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On the Derivatives of a Family of Analytic Functions

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Summary. - *For $\beta < 1$, $n = 0, 1, 2, \ldots$, and $-\pi < \alpha \leq \pi$, we let $M_n(\alpha, \beta)$ denote the family of functions $f(z) = z + \cdots$ that are analytic in the unit disk and satisfy there the condition*

$$Re\left\{(D^n f)' + \frac{1 + e^{i\alpha}}{2(n+1)} z(D^n f)''\right\} > \beta,$$

where $D^n f(z)$ is the Hadamard product or convolution of f with $z/(1-z)^{n+1}$. We prove the inclusion relations $M_{n+1}(\alpha, \beta) \subset M_n(\alpha, \beta)$, and $M_n(\alpha, \beta) < M_n(\pi, \beta), \beta < 1$. Extreme points, as well as integral and convolution characterizations, are found. This leads to coefficient bounds and other extremal properties. The special cases $M_0(\alpha, 0) \equiv \mathcal{L}_\alpha, M_n(\pi, \beta) \equiv M_n(\beta)$ have previously been studied [16], [1].

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

that are analytic in the unit disk $\Delta = \{z : |z| < 1\}$. Denote by $M_n(\alpha, \beta), \beta < 1$, $n = 0, 1, 2, \ldots, -\pi < \alpha \leq \pi$, the subfamily of $\mathcal{A}$

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consisting of functions f of the form (1) for which

$$\operatorname{Re}\left\{(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)} z (D^n f)''\right\} > \beta \quad \text{in } \Delta,$$

where $D^n f$ is the Ruscheweyh derivative [12] of f defined by

$$D^n f(z) = z(z^{n-1} f(z))^{(n)}/n! = f(z) * \left(z/(1-z)^{n+1}\right).$$

The operator $*$ stands for the Hadamard product or convolution of two power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, that is, $(f * g)(z) = f(z) * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k$. It is obvious that $M_n(\alpha, \beta) \subset M_n(\alpha, \gamma)$ if $\beta > \gamma$. We also know that $M_{n+1}(0, \beta) \subset M_n(0, \beta)$ [9]. Alexander [3] showed that $M_0(\pi, 0)$ is a subfamily of analytic univalent functions. Note that, for $\beta < 0, M_n(\alpha, \beta)$ need not be univalent in Δ. Singh and Singh [17] proved that the functions in $M_0(0, 0)$ are starlike in Δ. Silverman and Silvia [16] found extreme points, coefficient bounds, and convolution characterizations for $M_0(\alpha, 0), -\pi < \alpha \leq \pi$. Also Silverman [15] showed that for $f \in M_1(\pi, \beta)$, the partial sums $S_m(z, f)$ satisfy $\operatorname{Re}(S_m(z, f))' > \beta$. Ahuja and Jahangiri [2] showed that the functions in $M_n(\pi, \beta)$ are invariant under convolution with convex functions and introduced a convolution characterization for functions in $M_n(\pi, \beta)$. Furthermore, they found [1] $\gamma = \gamma(n, \beta) \geq \beta$ so that for f and g in $M_n(\pi, \beta)$, their convolution is in $M_n(\pi, \gamma)$. In this note we extend most of their results to more general case $M_n(\alpha, \beta)$. Finally, we will state some results as improvement to the previous results and we proved the inclusion relation $M_{n+1}(\alpha, \beta) \subset M_n(\alpha, \beta)$.

2. Main results

THEOREM 2.1. *$M_{n+1}(\alpha, \beta) \subset M_n(\alpha, \beta)$ for each $n \in N_0, \beta < 1$, and $-\pi < \alpha \leq \pi$.*

To prove this theorem we shall need the following lemma, which is due to Jack [7].

LEMMA 2.2. *Let w be an analytic function in Δ satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in \Delta$. Then if $|w|$ assumes its maximum value on the circle $|z| = r$ at a point z_1, we can write*

$$z_1 w'(z_1) = k w(z_1),$$

where k is real and $k \geq 1$.

Proof of theorem (2.1). Let $f \in M_{n+1}(\alpha, \beta)$. Then

$$\operatorname{Re}\left\{ \left(D^{n+1} f(z)\right)' + \frac{1+e^{i\alpha}}{2(n+2)} z \left(D^{n+1} f(z)\right)'' \right\} > \beta. \tag{2}$$

We define an analytic function $w(z)$ in Δ such that

$$(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)} z (D^n f)'' = \frac{1+(2\beta-1)w(z)}{1+w(z)}, \tag{3}$$

where $w(0) = 0$ and $w(z) \neq -1$ [10]. We shall show that $|w(z)| < 1$. From (3) we have

$$z(D^n f)'' = \frac{2(n+1)}{1+e^{i\alpha}} \left[\frac{1+(2\beta-1)w(z)}{1+w(z)} - (D^n f)' \right]. \tag{4}$$

Using the known identity

$$z(D^n f)' = (n+1)D^{n+1} f - n D^n f, \tag{5}$$

we get

$$(D^{n+1} f)' = \frac{2}{1+e^{i\alpha}} \frac{1+(2\beta-1)w(z)}{1+w(z)} - \frac{1-e^{i\alpha}}{1+e^{i\alpha}} (D^n f)'. \tag{6}$$

Now from (6) we conclude that

$$(D^{n+1} f)'' = \frac{2}{1+e^{i\alpha}} \frac{2(\beta-1)w'(z)}{(1+w(z))^2} - \frac{1-e^{i\alpha}}{1+e^{i\alpha}} (D^n f)''. \tag{7}$$

Suppose that for $z_0 \in \Delta$

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Using the lemma and setting $w(z_0) = e^{i\theta_0}$ in (6) and (7), we obtain

$$\begin{aligned} &\mathrm{Re}\left[(D^{n+1}f(z_0))' + \frac{1+e^{i\alpha}}{2(n+1)} z_0 (D^{n+1}f(z_0))''\right] \\ &= \beta + \frac{2(\beta-1)k}{n+1}\,\mathrm{Re}\frac{e^{i\theta_0}}{(1+e^{i\theta_0})^2} \\ &= \beta + \frac{(\beta-1)k}{(n+1)(1+\cos\theta_0)}. \end{aligned}$$

See that $(n+1)(1+\cos\theta_0) > 0$ for each n and $\theta_0 \neq \pi$, then

$$\mathrm{Re}\left[(D^{n+1}f(z_0))' + \frac{1+e^{i\alpha}}{2(n+1)} z_0 (D^{n+1}f(z_0))'' - \beta\right] < 0,$$

where $\beta < 1, k \geq 1, n \in N_0$, which is a contradiction to our hypothesis that $f \in M_{n+1}(\alpha, \beta)$. Thus $|w(z)| < 1$ and from (3) we conclude that $f \in M_n(\alpha, \beta)$. □

LEMMA 2.3. *[8] Let λ be a function that is defined on Δ with $Re\lambda(z) \geq 0$ for $z \in \Delta$. If p is analytic in Δ and $Re[p(z) + \lambda(z)zp'(z)] > 0$ for $z \in \Delta$, then $Rep(z) > 0$ for $z \in \Delta$.*

By taking $p = (D^n f)' - \beta$ and $\lambda(z) = \frac{1+e^{i\alpha}}{2(n+1)}$ in (2.3), we have

THEOREM 2.4. *For each $n \in N_0$, $-\pi < \alpha \leq \pi$, $\beta < 1, M_n(\alpha, \beta) \subset M_n(\pi, \beta)$.*

THEOREM 2.5. *The extreme points of $M_n(\alpha, \beta)$ are*

$$f_x(z) = z + 4(1-\beta)\sum_{k=2}^{\infty} \frac{n!(k-1)!}{(k+n-1)!k(k+1+(k-1)e^{i\alpha})} x^{k-1} z^k, \tag{8}$$

where $|x| = 1, z \in \Delta$.

Proof. From the definition of $M_n(\alpha, \beta)$ it follows that $f \in M_n(\alpha, \beta)$ if and only if $D^n f \in M_0(\alpha, \beta)$. Therefore the operator D^n is a linear homeomorphism from $M_0(\alpha, \beta)$ to $M_n(\alpha, \beta)$ and thus preserves extreme points. Now to find the extreme points of $\mathrm{clco}\, M_0(\alpha, \beta)$.

Let $f \in M_0(\alpha,\beta)$, then $\operatorname{Re}\left\{f'(z)+\frac{1+e^{i\alpha}}{2}zf''(z)\right\}>\beta, z\in\Delta, \beta<1, -\pi<\alpha\le\pi$, so there exists a $p\in\mathcal{P}$, the class of functions in the form $p(z)=1+\sum_{k=1}^{\infty}c_kz^k$ and $\operatorname{Re} p(z)>0, z\in\Delta$, such that

$$\begin{aligned} f'(z)+\frac{1+e^{i\alpha}}{2}zf''(z) &= \frac{1-e^{i\alpha}}{2}f'(z)+\frac{1+e^{i\alpha}}{2}(zf'(z))' \\ &= \beta+(1-\beta)p(z), \quad z\in\Delta. \end{aligned}$$

It follows that $\left(\frac{1-e^{i\alpha}}{1+e^{i\alpha}}\right)f'(z)+(zf'(z))'=\frac{2}{1+e^{i\alpha}}\left\{\beta+(1-\beta)p(z)\right\},$ which is equivalent to

$$cz^cf'(z)+z^c(zf'(z))'=\frac{2}{1+e^{i\alpha}}z^c\left\{\beta+(1-\beta)p(z)\right\},$$

where $c=\frac{1-e^{i\alpha}}{1+e^{i\alpha}}, \alpha\ne\pi$. Then we can write

$$\left[z^c(zf'(z))\right]'=\frac{2}{1+e^{i\alpha}}z^c\left\{\beta+(1-\beta)p(z)\right\}.$$

We conclude that

$$z^{c+1}f'(z)=(c+1)\int_0^z\xi^c\left\{\beta+(1-\beta)p(\xi)\right\}d\xi. \tag{9}$$

From Hergoltz's Theorem, see page 21 of [4], $p\in\mathcal{P}$ if and only if

$$p(z)=\int_X\frac{1+xz}{1-xz}d\mu(x)$$

for some probability measure μ. Substituting this into (9) leads to

$$f'(z)=\frac{c+1}{z^{c+1}}\int_0^z\xi^c\left[\beta+(1-\beta)\int_X\left(1+2\sum_{k=2}^{\infty}x^{k-1}\xi^{k-1}\right)d\mu(x)\right]d\xi.$$

Upon reversing the order of integration and integrating with respect to ξ, we obtain

$$f'(z)=\beta+(1-\beta)\int_X\left[1+2\sum_{k=2}^{\infty}\frac{c+1}{c+k}x^{k-1}z^{k-1}\right]d\mu(x),$$

then

$$f(z) = \beta z+(1-\beta)\int_X \left[z + 4\sum_{k=2}^{\infty} \frac{1}{k[k+1+(k-1)e^{i\alpha}]} x^{k-1}z^k\right] d\mu(x).$$

So the extreme points of clco$M_0(\alpha, \beta)$ are given by

$$z + 4(1-\beta)\sum_{k=2}^{\infty} \frac{1}{k[k+1+(k-1)e^{i\alpha}]} x^{k-1}z^k, \quad |x| = 1, z \in \Delta.$$

Also, note that

$$\begin{aligned} D^n f(z) &= \left(z + \sum_{k=2}^{\infty} a_k z^k\right) * \frac{z}{(1-z)^{n+1}} \\ &= z + \sum_{k=2}^{\infty} \begin{pmatrix} k+n-1 \\ n \end{pmatrix} a_k z^k. \end{aligned}$$

Thus the extreme points of clco $M_n(\alpha, \beta)$ are given by

$$z+\sum_{k=2}^{\infty} \frac{4(1-\beta)}{k[k+1+(k-1)e^{i\alpha}]} \begin{pmatrix} k+n-1 \\ n \end{pmatrix}^{-1} x^{k-1}z^k,\ |x| = 1, z \in \Delta$$

which simplifies to (8). Since the family $M_n(\alpha, \beta)$ is convex and therefore is equal to its convex hull, (8) gives the extreme points of $M_n(\alpha, \beta)$. For the cases $n = 0, \beta = 0, -\pi < \alpha \leq \pi$ and $n \in N_0, \beta < 1, \alpha = \pi$, (2.5) gives the extreme points of $M_0(\alpha, 0)$ and $M_n(\pi, \beta)$ obtained in [16] and [1], respectively. □

Corollary 2.6. *$f \in M_n(\alpha, \beta)$ if and only if f can be expressed as*

$$f(z) = \int_X z + \sum_{k=2}^{\infty} \left[\frac{4(1-\beta)n!(k-1)!}{(k+n-1)!k(k+1+(k-1)e^{i\alpha})} x^{k-1}z^k\right] d\mu(x),$$

where μ is a probability measure defined on the unit circle X.

Since the coefficient bounds are maximized at an extreme point, as an application of (2.5), we have

Corollary 2.7. *If* $f \in M_n(\alpha, \beta)$*, then*

$$|a_k| \le \frac{4(1-\beta)n!(k-1)!}{(k+n-1)!k|k+1+(k-1)e^{i\alpha}|}, \quad k \ge 2,\ -\pi < \alpha \le \pi.$$

Equality occurs for $f_x(z)$ *defined by* (8).

From (8) we see for $f \in M_n(\alpha, \beta)$ and $|z| = r < 1$ that

$$|f(z)| \le r + 4(1-\beta)\sum_{k=2}^{\infty} \frac{n!(k-1)!}{(k+n-1)!k|k+1+(k-1)e^{i\alpha}|} r^k.$$

By letting $r \to 1$ we obtain

$$|f(z)| \le 1 + 4(1-\beta)\sum_{k=2}^{\infty} \frac{n!(k-1)!}{(k+n-1)!k|k+1+(k-1)e^{i\alpha}|}. \quad (10)$$

Also, since for $n \ge 1, -\pi < \alpha \le \pi, M_n(\alpha,\beta) \subset M_1(\pi,\beta)$, we let $n = 1, \alpha = \pi$ in (10) to get

$$|f(z)| \le 1 + 2(1-\beta)\left(\frac{\pi^2}{6} - 1\right).$$

This shows that the family $M_n(\alpha, \beta), n \ge 1, -\pi < \alpha \le \pi$ is bounded in Δ for all $\beta, \beta < 1$. For $n = 0$, (10) becomes

$$\begin{aligned} |f(z)| &\le 1 + 4(1-\beta)\left(\sum_{k=2}^{\infty} 1/k|k+1+(k-1)e^{i\alpha}|\right) \\ &\le 1 + \frac{2(1-\beta)}{\cos(\alpha/2)}\left(\frac{\pi^2}{6} - 1\right). \end{aligned}$$

So the functions in $M_n(\alpha, \beta)$ are bounded in Δ for each $-\pi < \alpha < \pi, \beta < 1$. The above result yields Theorem 7 by Silverman and Silvia [16] for $\beta = 0$. Our next theorem is on the partial sums of the functions in $M_n(\alpha, \beta)$ which for the case $\alpha = \pi$ gives Theorem 2 by Ahuja and Jahangiri [1].

Theorem 2.8. *Let* $S_m(z, f)$ *denote the* m*-th partial sum of a function* f *in* $M_n(\alpha, \beta)$*. If* $f \in M_n(\alpha, \beta)$ *and if* $1 \le n \le 4$*, then*

$$S_m(z, f) \in M_{n-1}\left(\alpha, \frac{2\beta n + 1 - n}{n+1}\right).$$

To prove this theorem we shall need the following lemmas, the first of which is due to Gasper [6].

Lemma 2.9. *Let R be the positive root of the equation*

$$9t^7 + 55t^6 - 14t^5 - 948t^4 - 3247t^3 - 5013t^2 - 3780t - 1134 = 0.$$

If $-1 < t \leq R \simeq 4.5678018,$ *then*

$$\sum_{k=1}^{m} \frac{\cos k\theta}{k+t} \geq -\frac{1}{1+t}, \quad m = 1, 2, \ldots$$

When $t = 1$, (2.9) confirms the estimate by Rogosinski and Szegö [11].

Lemma 2.10. *Let* $-1 < t \leq R \simeq 4.5678018.$ *Then*

$$Re\left(\sum_{k=2}^{m} \frac{z^{k-1}}{k+t-1}\right) > \frac{-1}{1+t}, \quad z \in \Delta.$$

Lemma 2.11. *Let $p(z)$ be analytic in $\Delta, p(0) = 1$, and $Re\ p(z) > 1/2$ in Δ. Then for any function F, analytic in Δ, the function $p * F$ takes values in the convex hull of the image of Δ under F.*

Proof of Theorem (2.8). Let $f \in M_n(\alpha, \beta)$ be of the form (1). Then we have

$$\text{Re}\left(1 + \sum_{k=2}^{\infty} k\left[1 + \frac{(k-1)(1+e^{i\alpha})}{2(n+1)}\right]\left(\begin{array}{c} k+n-1 \\ n \end{array}\right) a_k z^{k-1}\right) > \beta \tag{11}$$

or

$$\text{Re}\left(1 + \sum_{k=2}^{\infty} \frac{2kn}{n+1}\left[1 + \frac{(k-1)(1+e^{i\alpha})}{2(n+1)}\right]\left(\begin{array}{c} k+n-1 \\ n \end{array}\right) a_k z^{k-1}\right)$$

$$> \frac{2\beta n + 1 - n}{n+1}.$$

For the m-th partial sum of f, we can write

$$(D^{n-1}S_m(z,f))' + \frac{1+e^{i\alpha}}{2(n+1)} z(D^{n-1}S_m(z,f))''$$
$$= 1 + \sum_{k=2}^{m} k\left[1 + \frac{(k-1)(1+e^{i\alpha})}{2(n+1)}\right]\binom{k+n-2}{n} a_k z^{k-1}$$
$$= \left(1 + \sum_{k=2}^{\infty} \frac{2kn}{n+1}\left[1 + \frac{(k-1)(1+e^{i\alpha})}{2(n+1)}\right]\binom{k+n-1}{n} a_k z^{k-1}\right)$$
$$* \left(1 + \sum_{k=2}^{m} \frac{n+1}{2(k+n-1)} z^{k-1}\right).$$

For $1 \le n \le 4$, it is clear by (2.10) that

$$\mathrm{Re}\left(1 + \sum_{k=2}^{m} \frac{n+1}{2(k+n-1)} z^{k-1}\right) > \frac{1}{2}, \quad z \in \Delta.$$

Now an application of (2.11) to

$$(D^{n-1}S_m(z,f))' + \frac{1+e^{i\alpha}}{2(n+1)} z(D^{n-1}S_m(z,f))''$$

concludes the theorem. □

Now we will denote the class

$$O_n(\alpha,\beta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left\{ (D^n f)' + \frac{e^{i\alpha}}{n+1} z(D^n f)'' \right\} > \beta,\ z \in \Delta \right\}.$$

For the case $n = \beta = 0$, we get the class which was introduced by Ruscheweyh [13]. Our next theorem gives a characterization condition for $O_n(\alpha,\beta)$ and $M_n(\alpha,\beta)$ in terms of convolutions.

Theorem 2.12. *1. A function $f \in \mathcal{A}$ is in $O_n(\alpha,\beta)$ if and only if*

$$Re\left\{ \frac{1}{z}\left[D^n f(z) * \left(\frac{e^{i\alpha} z(1+(1-2\beta)z)}{(n+1)(1-z)^{3-2\beta}} + \frac{(n+1-e^{i\alpha})z(1-z)^{1-\beta}(1-\beta z)}{(n+1)(1-z)^{3-2\beta}} \right)\right]\right\} > \beta,$$

with $-\pi < \alpha \le \pi, z \in \Delta, n \in N_0$.

2. $-\pi < \alpha \le \pi, n \in N_0, f \in M_n(\alpha,\beta)$ *if and only if*

$$Re\left\{\frac{1}{z}\left[D^n f(z) * \left(\frac{(1+e^{i\alpha})z(1+(1-\beta)z)}{2(n+1)(1-z)^{3-2\beta}} + \frac{(2n+1-e^{i\alpha})z(1-z)^{1-\beta}(1-\beta z)}{2(n+1)(1-z)^{3-2\beta}}\right)\right]\right\} > \beta,$$

where $z \in \Delta$.

3. A function $f \in \mathcal{A}$ *is in* $M_n(\alpha,\beta)$ *if and only if*

$$\frac{1}{z}\left[f(z) * \left(\frac{(1+x)\{(z+nz^2)(1-(\frac{1-e^{i\alpha}}{2})z) + (\frac{1+e^{i\alpha}}{2})z^2\}}{(1-z)^{n+3}} + \frac{(1-2\beta-x)z(1-z)^{n+3}}{(1-z)^{n+3}}\right)\right] \neq 0,$$

with $|x| = 1, z \in \Delta, -\pi < \alpha \le \pi$.

Proof of Theorem (2.12), *1.* We know that $f \in O_n(\alpha,\beta)$ if and only if $\mathrm{Re}\left\{(D^n f)' + \frac{e^{i\alpha}}{n+1} z (D^n f)''\right\} > \beta$ for all $n \in N_0, -\pi < \alpha \le \pi, \beta < 1$. On the other hand,

$$\begin{aligned}(D^n f)' + \frac{e^{i\alpha}}{n+1} z (D^n f)'' &= \left[\left(1 - \frac{e^{i\alpha}}{n+1}\right) D^n f + \frac{e^{i\alpha}}{n+1} z (D^n f)'\right]' \\ &= \left(D^n f * \left[\left(1 - \frac{e^{i\alpha}}{n+1}\right)\frac{z}{(1-z)^{1-\beta}} + \frac{e^{i\alpha}}{n+1}\frac{z}{(1-z)^{2(1-\beta)}}\right]\right)' \\ &= \left(D^n f + \frac{e^{i\alpha} z + (n+1-e^{i\alpha}) z (1-z)^{1-\beta}}{(n+1)(1-z)^{2(1-\beta)}}\right)'.\end{aligned}$$

For F and G normalized by $F(0) = G(0) = F'(0)-1 = G'(0)-1 = 0$, we have that $(F*G)' = \frac{(F*zG')}{z}$. The result now follows upon taking $F = f, G(z) = \frac{e^{i\alpha}z + (n+1-e^{i\alpha})z(1-z)^{1-\beta}}{(n+1)(1-z)^{2(1-\beta)}}$, and noting that

$$zG'(z) = \frac{e^{i\alpha} z (1 + (1-2\beta)z) + (n+1-e^{i\alpha}) z (1-z)^{1-\beta}(1-\beta z)}{(n+1)(1-z)^{3-2\beta}}.$$

□

Proof of Theorem (2.12), *2.* We know that $f \in M_n(\alpha,\beta)$ if and only if $\operatorname{Re}\left\{(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)} z(D^n f)''\right] > \beta$. Furthermore, since $(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)} z(D^n f)'' = \frac{2n+1-e^{i\alpha}}{2(n+1)}(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)}(z(D^n f)')'$, it follows that

$$\begin{aligned}(D^n f)' &+ \frac{1+e^{i\alpha}}{2(n+1)} z(D^n f)'' \\ &= \left(D^n f * \left[\frac{2n+1-e^{i\alpha}}{2(n+1)}\frac{z}{(1-z)^{1-\beta}} + \frac{1+e^{i\alpha}}{2(n+1)}\frac{z}{(1-z)^{2(1-\beta)}}\right]\right)' \\ &= \left(D^n f * \left[\frac{(2n+1-e^{i\alpha})z(1-z)^{1-\beta} + (1+e^{i\alpha})z}{2(n+1)(1-z)^{2(1-\beta)}}\right]\right)'\end{aligned}$$

this is equal to

$$\frac{1}{z}\left\{D^n f * \left(\frac{(1+e^{i\alpha})z(1+(1-\beta)z)}{2(n+1)(1-z)^{3-2\beta}} + \frac{2n+1-e^{i\alpha})z(1-z)^{1-\beta}(1-\beta z)}{2(n+1)(1-z)^{3-2\beta}}\right)\right\},$$

and (2.12), 2 follows. □

Proof of Theorem (2.12), *3.* Let $f \in M_n(\alpha,\beta)$, then

$$\operatorname{Re}\left[(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)} z(D^n f)''\right] > \beta$$

or

$$\operatorname{Re}\left[\left(\frac{1+e^{i\alpha}}{2}\right) D^{n+1} f + \frac{1-e^{i\alpha}}{2} D^n f\right]' > \beta.$$

Since $\left[\frac{1+e^{i\alpha}}{2} D^{n+1} f + \frac{1-e^{i\alpha}}{2} D^n f\right]' = 1$ at the origin, we can write $f \in M_n(\alpha,\beta)$ if and only if

$$\frac{[(\frac{1+e^{i\alpha}}{2})D^{n+1} f + (\frac{1-e^{i\alpha}}{2})D^n f]' - \beta}{1-\beta} \neq \frac{x-1}{x+1}, \quad |x| = 1, z \in \Delta.$$

This is equivalent to

$$(1+x)\left[\left(\frac{1+e^{i\alpha}}{2}\right) D^{n+1} f + \left(\frac{1-e^{i\alpha}}{2}\right) D^n f\right]'(1-2\beta-x) \neq 0. \quad (12)$$

Writing $g(z) = z/(1-z)^{n+1}$, we observe that

$$
\begin{aligned}
z & \left[\left(\frac{1+e^{i\alpha}}{2} \right) D^{n+1} f + \left(\frac{1-e^{i\alpha}}{2} \right) D^n f \right]' \\
& = z \left(\left(\frac{1+e^{i\alpha}}{2} \right) \left(f * \frac{g}{1-z} \right) + \left(\frac{1-e^{i\alpha}}{2} \right) (f * g) \right)' \\
& = \frac{1+e^{i\alpha}}{2} \left(f * z \left(\frac{g}{1-z} \right)' \right) + \frac{1-e^{i\alpha}}{2} (f * zg').
\end{aligned}
$$

From this and (12), we conclude that $f \in M_n(\alpha, \beta)$ if and only if

$$
\begin{aligned}
\frac{1}{z} \Bigg[f * \Bigg\{ (1+x) \left(\frac{1+e^{i\alpha}}{2} \right) z \left(\frac{g}{1-z} \right)' + (1+x) \left(\frac{1-e^{i\alpha}}{2} \right) zg' \\
+ (1 - 2\beta - x) z \Bigg\} \Bigg] \neq 0,
\end{aligned}
$$

or if and only if

$$
\begin{aligned}
\frac{1}{z} \Bigg[f * \Bigg\{ \frac{(1+x)z[(1+nz)(1 - (\frac{1-e^{i\alpha}}{2})z) + (\frac{1+e^{i\alpha}}{2})z]}{(1-z)^{n+3}} \\
+ \frac{(1-2\beta-x)z(1-z)^{n+3}}{(1-z)^{n+3}} \Bigg\} \Bigg] \neq 0
\end{aligned}
$$

which implies the theorem. □

For the cases $n = 0, \beta = 0$, Theorem (2.12) 1-2, gives Theorem 3 obtained in [16]. And for $\alpha = \pi$, Theorem (2.12) 3, gives Theorem 2.6 obtained in [2].

THEOREM 2.13. *1. Let $0 \le \gamma < 1$. If $\beta \le \beta_0 = \frac{41+23\gamma}{64}$ and if $n \ge n_0 = \frac{15+\gamma-16\beta}{1-\gamma}$, then $M_n(\alpha,\beta) \subset K(\gamma)$, where $K(\gamma)$ is the well-known class of convex functions of order γ.*

2. $O_n(\alpha, \beta) \subset \bigcap_\alpha M_n(\alpha, \beta)$.

3. For each $\alpha, -\pi < \alpha \le \pi, \alpha \neq 0$, $M_n(\alpha,\beta) - K(\beta)$ is nonempty.

To prove this theorem we shall need the following lemma, which is due to Ahuja and Jahangiri [2].

LEMMA 2.14. *Let* $0 \le \gamma < 1$. *If* $\beta \le \beta_0 = \frac{41+23\gamma}{64}$ *and if* $n \ge n_0 = \frac{15+\gamma-16\beta}{1-\gamma}$, *then* $M_n(\beta) \subset K(\gamma)$.

Proof of Theorem (2.13), *1.* From (2.4) we found that $M_n(\alpha,\beta) \subset M_n(\pi,\beta)$, and using Lemma (2.14) [2], the result follows. □

Proof of Theorem (2.13), *2.* If $f \in O_n(\alpha,\beta)$, then

$$\operatorname{Re}\left\{(D^n f)' + \frac{e^{i\alpha}}{n+1} z (D^n f)''\right\} > \beta$$

for all $z \in \Delta$. Since $\left|\frac{1+e^{i\alpha}}{2(n+1)}\right| \le 1$ for all $-\pi < \alpha \le \pi$, it follows that $\operatorname{Re}\left\{(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)} z (D^n f)''\right\} > \beta$ for $z \in \Delta$, $-\pi < \alpha \le \pi$. We conclude that $f \in \bigcap_\alpha M_n(\alpha,\beta)$. □

Proof of Theorem (2.13), *3.* Consider the function

$$f_\alpha(z) = z + \frac{1-\beta}{3+e^{i\alpha}} z^2.$$

Since

$$(D^n f)' + \frac{1+e^{i\alpha}}{2(n+1)} z (D^n f)'' - \beta = \\ (1-\beta) + \frac{2(1-\beta)(n+1)}{3+e^{i\alpha}}\left[1 + \frac{1+e^{i\alpha}}{2(n+1)}\right] z,$$

$f_\alpha \in M_n(\alpha,\beta) - K(\beta)$. Because $\frac{1-\beta}{|3+e^{i\alpha}|} > \frac{1-\beta}{4}$ for $\alpha \ne 0$. □

The following theorem gives the necessary and sufficient condition for the integral operator $\frac{(n+1)}{z^n}\int_0^z t^{n-1} f(t)dt$ to be in $M_{n+1}(\alpha,\beta)$.

THEOREM 2.15. *et* $J : \mathcal{A} \to \mathcal{A}$ *be an integral operator defined by*

$$J_f(z) = \frac{(n+1)}{z^n}\int_0^z t^{n-1} f(t)dt. \tag{13}$$

Then $J_f(z) \in M_{n+1}(\alpha,\beta)$ *if and only if* $f(z) \in M_n(\alpha,\beta)$.

Proof. It is sufficient to show that

$$D^n f(z) = D^{n+1} J_f(z) \tag{14}$$

From (13) we get

$$D^n J_f(z) = \frac{n+1}{z^n} \int_0^z t^{n-1} D^n f(t) dt. \tag{15}$$

By differentiating (15) and using (5) we obtain (14). □

THEOREM 2.16. *$M_n(\alpha, \beta)$ is closed under convolution with convex functions.*

For proving this theorem we shall use the following lemma which is due to Ruscheweyh and Sheil-Small [14].

LEMMA 2.17. *If $\phi \in K(0)$ and if $g \in \mathcal{A}$ is starlike in Δ, then the function $(\phi * gF)/(\phi * g)$ takes values in the convex hull of $F(\Delta)$ for every function F in $\mathcal{A}$.*

Proof of Theorem (2.16). Let $g(z) = z$ and

$$F(z) = \left[\left(\frac{1+e^{i\alpha}}{2}\right) D^{n+1} f + \left(\frac{1-e^{i\alpha}}{2}\right) D^n f\right]'.$$

Then for $\phi \in K(0)$, we have

$$\begin{aligned} \frac{\phi * zF}{\phi * z} &= \frac{\phi * z\left[\left(\frac{1+e^{i\alpha}}{2}\right) D^{n+1} f + \left(\frac{1-e^{i\alpha}}{2}\right) D^n f\right]'}{z} \\ &= \left(\phi * \left[\left(\frac{1+e^{i\alpha}}{2}\right) D^{n+1} f + \frac{1-e^{i\alpha}}{2} D^n f\right]\right)' \\ &= \left(\left(\frac{1+e^{i\alpha}}{2}\right) D^{n+1}(\phi * f) + \left(\frac{1-e^{i\alpha}}{2}\right) D^n(\phi * f)\right)'. \end{aligned}$$

By (2.17), we conclude that

$$\left\{\left(\frac{1+e^{i\alpha}}{2}\right) D^{n+1}(\phi * f) + \left(\frac{1-e^{i\alpha}}{2}\right) D^n(\phi * f)\right\} \in M_0(\pi, \beta).$$

This means that $\phi * f \in M_n(\alpha, \beta)$. So the proof is complete. □

The last theorem is on the convolution of functions in $M_n(\alpha, \beta)$ with functions in $M_n(\pi, \beta)$. We shall use the following lemma, due to Fejér [5], to prove this theorem. A sequence $\{c_k\}_{k=0}^{\infty}$ of non-negative real numbers is said to be a convex null sequence if $c_k \to 0$ as $k \to \infty$, and $c_0 - c_1 \geq c_1 - c_2 \geq \cdots \geq c_{k-1} - c_k \geq \cdots \geq 0$.

LEMMA 2.18. *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function $p(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k$, $z \in \Delta$, is analytic and Re $p(z) > 0$ in Δ.*

THEOREM 2.19. *Let $f \in M_n(\pi, \beta)$ and $g \in M_n(\alpha, \beta)$. Then $f * g \in M_n(\alpha, \gamma)$ if*

$$\gamma = \frac{n(2\beta + 1) + 4\beta - 1}{2(n+1)} \geq \beta. \tag{16}$$

Proof. For $c_0 = 1$ and

$$c_k = \frac{n+1}{(k+1)\begin{pmatrix} k+n \\ n \end{pmatrix}}, \quad k \geq 1,$$

we see that $\{c_k\}_{k=0}^{\infty}$ is a convex null sequence. Therefore, by (2.18), we have

$$\text{Re}\left(1 + \sum_{k=2}^{\infty} \frac{n+1}{k\begin{pmatrix} k+n-1 \\ n \end{pmatrix}} z^{k-1}\right) > \frac{1}{2}. \tag{17}$$

Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be in $M_n(\alpha, \beta)$. By (11) we have

$$\text{Re}\left(1 + \sum_{k=2}^{\infty} k\left[1 + \frac{(k-1)(1+e^{i\alpha})}{2(n+1)}\right]\begin{pmatrix} k+n-1 \\ n \end{pmatrix} b_k z^{k-1}\right) > \beta. \tag{18}$$

Now we convolve (17) and (18) and apply (2.11) to obtain

$$\text{Re}\left(1 + \sum_{k=2}^{\infty}(n+1)\left[1 + \frac{(k-1)(1+e^{i\alpha})}{2(n+1)}\right] b_k z^{k-1}\right) > \beta$$

or

$$\text{Re}\left\{\frac{g}{z} + \frac{1+e^{i\alpha}}{2(n+1)}\left(\frac{zg' - g}{z}\right)\right\} > \frac{n+\beta}{n+1}$$

or

$$\operatorname{Re}\left\{\frac{g}{z}+\frac{1+e^{i\alpha}}{2(n+1)}\left(\frac{zg'-g}{z}\right)-\frac{2\beta+n-1}{2(n+1)}\right\}>\frac{1}{2}.$$

Since $\operatorname{Re}(D^n f)'>\beta$, we once again use (2.11) to obtain

$$\operatorname{Re}\left((D^n f)'*\left[\frac{g}{z}+\frac{1+e^{i\alpha}}{2(n+1)}\left(\frac{zg'-g}{z}\right)-\frac{2\beta+n-1}{2(n+1)}\right]\right)>\beta$$

or

$$\operatorname{Re}\left\{\left((D^n f)'*\frac{g}{z}\right)+\frac{1+e^{i\alpha}}{2(n+1)}\left((D^n f)'*\frac{zg'-g}{z}\right)\right\}$$
$$>\frac{n(2\beta+1)4\beta-1}{2(n+1)}=\gamma.$$

Using the fact that

$$\begin{aligned}(D^n(f*g))' &= (D^n f)'*(g(z)/z)\\ z\,(D^n(f*g))'' &= z\left((D^n f)'*(g(z)/z)\right)'\\ &= (D^n f)'*z\,(g(z)/z)'\end{aligned}$$

conclude the theorem. □

For $\alpha=\pi$, (2.19) gives the corresponding result in Theorem 4 by Ahuja and Jahangiri [1].

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