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Asymptoptic Behaviour of Sobolev Constants for Thin Curved Rods or Pipes

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Summary. - *We study the Sobolev imbedding inequality in a curved rod or pipe with a smooth central curve γ. Using the variational approach and the two-scale convergence for thin domains we find the limit of the Sobolev imbedding constant $W^{1,r} \hookrightarrow L^q$ as ε, the ratio between cross section diameter and the length of the rod, tends to* 0.

1. Introduction

The study of thin (or long) curved rods or pipes has been subject of many papers due to their various applications (see e.g. [1], [2], [4], [6]). Since the geometry of such objects is rather complicated the main goal of most of those papers was to find the appropriate 1-dimensional models. For that purpose, the asymptotic analysis with respect to the small parameter ε, the ration between the width and the length of the rod, was used. To establish the asymptotic behaviour of the solution of some boundary value problem in such domain it is very important to know the exact asymptotic behaviour of the corresponding Sobolev imbedding constants for such domains. In the present paper we prove that the imbedding constant $C^{\varepsilon}(r,q)$ for $W^{1,r} \hookrightarrow L^q$ with $1 \leq q < r^* = \frac{3r}{3-r}$, $r < 3$ on the curved rod P_ε , multiplied by $\varepsilon^{2/r - 2/q}$, converges to the Sobolev's imbedding

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constant $C^0(r,q)$ for $W^{1,r}_{|S|}(0,\ell) \hookrightarrow L^q_{|S|}(0,\ell)$ [1], where $|S|$ stands for spaces defined with respect to the weighted measure $|S(t)|dt$, $S(t)$ is the cross section of the rod at level t and $|S(t)|$ is its area. From that result we can conclude that $C^\varepsilon(r,q) = \varepsilon^{2/q-2/r}\,(C^0(r,q) + o(\varepsilon))$.

2. The Geometry

We suppose that γ, the central curve of the rod, is a smooth generic curve of class C^4. For simplicity we assume that γ is parameterized by its arc length $y_1 \in [0,\ell]$. We denote by $\mathbf{x} : [0,\ell] \to \mathbf{R}^3$ its natural parameterization. In each point $\mathbf{x}(y_1)$, $y_1 \in [0,\ell]$ of the curve γ we define its curvature as $\kappa(y_1) = |\mathbf{x}''(y_1)|$ and the Frenet's basis, $\mathbf{t}(y_1) = \mathbf{x}'(y_1)$ (the tangent), $\mathbf{n}(y_1) = \frac{\mathbf{x}''(y_1)}{\kappa(y_1)}$ [2] (the normal) and $\mathbf{b}(y_1) = \mathbf{t}(y_1) \times \mathbf{n}(y_1)$ (the binormal) . As usual, we denote by $\tau(y_1) = -|\mathbf{b}'(y_1)|$ the torsion.
For a smooth 1-parameter family of subsets $S(\alpha) \subset \mathbf{R}^2$ (the cross section at point $\mathbf{x}(\alpha)$) and a small parameter $0 < \varepsilon \ll 1$ we define a straight rod as

$$\Omega_\varepsilon = \{y = (y_1,y_2,y_3) \in \mathbf{R}^3 \; ; \; y_1 \in \; [0,\ell] \; , \; (y_2,y_3) \in \; \varepsilon S(y_1) \; \}$$

such that, denoting $z_\alpha = \frac{y_\alpha}{\varepsilon}$, $\alpha = 2,3$, the standard assumptions

$$\int_{S(y_1)} z_2\, dz_2\, dz_3 = \int_{S(y_1)} z_3\, dz_2\, dz_3 = 0$$

hold (i.e. that the origin 0 is placed in the centroid of $S(y_1)$). We assume that Ω_ε is locally Lipschitz. We define the mapping $\Phi : \Omega_\varepsilon \to \mathbf{R}^3$ by

$$\Phi(y) = \mathbf{x}(y_1) + y_2\mathbf{n}(y_1) + y_3\mathbf{b}(y_1) \;\; .$$

In order to have the local injectivity of Φ, we suppose that ε is sufficiently small, more precisely, we assume that

$$\varepsilon\,|\kappa(\alpha)|\;\mathrm{diam}\, S(\alpha) < 1 \; , \;\; \alpha \in [0,\ell] \; , \qquad (1)$$

Indeed, it is easy to see (see e.g. [2] or [6]) that $\det \nabla\Phi = 1 - y_2\kappa(y_1)$ so that the supposition (1) assures the local injectivity of Φ. We are

[1] ℓ is the rod's length

[2] Assuming that $\mathbf{n}$ is extended by continuity in points where curvature is 0

now ready to define the curved rod with the central curve γ and the variable cross section $\varepsilon\, S(y_1)$ by

$$P_\varepsilon = \Phi(\Omega_\varepsilon)\ .$$

Obviously, the curve γ is passing the centroid of each cross section of the rod. For more details on the geometric tools see [2], [1] and [6]. In particular an interesting generalization to nongeneric curves was done in [2].

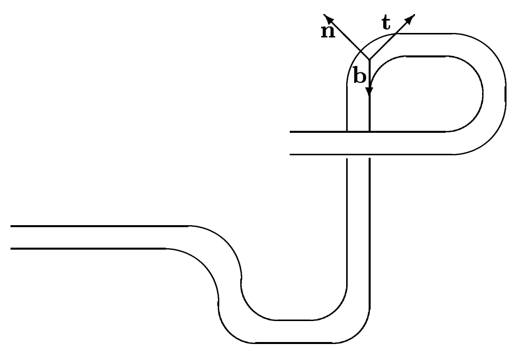

Figure 1: Curved rod or pipe

3. The Variational Approach

As in [3], we use the variational characterization of $C^\varepsilon(r,q)$ and then we study the convergence of the corresponding minima.
Let $r \in [1,3[$. Due to the Sobolev theorem there exist constants $C^\varepsilon(r,q)$ such that

$$|\varphi|_{L^q(P_\varepsilon)} \le C^\varepsilon(r,q)|\varphi|_{W^{1,r}(P_\varepsilon)} \tag{2}$$

for all $q \in [1, r^*]$, $r^* = \frac{3r}{3-r}$. Here, and in the sequel we denote

$$|\varphi|_{W^{1,r}(\mathcal{O})} = |\varphi|_{L^r(\mathcal{O})} + |\nabla\varphi|_{L^r(\mathcal{O})}\ .$$

Relich-Kondrachev's theorem implies that the above imbedding

$$W^{1,r}(P_\varepsilon) \hookrightarrow L^q(P_\varepsilon)$$

is compact for all $q \in [1, r^*[$. In that case $C^\varepsilon(r,q)$ can be seen as an eigenvalue on the nonlinear problem

$$C^\varepsilon(r,q)^{-1} = \inf\left\{ \frac{|\varphi|_{W^{1,r}(P_\varepsilon)}}{|\varphi|_{L^q(P_\varepsilon)}} \ ;\ \varphi \in W^{1,r}(P_\varepsilon)\ ,\ \varphi \neq 0 \right\} \ . \qquad (3)$$

Such $C^\varepsilon(r,q)$ is sharp. Indeed, we have:

LEMMA 3.1. *The infimum in (3) is attained and* $C^\varepsilon(r,q)^{-1} > 0$.

Since the mapping $\varphi \mapsto |\varphi|_{W^{1,r}(P_\varepsilon)}$ is convex and $\varphi \mapsto |\varphi|_{L^q(P_\varepsilon)}$ is continuous in $W^{1,r}(P_\varepsilon)$ topology, the proof of this lemma is an elementary exercise from the calculus of variations and can be found in [3].

We can now state our main result:

THEOREM 3.2. *Let*

$$C^0(r,q)^{-1} =$$
$$= inf \left\{ \frac{|\,|S(\cdot)|^{1/r}\,\varphi|_{L^r(0,\ell)} + |\,|S(\cdot)|^{1/r}\,\varphi'|_{L^r(0,\ell)}}{|\,|S(\cdot)|^{1/q}\,\varphi|_{L^q(0,\ell)}} \ ;\ \varphi \in W^{1,r}(0,\ell), \varphi \neq 0 \right\} \quad (4)$$

Then $C_0(r,q)^{-1} > 0$ *and*

$$\lim_{\varepsilon\to 0} \varepsilon^{2(\frac{1}{r}-\frac{1}{q})} C^\varepsilon(r,q) = C^0(r,q) \ .$$

The interesting part is that only the arc length of our curve γ turns out to be important in the asymptotic study of $C^\varepsilon(r,q)$. Indeed, there are no effects of the curvedness of γ and the limit of the Sobolev constant for the curved rod P_ε is the same as for the Sobolev constant for the straight rod Ω_ε. The effects of curvedness are expected in some higher order correctors.

To prove that the above result has a sense we first have the following lemma:

LEMMA 3.3. *The infimum in (4) is attained and* $C^0(r,q)^{-1} > 0$.

The proof is the same as the proof of lemma 3.1.

4. Proof of Theorem 3.2

4.1. Geometric tools

In this section we study the curve γ and we recall some basic geometric tools that are used in our method. The basic idea for studying the asymptotic bahaviour of curved thin structures, used in [1], [2], [4] or [6], is to define the appropriate curvilinear coordinate system in P_ε and write the variational problem (3) in such coordinates.

We start with covariant basis which is, in fact, the gradient of the mapping Φ, i.e. it consists of the vectors $\mathbf{a}_i(y) = \frac{\partial \Phi}{\partial y_i}(y)$. In our case, we have

$$\begin{aligned} \mathbf{a}_1(y) &= [1 - y_2\kappa(y_1)]\mathbf{t}(y_1) - y_3\tau(y_1)\mathbf{n}(y_1) + y_2\tau(y_1)\mathbf{b}(y_1) \\ &= \mathbf{t}(y_1) + O(\varepsilon) \ , \end{aligned} \tag{5}$$

while $\mathbf{a}_2(y) = \mathbf{n}(y_1)$, $\mathbf{a}_3(y) = \mathbf{b}(y_1)$. The covariant metric tensor $[\underline{\mathbf{M}}]_{ij} = g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$ is now easily computed as

$$\underline{\mathbf{M}}(y) =$$

$$\begin{bmatrix} 1-2y_2\kappa(y_1)+y_2^2\kappa(y_1)^2+y_3^2\tau(y_1)^2+y_2^2\tau(y_1)^2 & -y_3\tau(y_1) & y_2\tau(y_1) \\ -y_3\tau(y_1) & 1 & 0 \\ y_2\tau(y_1) & 0 & 1 \end{bmatrix} =$$

$$= \mathbf{I} + O(\varepsilon)$$

and its determinant equals

$$g_\varepsilon(y) = \det \underline{\mathbf{M}}(y) = [1 - y_2\kappa(y_1)]^2 = 1 + O(\varepsilon) \ .$$

The contravariant metric tensor $\overline{\mathbf{M}}(y) = [g^{ij}]$ is defined by $\overline{\mathbf{M}}(y) = \underline{\mathbf{M}}(y)^{-1}$, so that $\overline{\mathbf{M}}(y) = \mathbf{I} + O(\varepsilon)$.

Let $\phi(y) = (\varphi \circ \Phi)(y) = \varphi(x)$. We need the expression for $\nabla_x \varphi \circ \Phi$. We have

$$\nabla_x \varphi \circ \Phi = \sum_{\ell,k}^{3} g^{k\ell} \frac{\partial \phi}{\partial y_k} \mathbf{a}_\ell \ . \tag{6}$$

After noticing that

$$\int_{P_\varepsilon} f(x)\, dx = \int_{\Omega_\varepsilon} (f \circ \Phi)(y) \sqrt{g_\varepsilon}\, dy \ ,$$

our problem (3) can be written in the form

$$C^\varepsilon(r,q)^{-1} = \\ = \inf\left\{ \frac{|\{(\nabla_x\varphi)\circ\Phi\}h^r_\varepsilon|_{L^r(\Omega_\varepsilon)}+|(\varphi\circ\Phi)h^r_\varepsilon|_{L^r(\Omega_\varepsilon)}}{|(\varphi\circ\Phi)\ h^q_\varepsilon|_{L^q(\Omega_\varepsilon)}}\ ;\ \begin{array}{l}\varphi\in W^{1,r}(P_\varepsilon)\ ,\\ \varphi\neq 0\end{array}\right\} = \\ = \inf\left\{ \frac{|\sum_{k,\ell=1}^3 g^{k\ell}\,\frac{\partial\phi}{\partial y_k}\,\mathbf{a}_\ell\ h^r_\varepsilon|_{L^r(\Omega_\varepsilon)}+|\phi\ h^r_\varepsilon|_{L^r(\Omega_\varepsilon)}}{|\phi\ h^q_\varepsilon|_{L^q(\Omega_\varepsilon)}}\ ;\ \begin{array}{l}\phi\in W^{1,r}(\Omega_\varepsilon)\ ,\\ \phi\neq 0\end{array}\right\}, \tag{7}$$

where $h^r_\varepsilon = (g_\varepsilon)^{1/2r}$.

4.2. Two-scale convergence

We recall the definition of the two-scale convergence from [5] :

DEFINITION 4.1. *Let*

$$\Omega = \{(y_1,z)\in\mathbf{R}^3\ ;\ y_1\in]0,\ell[\ ,\ z=(z_2,z_3)\in S(y_1)\ \}\ .$$

We say that a sequence $\{v^\varepsilon\}_{\varepsilon>0}$, *such that* $v^\varepsilon\in L^r(\Omega_\varepsilon)$, L^r*-two-scale converges to a function* $V\in L^r(\Omega)$ *(we use the notation* L^r-2s *convergence in the sequel) if*

$$\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} v^\varepsilon(y)\phi(y_1,\frac{y'}{\varepsilon})dy \to \int_\Omega V(y_1,z)\phi(y_1,z)dy_1\,dz\ \ ,\ y'=(y_2,y_3) \tag{8}$$

for any $\phi\in L^{r'}(\Omega)$, *where* $1/r+1/r'=1$ *if* $1<r<\infty$ *and* $r'=1$ *if* $r=\infty$, $r'=\infty$ *if* $r=1$.
We, also, say that a sequence $\{v^\varepsilon\}_{\varepsilon>0}$, *such that* $v^\varepsilon\in L^r(\Omega_\varepsilon)$ *strongly* L^r*-two-scale converges (notation* $s{-}L^r{-}2s$*) to a function* $V\in L^r(\Omega)$ *if*

$$\frac{1}{\varepsilon^{2/r}}|v^\varepsilon(y)-V(y_1,\frac{y'}{\varepsilon})|_{L^r(\Omega_\varepsilon)}\to 0\ \ . \tag{9}$$

For the properties of such convergence we refer to [5]. Theorem 1 from [5] and proposition 3 from [3] contain the compactness result that will be needed in the sequel:

PROPOSITION 4.2. *Suppose that the sequence* $\{\varphi_\varepsilon\}$, $\varphi_\varepsilon \in W^{1,r}(\Omega_\varepsilon)$, $r < 3$ *is such that*

$$\varepsilon^{-2/r} |\varphi_\varepsilon|_{W^{1,r}(\Omega_\varepsilon)} \le C \quad . \tag{10}$$

Then there exists a subsequence, denoted for simplicity by $\{\varphi_\varepsilon\}$, *and* $\varphi_0 \in W^{1,r}(0,\ell)$, $\psi_0 \in Y^r = \{\psi \in L^r(\Omega)\ ;\ \nabla_z \psi \in L^r(\Omega)\ \}$ *such that*

$$\varphi_\varepsilon \to \varphi_0 \qquad s - L^q - 2s \ \ , \ \ 1 \le q < r^* = \frac{3r}{3-r} \tag{11}$$

$$\nabla \varphi_\varepsilon \to \frac{\partial \varphi_0}{\partial y_1} \mathbf{e}_1 + \nabla_z \psi_0 \qquad L^r - 2s \ \ , \tag{12}$$

where $\nabla_z \psi_0 = \frac{\partial \psi_0}{\partial z_2} \mathbf{e}_2 + \frac{\partial \psi_0}{\partial z_3} \mathbf{e}_3$.
Furthermore, ψ_0 *can be chosen such that*

$$\int_{\partial S(y_1)} \psi_0(y_1, z)\ \mathbf{n}(y_1, z)\ dS_z = 0 \ \ \textit{for (a.e.)}\ \ y_1 \in]0, \ell[\ \ , \tag{13}$$

where $\mathbf{n}(y_1, \cdot)$ *is the exterior unit normal on* $\partial S(y_1)$.

4.3. Proof of theorem 3.2

Let F_ε be defined by

$$F_\varepsilon(\phi) = \begin{cases} \dfrac{|\sum_{k,\ell=1}^3 g^{k\ell} \frac{\partial \phi}{\partial y_\ell} \mathbf{a}_\ell h_\varepsilon^r|_{L^r(\Omega_\varepsilon)} + |\phi\ h_\varepsilon^r|_{L^r(\Omega_\varepsilon)}}{|\phi\ h_\varepsilon^q|_{L^q(\Omega_\varepsilon)}} \ ; & \phi \in W^{1,r}(\Omega_\varepsilon)\ , \ \phi \ne 0 \\ +\infty \quad \text{otherwise} & \end{cases}$$

and F_0 by

$$F_0(\varphi) = \begin{cases} \dfrac{|\ |S(\cdot)|^{1/r}\ \varphi|_{L^r(0,\ell)} + |\ |S(\cdot)|^{1/r}\ \varphi'|_{L^r(0,\ell)}}{|\ |S(\,\cdot\,)|^{1/q}\ \varphi|_{L^q(0,\ell)}} \ ; & \varphi \in W^{1,r}(0,\ell)\ , \ \varphi \ne 0 \\ +\infty \ \ \text{otherwise} & \end{cases} .$$

We begin by noticing that both infima of F_ε and of F_0 are attained at some points $\phi_\varepsilon \in W^{1,r}(\Omega_\varepsilon)$ and $\varphi_0 \in W^{1,r}(0,\ell)$, respectively. Furthermore, due to the homogeneity of F_ε we can assume that

$$\varepsilon^{-2/r} |\phi_\varepsilon|_{W^{1,r}(\Omega_\varepsilon)} = 1 \quad .$$

Using the asymptotic behaviour of $g^{k\ell} = \delta_{k\ell} + O(\varepsilon)$, and $\mathbf{a}_1 = \mathbf{t} + O(\varepsilon)$, a simple computation gives

$$\varepsilon^{2(q^{-1}-r^{-1})} F_\varepsilon(\phi_\varepsilon) \le \varepsilon^{2(q^{-1}-r^{-1})} F_\varepsilon(\varphi_0) = F_0(\varphi_0) + o(\varepsilon)$$

so that

$$\limsup_{\varepsilon\to 0}\ \varepsilon^{2(q^{-1}-r^{-1})} F_\varepsilon(\phi_\varepsilon) \le F_0(\varphi_0)\ . \tag{14}$$

Due to the proposition 4.2, there exist $p \in W^{1,r}(0,\ell)$, $w \in Y^r$ such that

$$\begin{aligned} &\phi_\varepsilon \to p \quad s - L^q - 2s \\ &\nabla \phi_\varepsilon \to \frac{\partial p}{\partial y_1}\,\mathbf{e}_1 + \nabla_z w \quad L^r - 2s \end{aligned}$$

and (13) holds for w. The convexity of L^r norm implies that

$$|p|_{L^r(\Omega)} + |\frac{\partial p}{\partial y_1}\,\mathbf{e}_1 + \nabla_z w|_{L^r(\Omega)} \le \liminf_{\varepsilon\to 0} \varepsilon^{-2/r} |\phi_\varepsilon|_{W^{1,r}(\Omega_\varepsilon)} = 1\ .$$

The strong two scale convergence implies that

$$\varepsilon^{-2/q} |\phi_\varepsilon|_{L^q(\Omega_\varepsilon)} \to |p|_{L^q(\Omega)} = |\ |S|^{1/q}\, p|_{L^q(0,\ell)}\ .$$

As

$$\begin{aligned} &\left|\frac{\partial p}{\partial y_1}\,\mathbf{e}_1 + \nabla_z w\right|^r_{L^r(\Omega)} = \\ &= \int_0^\ell \int_{S(y_1)} |\frac{\partial p}{\partial y_1}\,\mathbf{e}_1 + \nabla_z w|^r \ge \\ &\int_0^\ell \left|\int_{S(y_1)} (\frac{\partial p}{\partial y_1}\,\mathbf{e}_1 + \nabla_z w)\, dz\right|^r |S(y_1)|^{1-r} dy_1 = \\ &\int_0^\ell |S(y_1)|^{1-r} \left|\ |S(y_1)|\ \frac{\partial p}{\partial y_1}(y_1)\,\mathbf{e}_1 + \int_{\partial S(y_1)} w(y_1,z)\ \mathbf{n}(y_1,z)\, dS_z\ \right|^r dy_1 = \\ &= (\text{condition (13)}) = \left|\ |S|^{1/r}\ \frac{\partial p}{\partial y_1}\right|^r_{L^r(0,\ell)} \end{aligned}$$

we obtain

$$\begin{aligned} F_0(p) &\leq \liminf_{\varepsilon\to 0} \varepsilon^{2(q^{-1}-r^{-1})} F_\varepsilon(\phi_\varepsilon) \leq \limsup_{\varepsilon\to 0} \varepsilon^{2(q^{-1}-r^{-1})} F_\varepsilon(\phi_\varepsilon) \leq \\ &\leq \text{(due to (14))} \leq F_0(\varphi_0) \ . \end{aligned}$$

Thus

$$\begin{aligned} C^0(r,q)^{-1} = F_0(p) &= F_0(\varphi_0) = \lim_{\varepsilon\to 0} F_\varepsilon(\phi_\varepsilon) = \\ &= \lim_{\varepsilon\to 0} \varepsilon^{2(q^{-1}-r^{-1})} C^\varepsilon(r,q)^{-1} \ . \end{aligned}$$

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