partial}{\partial y} U_n(x, y; \alpha) = \frac{\partial}{\partial \alpha} U_{n-2}(x, y; \alpha), \tag{20}$$

$$\frac{\partial}{\partial x} U_n(x, y; \alpha) = -2\frac{\partial}{\partial \alpha} U_{n-1}(x, y; \alpha), \tag{21}$$

which can be combined to give

$$\frac{\partial^2}{\partial x^2} U_n(x, y; \alpha) = 4\frac{\partial^2}{\partial \alpha \partial y} U_n(x, y; \alpha). \tag{22}$$

This last identity and the fact that

$$U_n(x, 0; \alpha) = \frac{(2x)^n}{\alpha^{n+1}} \tag{23}$$

allows to define the $U_n(x, y; \alpha)$ as

$$\begin{aligned} U_n(x, y; \alpha) &= e^{\frac{y}{4}\hat{D}_\alpha^{-1}\frac{\partial^2}{\partial x^2}} \frac{(2x)^n}{\alpha^{n+1}} \\ &= \sum_{s=0}^{[n/2]} \frac{(-y)^s (n-s)! (2x)^{n-2s}}{s!(n-2s)!\alpha^{n+1-s}} \end{aligned} \tag{24}$$

where we have denoted by D_α^{-1} the inverse of the derivative operator, the rules concerning the use of this operator will be discussed in the concluding section.

The polynomials $U_n(x,y;\alpha)$ can be specified by integral representation

$$U_n(x,y;\alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-\alpha t} t^n H_n\left(2x, -\frac{y}{t}\right) dt \tag{25}$$

which suggests the introduction of the polynomials $T_n(x,y;\alpha)$ and $W_n(x,y;\alpha)$ (see equation (5)) linked by

$$\frac{\partial}{\partial\alpha} U_n(x,y;\alpha) = -\frac{1}{2}(n+1)W_n(x,y;\alpha), \tag{26}$$

$$\frac{\partial}{\partial\alpha} T_n(x,y;\alpha) = -\frac{n}{2} U_n(x,y;\alpha). \tag{27}$$

Chebyshev polynomials are particular cases of the Gegenbauer polynomials specified by the series [1, 5]

$$C_n^{(\mu)}(x) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k} \Gamma(n-k+\mu)}{k!(n-2k)!}. \tag{28}$$

By using the integral representation of the Euler function [1, 5]

$$\Gamma(\nu) = \int_0^{+\infty} e^{-t} t^{\nu-1} dt \tag{29}$$

and the same arguments exploited for the Chebyshev case, we can introduce the polynomials

$$\begin{aligned} C_n^{(\mu)}(x,y;\alpha) &= \frac{1}{\Gamma(\mu)} \sum_{s=0}^{[n/2]} \frac{(-y)^s (2x)^{n-2s} \Gamma(n+\mu-s)}{s!(n-2s)!\alpha^{n+\mu-s}} \\ &= \frac{1}{n!\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu-1} H_n\left(2x, -\frac{y}{t}\right) dt. \end{aligned} \tag{30}$$

The relevant generating function is readily obtained by means of the same procedure leading to (14) and (15) thus getting indeed ($\mu \neq 0$)

$$\sum_{n=0}^{\infty} \xi^n C_n^{(\mu)}(x,y;\alpha) = \frac{1}{[\alpha - 2x\xi + y\xi^2]^\mu}. \tag{31}$$

The integral representation (30) is very flexible and can be exploited for different purposes. It is e.g. evident that

$$(-1)^m \frac{\partial^m}{\partial \alpha^m} U_n(x,y;\alpha) = m! C_n^{(m+1)}(x,y;\alpha), \tag{32}$$

and

$$\begin{aligned} \sum_{m=1}^{\infty} (-\xi)^{m-1} C_n^{(m)}(x,y;\alpha) &= C_n^{(1)}(x,y;\alpha+\xi) \\ &= U_n(x,y;\alpha+\xi). \end{aligned} \tag{33}$$

The recurrences

$$\frac{n+1}{\mu} C_{n+1}^{(\mu)}(x,y;\alpha) = x C_n^{(\mu+1)}(x,y;\alpha) - y C_{n-1}^{(\mu+1)}(x,y;\alpha) \tag{34}$$

and

$$\frac{\partial}{\partial y} C_n^{(\mu)}(x,y;\alpha) = -\mu C_{n-2}^{(\mu+1)}(x,y;\alpha), \tag{35}$$

are just a consequence of (30) and of the identities (7) and (19). The use of the operational relation

$$H_{n+m}(x,y) = \left(x + 2y \frac{\partial}{\partial x} \right)^m H_n(x,y) \tag{36}$$

has been used to generalize the Rainville identity [10]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+l}(x,y) = e^{xt+yt^2} H_l(x+2yt,y). \tag{37}$$

The previous relations allow the derivation of a wealth of generating functions for the $C_n^{(\mu)}(x,y;\alpha)$ polynomials, we find indeed that $(\mu > 0)$

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(n+m)!}{n!} \xi^n C_{n+m}^{(\mu)}(x,y;\alpha) = \\ &\quad = \frac{1}{\Gamma(\mu)} \int_0^{+\infty} e^{-\alpha t} t^{m+\mu-1} \left(2x - 2\frac{y}{t}\frac{\partial}{\partial x} \right)^m e^{-t(\alpha - 2x\xi + y\xi^2)} dt. \end{aligned} \tag{38}$$

can be written as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n+m)!}{n!m!}\xi^n C_{n+m}^{(\mu)}(x,y;\alpha)=\\ =\frac{1}{[F(x,y;\alpha,\xi)]^{\mu+m}}C_m^{(\mu)}(x-y\xi,F(x,y;\alpha,\xi)y;1).\end{aligned} \tag{39a}$$

where

$$F(x,y;\alpha,\xi)=\alpha-2x\xi+y\xi^2 \tag{39b}$$

In this section we have provided further elements proving the usefulness of the link between Hermite and Gegenbauer polynomials, in the following we will show that we can go much further using the wealth of Hermite forms which will allow us to get further extensions.

3. Hermite and Humbert polynomials

The Kampé de Fériét Hermite polynomials of order m are specified by the series definition [4]

$$H_n^{(m)}(x,y)=n!\sum_{r=0}^{[n/m]}\frac{y^r x^{n-mr}}{r!(n-mr)!}. \tag{39}$$

and by the generating function

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}H_n^{(m)}(x,y)=e^{x\xi+y\xi^m}. \tag{40}$$

we can therefore suggest the generalization of the second kind Chebyshev polynomials by introducing the polynomial

$${}_{(m)}U_n(x,y;\alpha)=\frac{1}{n!}\int_0^{+\infty}e^{-\alpha t}t^n H_n^{(m)}\left(mx,-\frac{y}{t^{m-1}}\right)dt \tag{41}$$

whose generating function can be easily obtained by employing the same procedure leading to (14) which yields

$$\sum_{n=0}^{\infty}\xi^n_{(m)}U_n(x,y;\alpha)=\frac{1}{\alpha-mx\xi+y\xi^m}. \tag{42}$$

The generalized Gegenbauer polynomials can be found in a similar way thus finding ($\mu > 0$)

$$\sum_{n=0}^{\infty} \xi^n_{(m)} C_n^{(\mu)}(x, y; \alpha) = \\ = \frac{1}{\Gamma(\mu) n!} \int_0^{+\infty} e^{-\alpha t} t^{n+\mu-1} H_n^{(m)} \left(mx, -\frac{y}{t^{m-1}} \right) dt. \tag{43}$$

with generating function ($\mu \neq 0$)

$$\sum_{n=0}^{\infty} \xi^n_{(m)} C_n^{(\mu)}(x, y; \alpha) = \frac{1}{[\alpha - mx\xi + y\xi^m]^{\mu}}. \tag{44}$$

The use of the identities

$$\begin{aligned} H_{n+1}^{(m)}(x, y) &= \left(x + my \frac{\partial^{m-1}}{\partial x^{m-1}} \right) H_n^{(m)}(x, y), \\ \frac{\partial}{\partial y} H_n^{(m)}(x, y) &= n(n-1) \cdots (n-m+1) H_{n-m}^{(m)}(x, y) \end{aligned} \tag{45}$$

yields the recurrences

$$\begin{aligned} \frac{n+1}{m\mu} \left({}_{(m)}C_{n+1}^{(\mu)}(x, y; \alpha) \right) &= x \left({}_{(m)}C_n^{(\mu+1)}(x, y; \alpha) \right) \\ &\quad - y \left({}_{(m)}C_{n-m+1}^{(\mu+1)}(x, y; \alpha) \right) \\ \frac{\partial}{\partial y} \left({}_{(m)}C_n^{(\mu)}(x, y; \alpha) \right) &= -\mu \left({}_{(m)}C_{n-m}^{(\mu+1)}(x, y; \alpha) \right) \end{aligned} \tag{46}$$

This last class of polynomials can be identified as the polynomials introduced by Humbert in ref. [8] within a different context. A systematic effort of generalization of the Gegenbauer polynomials was also undertaken by Gould [7]. The strategy of generalization we are developing is complementary to that of Gould and benefits from the variety of existing Hermite polynomials.

To give a further example we note that the polynomial

$$H_n(x, y, z) = n! \sum_{r=0}^{[n/3]} \frac{z^r H_{n-3r}(x, y)}{r!(n-3r)!}. \tag{47}$$

can be exploited to define ($\mu > 0$)

$$C_n^{(\mu)}(x,y,z;\alpha) = \frac{1}{\Gamma(\mu)n!}\int_0^{+\infty} e^{-\alpha t} t^{n+\mu-1} H_n\left(3x, -\frac{2y}{t}, \frac{z}{t^2}\right) dt \tag{48}$$

and the generating function

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} H_n(x,y,z) = e^{x\xi + y\xi^2 + z\xi^3} \tag{49}$$

can be exploited to prove that ($\mu \neq 0$)

$$\sum_{n=0}^{\infty} \xi^n C_n^{(\mu)}(x,y,z;\alpha) = \frac{1}{[\alpha - 3x\xi + 2y\xi^2 + z\xi^3]^\mu}. \tag{50}$$

This function too has interesting properties which will be more deeply investigated in a forthcoming paper.

In this section we have outlined how a systematic effort to generalize Gegenbauer polynomials can be achieved in a fairly straightforward way from the properties of the Hermite polynomials; in the forthcoming section we will discuss further examples and discuss future direction of the present line of research.

4. Concluding Remarks

We have left open some points in the previous section which we will try to clarify in this concluding section.

We have introduced the polynomials $W_n(x)$ which have been indicated as *third kind* Chebyshev polynomials. It is clear that in terms of Gegenbauer polynomials they can be recognized as

$$W_n(x) = \frac{2}{n+1} C_n^{(2)}(x), \tag{51}$$

furthermore

$$\begin{aligned} T_n(x) &= \cos(n\cos^{-1}(x)), \\ U_n(x) &= \frac{\sin[(n+1)\cos^{-1}(x)]}{\sqrt{1-x^2}}. \end{aligned} \tag{52}$$

we also find

$$(n+1)W_n(x) = -\frac{(n+1)x}{1-x^2}\frac{\sin[(n+2)\cos^{-1}(x)]}{\sqrt{1-x^2}} - \frac{(n+2)}{\sqrt{(1-x^2)^3}}\cos[(n+1)\cos^{-1}(x)]. \quad (53)$$

It is also worth noting that

$$U_n(x,y;\alpha) = \frac{1}{\alpha}\left(\frac{y}{\alpha}\right)^{n/2} U_n\left(\frac{x}{\sqrt{\alpha y}}\right) \quad (54)$$

which can be exploited to get $U_n(x,y;\alpha)$ and $T_n(x,y;\alpha)$ in terms of circular functions.

The use of the generating function [2]

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}T_n(x) = e^{x\xi}\cos\left(\xi\sqrt{1-x^2}\right) \quad (55)$$

which follows directly from (52) (see [2]) can also be exploited to derive the integral representation

$$T_n(x) = \frac{n!}{2\pi}\int_0^{2\pi} e^{xe^{i\theta}}\cos\left[e^{i\theta}\sqrt{1-x^2}\right]e^{-in\theta}d\theta. \quad (56)$$

Furthermore, generating functions of the type [2]

$$\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\left[T_n(x)\right]^2 = \frac{1}{2}e^{\xi}\left[1+\exp\left(2\xi(x^2-1)\right)\cos\left(2x\xi\sqrt{1-x^2}\right)\right] \quad (57)$$

can be exploited to state further integral transforms concerning the products of Chebyshev polynomials.

These last equations can be used as a useful complement of the generating functions given in (14) and (15).

The use of the method of the integral transform can be usefully extended to other families of polynomials, a noticeable example is provided by the Bessel polynomials (7), defined by

$$y_n(x) = \sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\left(\frac{x}{2}\right)^k \quad (58)$$

or, equivalently, by the integral representation

$$y_n(x) = \frac{1}{n!}\int_0^{+\infty} e^{-t}\left[t + \frac{xt^2}{2}\right]^n dt. \tag{59}$$

This last identity suggests the following generalized forms of Bessel polynomials ($\alpha \in \mathbb{R}$)

$$y_n(x, w; \alpha) = \frac{1}{n!}\int_0^{+\infty} e^{-t}t^{\alpha}\left[wt + \frac{xt^2}{2}\right]^n dt \tag{60}$$

with the interesting properties

$$\begin{aligned} \frac{\partial}{\partial x} y_n(x, w; \alpha) &= \frac{1}{2} y_{n-1}(x, w; \alpha + 2) \\ \frac{\partial}{\partial w} y_n(x, w; \alpha) &= y_{n-1}(x, w; \alpha + 1) \\ (n+1)\ y_{n+1}(x, w; \alpha) &= w\ y_n(x, w; \alpha + 1) + \frac{1}{2}\ y_n(x, w; \alpha + 2) \end{aligned} \tag{61}$$

and

$$\sum_{m=0}^{\infty} \frac{\xi^m}{m!} y_n(x, w; m) = \frac{1}{1-\xi}\ y_n\left(\frac{x}{(1-\xi)^2}, \frac{w}{1-\xi}; 0\right). \tag{62}$$

Regarding the use of the operator D_α^{-1} we note that it acts on functions of the type $f(\alpha) = \alpha^{-m}$ so that we find

$$D_\alpha^{-s}\alpha^{-m+1} = \frac{(m-s)!}{m!}\alpha^{-m+1-s}. \tag{63}$$

We have not specified the lower limit of integration which is tacitly assumed to be infinite.

The results we have presented in this paper have been shown to be a versatile tool of analysis to study the properties of classical and generalized polynomials. In a forthcoming investigation we will benefit from the present results to go more deeply into the properties and the applications of the discussed polynomials.

References

[1] L. C. Andrews, *Special functions for engineers and applied mathematics*, MacMillan, New York, 1958.

[2] G. Dattoli, C. Cesarano and D. Sacchetti, *A note on Chebyshev polynomials*, Ann. Univ. Ferrara **7** (2001), no. 47, 107–115.

[3] G. Dattoli, S. Lorenzutta and A. Torre, *The generating functions method and evaluation of integrals involving combinations of special functions*, Quaderni del Gruppo Naz. per l'Informatica Matematica, vol. 8, C.N.R., Firenze, 1997.

[4] G. Dattoli and A. Torre, *Theory and applications of generalized Bessel functions*, Aracne, Roma, 1996.

[5] P. Davis, *Interpolation and approximation*, Dover, New York, 1975.

[6] A. Erdéli, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher trascendental functions*, vol. 10, Mc-Graw Hill, New York, 1953, page 172.

[7] H. W. Gould, *Inverse series relations and other expansions involving Humbert polynomials*, Duke Math. J. **32** (1965), 697–711.

[8] P. Humbert, *Sur une généralization de l'équation de Laplace*, J. de Mathématiques Pures et Appliquées **39** (1921), 145–159.

[9] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc. **65** (1948), 100–105.

[10] E. D. Rainville, *Special functions*, MacMillan, New York, 1960.

Received May 17, 2001.