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On the Symmetry Group of a Differential Equation and the Liouville-Gelfand Problem

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Summary. - *The Lie point symmetry group of the Liouville-Gelfand Problem for quasilinear differential equations is calculated. The corresponding Nöther symmetries are found and used to obtain first integrals and explicit solutions.*

1. Introduction

Let Ω be the ball in $\mathbb{R}^N$ with centre at the origin and radius R. The problem of finding solutions of

$$\begin{cases} -\Delta u = \lambda\, e^{u} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u > 0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where λ is a real parameter, is called the Liouville-Gelfand Problem. It has been studied by Liouville [7] who found an explicit solution in dimension $N = 1$, and also (in the case $N = 2$) a general solution

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of the equation in (1) in terms of an arbitrary harmonic function. Further Bratu [1] found two explicit solutions of (1) if $0 < \lambda < 2/R^2$ and $N = 2$. The next substantial result concerning (1) was obtained by Gelfand in [3] who considered, among other things, the problem of thermal self-ignition of a chemically active mixture of gases in plane, cylindrical and spherical vessels. In particular for $N = 3$ he investigated the values of λ for which the problem has a solution and studied the multiplicity of such solutions.

It can be easily seen that if $\lambda \leq 0$ or if λ is greater than a certain positive constant λ^* there is no solution of the boundary value problem (1). Moreover, by the well known results of Gidas, Ni and Nirenberg [4] it follows that if a solution exists, then it must be radially symmetric. Thus (1) can be reduced to the study of the following problem for an ordinary differential equation:

$$\begin{cases} -y'' - \frac{N-1}{x} y' = \lambda\, e^{y} & \text{in} \quad (0, R), \\ y'(0) = y(R) = 0, & \\ y > 0 & \text{in} \quad [0, R), \end{cases} \tag{2}$$

where $x \geq 0$ is the Euclidean norm of a vector in $\mathbb{R}^N$ and $y = y(x)$ is a radial function in Ω. In this way the problem becomes more tractable. This suggests one of the possible generalizations of the Liouville-Gelfand Problem. Namely, there is a general class of radial quasilinear operators given by

$$Ly = -(x^{\alpha}|y'|^{\beta} y')' x^{-\gamma}, \tag{3}$$

considered in [2]. In the definition (3) the parameters α, β and γ are real numbers satisfying some relations which will be specified below. One can observe that this class contains the following operators acting on radial functions defined in Ω:

1.) Laplace operator if $\alpha = \gamma = N - 1, \beta = 0$;
2.) $p-$Laplace operator if $\alpha = \gamma = N - 1, \beta = p - 2$;
3.) $k-$Hessian operator if $\alpha = N - k, \gamma = N - 1, \beta = k - 1$.

In [2] Clément, De Figueiredo and Mitidieri proposed the following generalization of (2) in the terms of L:

$$\begin{cases} -(x^{\alpha}|y'|^{\beta}y')' = \lambda\, x^{\gamma}e^{y} & \text{in} \quad (0,R), \\ y'(0) = y(R) = 0, & \\ y > 0 & \text{in} \quad [0,R), \end{cases} \tag{4}$$

where

$$\alpha - \beta - 1 = 0, \quad \beta > -1, \quad \gamma > -1. \tag{5}$$

They found the above mentioned constant λ^*, proved that there exists only one solution of (4) if $\lambda = \lambda^*$, and that there are exactly two solutions if $0 < \lambda < \lambda^*$. Moreover, using the method of first integrals, they found the explicit formulas for these solutions.

The relation $\alpha = \beta + 1$ was essentially used in [2]. This condition corresponds to the so-called Pohozaev-Trudinger case and is connected with a loss of compactness of exponential type of certain embeddings. See [2]. In this regard the following questions cannot be avoided: What is the relationship between the condition $\alpha = \beta + 1$ and the nature of the problem (4)? In what context it appears? Why it was possible to find the solutions explicitly?

The aim of the present paper is to answer these questions. For this purpose we shall enlighten the problem from the point of view of the *Lie Symmetry Theory* of differential equations [8]. Actually our strategy is very simple: we just calculate the Lie point symmetries of (4) - Theorems 2.1 and 3.1. Further we look for those which are Noether symmetries. We prove our main result, Theorem 4.2, which states that a Lie point symmetry of the problem (4) is a Noether symmetry if and only if $\alpha = \beta + 1$. Then with the help of the Noether symmetry we find one of the first integrals used in [2], from which we immediately derive the solution(s), Theorem 4.3, without use of other first integrals. In this way we show that the explicit solutions can be accounted for by symmetry techniques which is the goal of this paper.

In sections 2 and 3 we calculate the Lie point symmetries of the equation in (4) with $\beta \neq 0$ and $\beta = 0$ correspondingly. In section 4 we study the Noether symmetries, then we find a first integral and hence the exact solution(s). We also comment on first integrals. In section 5 we briefly discuss the case $N = 2$.

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2. The Lie point symmetries: $\beta \neq 0$

In this section we calculate the Lie point symmetries of (4) supposing that

$$\beta > -1, \quad \beta \neq 0, \quad \gamma > -1. \tag{6}$$

Recall that a Lie point symmetry is determined by its infinitesimal generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

Thus we have to find the functions ξ and η.

To begin with, we write the equation in (4) in the form

$$y'' = -\frac{\alpha}{\beta+1}\frac{y'}{x} - \frac{\lambda}{\beta+1}x^{\gamma-\alpha}|y'|^{-\beta}e^{y} =: f. \tag{7}$$

Note that the solution of (4), if it exists, must satisfy $y' < 0$ in $(0, R)$ ([2]).

Then we substitute

$$f_x = \frac{\alpha}{\beta+1}\,\frac{y'}{x^2} + \frac{\lambda(\alpha-\gamma)}{\beta+1}\,x^{\gamma-\alpha-1}|y'|^{-\beta}e^{y},$$

$$f_y = -\frac{\lambda}{\beta+1}\,x^{\gamma-\alpha}|y'|^{-\beta}e^{y},$$

$$f_{y'} = -\frac{\alpha}{\beta+1}\frac{1}{x} - \frac{\lambda\beta}{\beta+1}x^{\gamma-\alpha}|y'|^{-\beta-1}e^{y}$$

into the identity (4.11), [9], p. 28, which determines the Lie point symmetries of a second order differential equation. We obtain

$$\frac{\alpha}{\beta+1}\,\frac{\eta_x}{x} + \eta_{xx} + y'\,[\frac{\alpha}{\beta+1}\,\frac{\xi_x}{x} - \frac{\alpha}{\beta+1}\,\frac{\xi}{x^2} + 2\eta_{xy} - \xi_{xx}]$$

$$+y'^2 \, [\frac{2\alpha}{\beta+1} \, \frac{\xi_y}{x} + \eta_{yy} - 2\xi_{xy}] - y'^3 \, \xi_{yy} + |y'|^{-\beta-1} \frac{\lambda\beta}{\beta+1} \, x^{\gamma-\alpha} e^y \eta_x$$

$$+|y'|^{-\beta}[-\lambda x^{\gamma-\alpha} e^y \eta_y + \frac{\lambda(\beta+2)}{\beta+1} x^{\gamma-\alpha} e^y \xi_x + \frac{\lambda(\gamma-\alpha)}{\beta+1} x^{\gamma-\alpha-1} e^y \xi$$

$$+\frac{\lambda}{\beta+1} x^{\gamma-\alpha} e^y \eta] - |y'|^{-\beta+1} \frac{\lambda(\beta+3)}{\beta+1} x^{\gamma-\alpha} e^y \xi_y = 0. \tag{8}$$

Since η and ξ do not depend on y' we shall equate to zero the coefficients of the powers of y'.

Since $\beta > -1$, $\beta \neq 0$ and $\lambda \neq 0$, equating to zero the coefficient of $|y'|^{-\beta-1}$ implies that

$$\eta_x = 0. \tag{9}$$

Therefore η is a function only of y and we can write

$$\eta = b(y). \tag{10}$$

Now we must be more careful. Since $\beta > -1$ and $\beta \neq 0$ it follows that $-\beta+1 \neq 3$, $-\beta+1 \neq 2$ and $-\beta+1 \neq 1$. However $-\beta+1$ may be 0. Therefore we must consider two cases: $\beta \neq 1$ and $\beta = 1$.

(i) $\beta \neq 1$. Hence $-\beta+1 \neq 0$. Since $\lambda \neq 0$ and $\beta+3 \neq 0$ equating to 0 the coefficient of $|y'|^{-\beta+1}$ gives that

$$\xi_y = 0, \tag{11}$$

that is, ξ is a function of x only:

$$\xi = a(x). \tag{12}$$

(ii) Suppose that $\beta = 1$. Therefore the free term in the identity (8) becomes

$$\frac{\alpha}{2} \, \frac{\eta_x}{x} + \eta_{xx} - 2\lambda x^{\gamma-\alpha} e^y \xi_y = -2\lambda x^{\gamma-\alpha} e^y \xi_y$$

since η is already a function of y only (see (10)). The last expression must vanish. Hence

$$\xi_y = 0$$

since $\lambda \neq 0$. Therefore (12) is again valid.

Further, by (10) and (12) the coefficient of y'^2 becomes η_{yy} and its vanishing implies that

$$\eta = b(y) = Ay + B, \tag{13}$$

where A and B are constants. Now substituting (12) and (13) into the coefficient of $|y'|^{-\beta}$, which also must vanish, implies that

$$A = 0.$$

Therefore

$$\eta = B = const. \tag{14}$$

and

$$(\beta + 2)a' + (\gamma - \alpha)\frac{a}{x} + B = 0. \tag{15}$$

Finally, by vanishing of the coefficient of y' we obtain

$$x^2 a'' - \frac{\alpha}{\beta + 1} x a' + \frac{\alpha}{\beta + 1} a = 0. \tag{16}$$

The equation (16) for a is an Euler Equation. Its indicial equation

$$r(r-1) - \frac{\alpha}{\beta + 1} r + \frac{\alpha}{\beta + 1} = 0 \tag{17}$$

has two rooths:

$$r_1 = 1$$

and

$$r_2 = \frac{\alpha}{\beta + 1}.$$

We have to consider various cases.

I.) $r_1 = r_2 = 1$. This condition means that the indicial equation (17) has a double rooth if and only if $\alpha = \beta + 1$. It is the first place where the condition $\alpha - \beta - 1 = 0$ appears!

The solution of (16) has the form

$$a(x) = c_1 x + c_2 x \ln x. \tag{18}$$

Then substituting $a(x)$ from (18) into (15) we obtain:

$$(\beta + 2)(c_1 + c_2) + (\gamma - \alpha)c_1 + B + c_2(\beta + 2 + \gamma - \alpha)\ln x = 0.$$

But $\beta+2+\gamma-\alpha=\alpha+1+\gamma-\alpha=1+\gamma>0$. Therefore $c_2=0$ and hence

$$c_1=-\frac{B}{\gamma-\alpha+\beta+2}=-\frac{B}{\gamma+1}.$$

Thus

$$\begin{cases}\xi = -\dfrac{B}{\gamma+1}\,x,\\ \\ \eta = B.\end{cases}$$

II.) $r_1\neq r_2$, that is $\alpha\neq\beta+1$. In this case

$$a(x)=c_1x+c_2x^{r_2}. \tag{19}$$

II.1.) Suppose that $\gamma-\alpha+\beta+2=0$. Then the general solution of (15) is given by

$$a(x)=cx-\frac{B}{\beta+2}\,x\ \ln x. \tag{20}$$

Comparing (19) and (20) we conclude that

$$\begin{cases}\xi = cx,\\ \eta = 0\end{cases}$$

since $r_2\neq 1$.

II.2.) Let $\gamma-\alpha+\beta+2\neq 0$. In this case the general solution of (15) is

$$a(x)=cx^{-\dfrac{\gamma-\alpha}{\beta+2}}-\frac{B}{\gamma-\alpha+\beta+2}\,x.$$

Hence and by (19) we have

$$cx^{-\dfrac{\gamma-\alpha}{\beta+2}}-\frac{B}{\gamma-\alpha+\beta+2}\,x=a(x)=c_1x+c_2x^{\dfrac{\alpha}{\beta+1}}. \tag{21}$$

II.2.1) Let $\alpha+\gamma+\beta\gamma=0$ in addition to II.2). Then by (21) we have that

$$\begin{cases}\xi = -\dfrac{B}{\gamma-\alpha+\beta+2}\,x+c_2x^{\dfrac{\alpha}{\beta+1}},\\ \\ \eta = B.\end{cases}$$

II.2.2) If $\alpha+\gamma+\beta\gamma\neq 0$ and $\gamma-\alpha+\beta+2\neq 0$, then

$$\begin{cases} \xi = -\dfrac{B}{\gamma-\alpha+\beta+2}\, x, \\ \eta = B. \end{cases}$$

Summarizing we have proved the following

THEOREM 2.1. *Let $\beta\neq 0$. Then the Lie point symmetry group of (4) is generated by*

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y},$$

where

I.) if $\alpha=\beta+1$ then

$$\begin{cases} \xi = -\dfrac{c}{\gamma+1}\, x, \\ \eta = c; \end{cases} \tag{22}$$

II.1.) if $\alpha\neq\beta+1$ and $\gamma-\alpha+\beta+2=0$, then

$$\begin{cases} \xi = cx, \\ \eta = 0; \end{cases} \tag{23}$$

II.2.1) if $\alpha\neq\beta+1$, $\gamma-\alpha+\beta+2\neq 0$ and $\alpha+\gamma+\beta\gamma=0$, then

$$\begin{cases} \xi = -\dfrac{c}{\gamma-\alpha+\beta+2}\, x + Cx^{\frac{\alpha}{\beta+1}}, \\ \eta = c; \end{cases} \tag{24}$$

II.2.2) if $\alpha\neq\beta+1$, $\gamma-\alpha+\beta+2\neq 0$ and $\alpha+\gamma+\beta\gamma\neq 0$, then

$$\begin{cases} \xi = -\dfrac{c}{\gamma-\alpha+\beta+2}\, x, \\ \eta = c. \end{cases} \tag{25}$$

Here c and C are arbitrary constants.

3. The Lie point symmetries: $\beta = 0$

In the previous section we calculated the symmetries supposing that $\beta \neq 0$. We have essentially used this condition - namely, we used it in concluding that $\eta_x = 0$. We assume throughout this section that

$$\beta = 0, \quad \gamma > -1. \tag{26}$$

In order to calculate the Lie point symmetries we shall proceed as before. We write the equation as

$$y'' = -\frac{\alpha}{x}\, y' - \lambda x^{\gamma-\alpha} e^{y}. \tag{27}$$

Then we substitute the corresponding derivatives of the right-hand side of (27) into the identity (4.11), [2], p. 28. We obtain in this way the following identity:

$$(-\frac{\alpha}{x}\, y' - \lambda x^{\gamma-\alpha} e^{y})(\eta_y - 2\xi_x - 3y'\xi_y) - \frac{\alpha}{x^2}\xi y' + (\gamma-\alpha)\lambda x^{\gamma-\alpha-1}\xi e^{y}$$

$$+\lambda x^{\gamma-\alpha}\eta e^{y} + \frac{\alpha}{x}[\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2]$$

$$+\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} = 0. \tag{28}$$

Equating to zero the ξ_{yy} - the coefficient of y'^3, implies

$$\xi = a(x) + p(x)\, y. \tag{29}$$

Then the coefficient of y'^2

$$\eta_{yy} - 2\xi_{xy} + \frac{2}{x}\alpha\, \xi_y = 0. \tag{30}$$

Substituting (29) into (30) we get that

$$\eta = (p' - p\alpha/x)\, y^2 + b(x)\, y + c(x). \tag{31}$$

Therefore ξ and η are polynomials of y.

Further we substitute ξ and η, given by (29) and (31) respectively, into the coefficient of y'. By equating to zero we obtain

$$[3p'' - 4(p\alpha/x)' - \frac{\alpha}{x^2}p + \frac{\alpha}{x}p']\, y + 2b' - a'' - \frac{\alpha}{x^2}a + \frac{\alpha}{x}a' + 3p\,\lambda\, x^{\gamma-\alpha}e^{y} = 0.$$

Hence $p = 0$ and

$$2b' - a'' - \frac{\alpha}{x^2}a + \frac{\alpha}{x}a' = 0.$$

Similarly the vanishing of the free term implies that $b = 0$. Thus

$$\xi = a(x), \quad \eta = c(x),$$

where the functions $a(x)$ and $c(x)$ satisfy the following equations:

$$x^2a'' - \alpha\, x\, a' + \alpha\, a = 0, \tag{32}$$

$$c'' + \frac{\alpha}{x}\, c' = 0, \tag{33}$$

$$a' + \frac{\gamma - \alpha}{2}\, \frac{1}{x}\, a = -\frac{c}{2}. \tag{34}$$

With regard to the homogeneous equation (33) there are two cases:

I.) $\alpha = 1$; then $c(x) = c_1 \ln x + c_2$;

II.) $\alpha \neq 1$; then $c(x) = c_1\, x^{1-\alpha} + c_2$, where c_1 and c_2 are arbitrary constants.

In regard to the Euler equation (32) we write its indicial equation

$$r(r-1) - \alpha r + \alpha = 0,$$

which has the rooths

$$r_1 = 1$$

and

$$r_2 = \alpha.$$

Again we have two cases:

(i) $\alpha = 1$, that is, $r = 1$ is a double rooth; then $a(x) = k_1 x + k_2 x \ln x$;

(ii) $\alpha \neq 1$, then $a(x) = k_1 x + k_2 x^{\alpha}$, where k_1 and k_2 are arbitrary constants.

Then combining (i)-I.) and(ii)-II.) and taking into account the fact that the functions a and c must satisfy the equation (34) we conclude that the following theorem holds.

THEOREM 3.1. *Let $\beta = 0$. Then the Lie point symmetry group of (4) is generated by*

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y},$$

where

I.) if $\alpha = 1$ then

$$\begin{cases} \xi = -\dfrac{c_1}{\gamma+1}\, x + \dfrac{c_2}{\gamma+1}\left(\dfrac{2}{\gamma+1}\, x - x\ln x\right), \\ \\ \eta = c_1 + c_2 \ln x; \end{cases} \tag{35}$$

II.1.) if $\alpha \neq 1$ and $\gamma \neq -\alpha$, then

$$\begin{cases} \xi = -\dfrac{c}{\gamma-\alpha+2}\, x, \\ \eta = c; \end{cases} \tag{36}$$

II.2.) if $\alpha \neq 1$ and $\gamma = -\alpha$, then

$$\begin{cases} \xi = -\dfrac{c_1}{2(\alpha-1)}\, x + c_2 x^{\alpha}, \\ \\ \eta = c_1. \end{cases} \tag{37}$$

Here c, c_1 and c_2 are arbitrary constants.

For the Laplace operator we have the following corollaries.

COROLLARY 3.2. *Let $\alpha = \gamma = N - 1$ and $N \geq 3$. Then*

$$\begin{cases} \xi = -\dfrac{c}{2}\, x, \\ \eta = c. \end{cases} \tag{38}$$

COROLLARY 3.3. *Let $N = 2$ and $\alpha = \gamma = 1$. Then*

$$\begin{cases} \xi = -\dfrac{c_1}{2}\, x + \dfrac{c_2}{2}(x - x\ln x), \\ \eta = c_1 + c_2 \ln x. \end{cases} \tag{39}$$

4. The Noether Symmetries

In this section we study the Noether symmetries of our problem (4). We first observe that it has a variational structure. Indeed the following lemma holds.

LEMMA 4.1. *The equation in* (4) *is the Euler-Lagrange equation of the functional*

$$J[y]=\int_0^R L(x,y,y')\,dx,$$

where the function of Lagrange $L=L(x,y,y')$ *is given by*

$$L(x,y,y')=\frac{1}{\beta+2}x^{\alpha}|y'|^{\beta+2}-\lambda\,x^{\gamma}e^{y}.$$

The proof of this lemma is by a straightforward calculation.

Now let us consider the cases $I.$) and $II.2.2$). The corresponding formulas (22) and (25) can be written as one:

$$\begin{cases}\xi &= -\dfrac{c}{k}\,x,\\[2ex] \eta &= c,\end{cases}\tag{40}$$

where $k=\gamma-\alpha+\beta+2$ in both cases since, obviously, for $I.$) we have $\alpha=\beta+1$ and hence $k=\gamma+1$, which appears in (22). Then the generator of such a symmetry

$$X=-\frac{x}{k}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$$

has the extension

$$\hat{X}=-\frac{x}{k}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{y'}{k}\frac{\partial}{\partial y'}.$$

We are going to verify the relation

$$\hat{X}L+(A\xi)L=A(V(x,y)),$$

where

$$A=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+f\frac{\partial}{\partial y'}$$

in order to check whether or not $\hat{X}$ determines a Noether symmetry (see [9], p. 98, (10.27)) . We calculate:

$$\hat{X}L = -\frac{1}{k}\left(1-\frac{\alpha}{\beta+2}\right)x^{\alpha}|y'|^{\beta+2} - \lambda(1-\gamma/k)x^{\gamma}e^{y}$$

and

$$(A\xi)L = -\frac{1}{k(\beta+2)}x^{\alpha}|y'|^{\beta+2} + \frac{1}{k}\lambda x^{\gamma}e^{y}.$$

Hence

$$\hat{X}L + (A\xi)L = \frac{1}{k}\left(1-\frac{\alpha}{\beta+2}-\frac{1}{\beta+2}\right)x^{\alpha}|y'|^{\beta+2}$$
$$-\lambda\left(1-\frac{\gamma}{k}-\frac{1}{k}\right)x^{\gamma}e^{y}.$$

Therefore $\hat{X}L + (A\xi)L$ is a total derivative of a function only of x and y if and only if

$$1-\frac{\alpha}{\beta+2}-\frac{1}{\beta+2} = \frac{\beta+1-\alpha}{\beta+2} = 0.$$

We thus conclude that $\hat{X}$ is a Noether symmetry if and only if $\alpha = \beta+1$. The other cases in the Theorem 2.1 can be treated in a similar way. This argument applies also to the cases in the Theorem 3.1, namely, if $\beta = 0$ then a Lie point symmetry is a Noether symmetry if and only if $\alpha = 1$ and $c_2 = 0$. That is, again $\alpha = \beta+1$ holds. The following theorem summarizes these considerations.

THEOREM 4.2. *A Lie point symmetry of the problem* (4) *is a Noether symmetry if and only if* $\alpha = \beta + 1$.

Actually the Noether symmetry is variational since if $\alpha = \beta+1$ then $k = \gamma + 1$ and

$$1-\frac{\gamma}{k}-\frac{1}{k} = 0,$$

which implies that $\hat{X}L + (A\xi)L = 0$. Henceforth we shall suppose $\alpha = \beta + 1$.

Now we can apply the Noether Theorem ([9], p. 99) which provides the existence of a first integral of the form

$$\phi = \xi(y'\frac{\partial L}{\partial y'} - L) - \eta\frac{\partial L}{\partial y'}$$

such that $\hat{X}\phi = 0$. Using $\alpha = \beta + 1$ and $|y'| = -y'$ ([2]), we obtain the following first integral

$$\phi(x, y, y') = -\frac{\alpha}{(\alpha+1)(\gamma+1)}x^{\alpha+1}|y'|^{\alpha+1}$$
$$-\frac{\lambda}{\gamma+1}x^{\gamma+1}e^{y} + x^{\alpha}|y'|^{\alpha} = const \tag{41}$$

whenever y is a solution of the equation (4). Since $\alpha = \beta + 1 > 0$, $\alpha+1 > 0$ and $\gamma+1 > 0$ by continuity it follows that $\phi = 0$. Therefore

$$\psi(x, y, y') := \alpha x^{\alpha+1}|y'|^{\alpha+1} - (\alpha+1)(\gamma+1)x^{\alpha}|y'|^{\alpha}$$
$$+\lambda(\alpha+1)x^{\gamma+1}e^{y} = 0. \tag{42}$$

Taking a look at [2] one can observe that the function ψ above is the same which appears in [2]. Now we express y from (42) as a function of x and y' and then substitute it into the equation (7). In this way we obtain that y' must satisfy the following Bernoulli Equation

$$y'' = \frac{(\gamma - \alpha + 1)}{\alpha}\frac{1}{x}y' + \frac{1}{\alpha+1}y'^2$$

which can be easily solved. Thus

$$y(x) = -(\alpha+1)\ln\left|c_1 - \frac{\alpha}{(\alpha+1)(\gamma+1)}x^{(\gamma+1)/\alpha}\right| + c_2.$$

By the boundary condition $y(R) = 0$ we obtain that

$$c_1 = -\frac{(\gamma+1)^{1/\alpha}}{\mu_i(\lambda\mu_i)^{1/\alpha}}$$

and

$$c_2 = (\alpha+1)\ln\frac{(\gamma+1)^{1/\alpha}}{(\lambda\mu_i)^{1/\alpha}},$$

where μ_i, $i = 1, 2$, is rooth of

$$H(\mu) := m\mu^{(\alpha+1)/\alpha} - \mu + 1 = 0, \quad m = \frac{\alpha\lambda^{1/\alpha}R^{(\gamma+1)/\alpha}}{(\alpha+1)(\gamma+1)^{(\alpha+1)/\alpha}}.$$

THEOREM 4.3. *The solution(s) of* (4) *can be represented in the following way:*

1.) *if* $0 < \lambda < \lambda^* := \dfrac{(\gamma+1)^{\alpha+1}}{(\alpha+1)R^{\gamma+1}}$ *then there are two solutions*

$$y_{1,2}(x) = -(\alpha+1)\ln\left(\frac{1}{\mu_{1,2}} + \frac{\alpha(\lambda\mu_{1,2})^{1/\alpha}}{(\alpha+1)(\gamma+1)^{(\alpha+1)/\alpha}}\, x^{(\gamma+1)/\alpha}\right);$$

2.) *if* $\lambda = \lambda^*$ *then there exists a unique solution*

$$y_0(x) = -(\alpha+1)\ln\left(\frac{1}{\mu_0} + \frac{\alpha{\mu_0}^{1/\alpha}}{(\alpha+1)^{(\alpha+1)/\alpha}R^{(\gamma+1)/\alpha}}\, x^{(\gamma+1)/\alpha}\right),$$

where μ_0 *is the unique solution of* $H(\mu) = 0$*;*

3.) *if* $\lambda \leq 0$ *or* $\lambda > \lambda^*$ *there is no solution.*

See [2] for further details. We conclude this section with the following observations.

(i) We have obtained the solutions found in [2] just by applying symmetry and variational methods. We did not need to use another first integral

$$\varphi := x^{(\sigma-\gamma-1)/\alpha}\frac{d}{dx}\left(e^{-y(x)/(\alpha+1)}\right).$$

See [2]. In fact, this φ corresponds to a *dynamical* symmetry.

(ii) Since our problem possesses a Lagrangian (Lemma 1) one can try to find out a first integral of the form

$$\frac{1}{\beta+2}A(x,y)|y'|^{\beta+2} + B(x,y)|y'|^{\beta+1} + C(x,y)y' + D(x,y)$$

using the method suggested in [9], p. 118. A straightforward calculation gives the following result: $C(x,y) = 0$ and

(a) if $\alpha = \beta + 1$ then the obtained first integral is ψ given by (42)(of course!);

(b) if $\alpha \neq \beta + 1$ and $\alpha + \gamma + \beta\gamma = 0$ then

$$\frac{1}{\beta+2}x^{\alpha-\gamma}|y'|^{\beta+2} + \lambda e^{y} = c_3 = const$$

is a first integral for the case II.2.1), which can be integrated directly. This and the other cases for which $\alpha \neq \beta+1$ will be treated elsewhere in more details.

(iii) If $\alpha = \beta + 1$ the symmetry assumes the form (40). Then by a standard procedure (see [8, 9]) one finds the following change of variables:

$$\begin{cases} y = v + cs, \\ x = e^{-cs/(\gamma+1)}. \end{cases} \tag{43}$$

Since $\hat{X}\psi = 0$ the equation (42) admits the (40) as a symmetry ([9], p. 100). Therefore we can use the change (43) in (42), which assumes the form

$$\alpha|v' + c|^{\alpha+1} - (\alpha+1)c|v' + c|^{\alpha} + \frac{\lambda(\alpha+1)c^{\alpha+1}}{(\gamma+1)^{\alpha+1}} e^{v} = 0, \tag{44}$$

where $v = v(s)$ and $v' = \frac{dv}{ds}$. If $N = 2$, $\beta = 0$ and $\alpha = \gamma = 1$ the equation (44) can be easily handled. This will be done in the next section.

5. The case $N = 2$

The group classification of linear second-order differential equations has been carried out in [5] in the case $N = 2$. The partial differential equations of the form $u_{x_1 x_2} = f(u)$ have been classified in [6].

For sake of completeness we present in this section the details for the two dimensional Liouville-Gelfand Problem. In particular we recover the solutions found in [1]. See also [2]. In fact, this is a *third* way to obtain the Bratu solution(s) - a way based on the *Lie Theory*.

Substituting $\alpha = \gamma = 1$ in (44) we obtain

$$(v' + c)^2 - 2c(v' + c) + \frac{\lambda c^2}{2} e^{v} = 0 \tag{45}$$

and hence

$$\frac{dv}{ds} = \sqrt{c^2 - \frac{\lambda c^2}{2} e^{v}}. \tag{46}$$

The equation (46) can also be obtained in the following way. Making the change of variables (43) with $\gamma = 1$, the Liouville equation

$$y'' + \frac{1}{x} y' = -\lambda e^{y}$$

becomes

$$v'' = -\frac{\lambda c^2}{4} e^v \tag{47}$$

from which we derive (46).

The equation (46) can be solved applying some standard calculus techniques of integration. We obtain

$$\ln \frac{c - \sqrt{c^2 - \frac{\lambda c^2}{2} e^v}}{c + \sqrt{c^2 - \frac{\lambda c^2}{2} e^v}} = c\, s + C, \tag{48}$$

where C is an arbitrary constant and we have supposed that $c > 0$. After some manipulations with (48), which we do not present here in order not to increase the volume of this paper, and going back to the variables x and y we finally obtain that

$$\lambda e^y = \frac{8\mu}{(\mu + x^2)^2}, \tag{49}$$

where μ is an arbitrary positive constant.

Then (49) and the boundary condition $y(R) = 0$ imply that necessarily

$$\lambda \mu^2 + 2(\lambda R^2 - 4)\mu + \lambda R^4 = 0. \tag{50}$$

The solution of the quadratic equation (50) is given by

$$\mu_{1,2} = (4 - \lambda R^2 \pm \sqrt{(\lambda R^2 - 4)^2 - \lambda^2 R^4})/\lambda. \tag{51}$$

The discriminant in (51) is $16 - 8\lambda R^2$. Thus:

(i) if $0 < \lambda < \lambda^* = 2/R^2$ there are two solutions;

(ii) if $\lambda = \lambda^*$ there is only one solution;

(iii) if $\lambda > \lambda^*$ there is no solution.

In the first case, by (49) and (51), the solutions to the Liouville-Gelfand Problem in dimension two are:

$$y_{1,2}(x) = \ln\,(8\mu_{1,2}/\lambda) - 2\ln\,(\mu_{1,2} + x^2).$$

If $\lambda = 2/R^2$ then

$$y_0(x) = -2\ln\left(\frac{1}{2} + \frac{x^2}{2R^2}\right)$$

is the unique solution.

In conclusion we would like to notice that it would be interesting to compare these solutions with some of the known numerical solutions of the Liouville-Gelfand Problem.

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