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Generalized Truncated Exponential Polynomials and Applications

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Summary. - *The use of the* monomiality principle *will allow us to obtain generalizations of the truncated exponential polynomials. We will derive the properties of the new families of polynomials and finally we will mention their applications in problems of practical interest.*

1. Introduction

The study of the properties of ordinary and generalised polynomials is simplified by the use of concepts associated with the monomiality principle, according to which a given polynomial $p_n(x)$ ($n \in \mathbf{N}$ and $x \in \mathbf{C}$) is defined a "quasi monomial" if two operators $\hat{P}$ and $\hat{M}$, called from now on "derivative" and "multiplicative" operator respectively, can be defined in such a way that [4]

$$\hat{P}(p_n(x)) = np_{n-1}(x), \tag{1}$$

$$\hat{M}(p_n(x)) = p_{n+1}(x). \tag{2}$$

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The $\hat{P}$ and $\hat{M}$ operators are shown to satisfy the relation of commutation

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}, \tag{3}$$

and thus display a Weyl group structure.
The properties of $p_n(x)$ can be deduced from those of the $\hat{P}$ and $\hat{M}$ operators.
If $\hat{P}$ and $\hat{M}$ possess a differential realization, then the polynomial $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}(p_n(x)) = np_n(x). \tag{4}$$

If $p_0(x) = 1$, then $p_n(x)$ can be explicitly costructed as

$$p_n(x) = \hat{M}^n(1). \tag{5}$$

The identity (5) implies that a generating function of $p_n(x)$ can be cast in the form

$$e^{t\hat{M}}(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x). \tag{6}$$

The Hermite and Laguerre polynomials are two examples of quasi monomials [4].
The monomiality principle allows us to define new families of functions by exploiting the corrispondence

$$\begin{aligned} \hat{M} &\longrightarrow x && (7)\\ \hat{P} &\longrightarrow \partial_x \\ p_n(x) &\longrightarrow x^n. && (8) \end{aligned}$$

According to such a correspondence, one can define families of *p-base* functions, namely functions defined in such a way that the quasi monomial $p_n(x)$ replaces x^n in the series expansion of a known function.
We will e.g. define p-base exponential and Gauss functions respectively as

$$ {}_p e(x) = \sum_{n=0}^{\infty} \frac{1}{n!} p_n(x), \tag{9}$$

$$ {}_p g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, p_{2n}(x)}{n!}. \tag{10}$$

Within such a context the p-base extension of Bessel functions has been shown to be of particular importance in various problems [3]. In this paper we will discuss the p-base extension of truncated polynomials, which play an important role in various problems in classical and quantum optics [1].
The ordinary truncated polynomials are defined by [1, 5]

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{11}$$

Most of the properties of this polynomials can be derived from the definition (11). We find indeed

$$e_n(x) = \frac{1}{n!} \int_0^\infty e^{-\alpha}(x+\alpha)^n \, d\alpha. \tag{12}$$

Eq. (12) allows the derivation of the generating function

$$\frac{e^{xt}}{1-t} = \sum_{n=0}^{\infty} t^n e_n(x). \tag{13}$$

By using eq. (13) we obtain the differential recursions

$$\begin{aligned} &\frac{d}{dx} e_n(x) = e_{n-1}(x), \\ &\left[1 + \frac{x}{n+1}\left(1 - \frac{d}{dx}\right)\right] e_n(x) = e_{n+1}(x) \end{aligned} \tag{14}$$

and we can define the shifting operators

$$\begin{aligned} \hat{E}_- &= \frac{d}{dx}, \\ \hat{E}_+ &= 1 + \frac{x}{\hat{n}+1}\left(1 - \frac{d}{dx}\right). \end{aligned} \tag{15}$$

Furthermore the use of the relation

$$\hat{E}_+ \left[\hat{E}_- e_n(x)\right] = e_n(x) \tag{16}$$

yields the differential equation

$$\left[x\frac{d^2}{dx^2}-(n+x)\frac{d}{dx}+n\right]e_n(x)=0. \tag{17}$$

The p-base truncated polynomials will be denoted by ${}_pe_n(x)$ and, according to the correspondences (7) and (8), we find

$${}_pe_n(x)=\sum_{k=0}^{n}\frac{p_k(x)}{k!}, \tag{18}$$

the generating function

$$\frac{e^{\hat{M}t}}{1-t}=\sum_{n=0}^{\infty}t^n\,{}_pe_n(x) \tag{19}$$

and the differential equation

$$\left[\hat{M}\hat{P}^2-(n+\hat{M})\hat{P}+n\right]{}_pe_n(x)=0, \tag{20}$$

which provides an isospectral problem to the ordinary truncated polynomials equation.
By using eq. (12) and the corrispondences (7), (8), we can write the integral representation of ${}_pe_n(x)$

$${}_pe_n(x)=\frac{1}{n!}\int_0^{\infty}e^{-\alpha}(\hat{M}+\alpha)^n\,d\alpha. \tag{21}$$

In this paper we will discuss the problems associated with the properties of Hermite and Laguerre based truncated polynomials.

2. Hermite based truncated exponential polynomials

In the case of Hermite-based polynomials eq. (18) reduces to

$${}_He_n(x,y)=\sum_{k=0}^{n}\frac{H_k(x,y)}{k!}, \tag{22}$$

where $H_k(x,y)$ are the Hermite polynomials [11, 7], and the generating function (19)

$$\frac{e^{xt+yt^2}}{1-t} = \sum_{n=0}^{\infty} {}_He_n(x,y)\ t^n, \tag{23}$$

which is a trivial consequence of the structure of the operator $\hat{M}$ and of the Weyl disentanglement rule.
By taking the derivative with respect to x, y and t of both sides of eq. (23) we can derive the recurrences

$$\partial_x\ {}_He_n(x,y) = {}_He_{n-1}(x,y), \tag{24}$$
$$\partial_y\ {}_He_n(x,y) = {}_He_{n-2}(x,y), \tag{25}$$
$$\left(1+\frac{x}{n+1}+\frac{2y-x}{n+1}\partial_x-\frac{2y}{n+1}\partial_x^2\right)\ {}_He_n(x,y) = {}_He_{n+1}(x,y). \tag{26}$$

Furthermore, by defining the shifting operators

$$\begin{aligned}\hat{E}_+ &= \left(1+\frac{x}{\hat{n}+1}+\frac{2y-x}{\hat{n}+1}\partial_x-\frac{2y}{\hat{n}+1}\partial_x^2\right), \\ \hat{E}_- &= \partial_x\end{aligned}$$

and by using the relation

$$\hat{E}_+\left(\hat{E}_-\ {}_He_n(x,y)\right) = {}_He_n(x,y), \tag{27}$$

we obtain the differential equation in the form

$$\left[2y\partial_x^3+(x+2y)\partial_x^2+(x-n)\partial_x+n\right]\ {}_He_n(x,y) = 0. \tag{28}$$

Furthermore by using (21) we find the integral representation

$$\begin{aligned}{}_He_n(x,y) &= \frac{1}{n!}\int_0^\infty e^{-\alpha}\,(x+2y\partial_x+\alpha)^n\ d\alpha\ + \\ &+ \frac{1}{n!}\int_0^\infty e^{-\alpha}H_n(x+\alpha,y)\,d\alpha.\end{aligned} \tag{29}$$

In section 3 we extend the truncated polynomials to the Laguerre family.

3. Laguerre based truncated exponential polynomials

Putting $p_n(x) = \mathcal{L}_n(x,y)$ in eq. (18), we obtain the Laguerre-based truncated polynomials defined as

$$ {}_{\mathcal{L}}e_n(x,y) = \sum_{k=0}^{n} \frac{\mathcal{L}_k(x,y)}{k!}, \qquad (30) $$

where $\mathcal{L}_k(x,y)$ are the Laguerre polynomials [9, 2].
We derive the generating function by eq. (31),

$$ \frac{e^{yt} C_0(xt)}{1-t} = \sum_{n=0}^{\infty} {}_{\mathcal{L}}e_n(x,y) t^n, \qquad (31) $$

where

$$ C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}, \qquad (32) $$

which is a direct consequence of the structure of the operator $\hat{M}$ and of the Weyl disentanglement rule.
By using eq. (30) we find the integral representation

$$ {}_{\mathcal{L}}e_n(x,y) = \frac{1}{n!} \int_0^{\infty} e^{-\alpha} \left(y - D_x^{-1} + \alpha\right)^n d\alpha. \qquad (33) $$

By taking the derivative with respect to $-\partial_x x \partial_x$, ∂_y and $-\partial_t t \partial_t$ of both sides of eq. (31), we can derive the recurrences

$$ (\hat{E}_-)_x \, {}_{\mathcal{L}}e_n(x,y) = {}_{\mathcal{L}}e_{n-1}(x,y) \qquad (34) $$
$$ (\hat{E}_-)_y \, {}_{\mathcal{L}}e_n(x,y) = {}_{\mathcal{L}}e_{n-1}(x,y) \qquad (35) $$
$$ \hat{E}_+ \, {}_{\mathcal{L}}e_n(x,y) = {}_{\mathcal{L}}e_{n+1}(x,y), \qquad (36) $$

where the shifting operators

$$ \hat{E}_+ = 1 - \frac{x}{(\hat{n}+1)^2} + \frac{y(1+2\hat{n})}{(\hat{n}+1)^2} + $$
$$ + \left(\frac{y^2}{(\hat{n}+1)^2} - \frac{x}{(\hat{n}+1)^2} + \frac{y(1+2\hat{n})}{(\hat{n}+1)^2} \right) \partial_x x \partial_x + \frac{y^2}{(\hat{n}+1)^2} \partial_x^2 x^2 \partial_x^2 $$

$$ (\hat{E}_-)_x = -\partial_x x \partial_x $$
$$ (\hat{E}_-)_y = \partial_y $$

can be embedded to give the relation

$$\hat{E}_+\left((\hat{E}_-)_x\,{}_{\mathcal{L}}e_n(x,y)\right) = {}_{\mathcal{L}}\,e_n(x,y), \tag{37}$$

which can be further manipulated to get the differential equation

$$\begin{aligned}&\left[x^2y\partial_x^4 + (4xy - x^2)\partial_x^3 + (y(2+x) + x(n-2))\partial_x^2 + \right.\\ &\left. +(n+y-x)\partial_x + n\right]\,{}_{\mathcal{L}}e_n(x,y) = 0.\end{aligned} \tag{38}$$

The last equation can also be derived by using eq. (20), where $\hat{M} = (y - D_x^{-1})$ and $\hat{P} = -\partial_x x \partial_x$. We obtain indeed

$$\left[(y - D_x^{-1})\partial_x^2 x^2 \partial_x^2 + (n + y - D_x^{-1})\partial_x x \partial_x + n\right]\,{}_{\mathcal{L}}e_n(x,y) = 0, \tag{39}$$

which easily reduces to eq. (38).

4. The generating function method

In the previous sections we have discussed p-base truncated polynomials, but the correspondences (7), (8) allows the introduction of the Hermite-Bessel and Laguerre-Bessel functions, the Hermite-Laguerre and Laguerre-Hermite polynomials.
These new functions are useful for the study of integrals involving combinations of special functions [9].
We can find the analytic form of these integrals by using the generating function method, which can be summarized as follows.

Consider the problem of computing the sequence of definite integrals

$$\int_a^b f_n(x)\,dx, \qquad (n \in \mathbf{N}_0 := \{0,1,2,...\}), \tag{40}$$

without knowing explicitly the primitives of the functions $f_n(x)$. Let $F(x,t)$ be the generating function of the set $\{f_n(x)\}_{n\in\mathbf{N}_0}$, namely

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n, \tag{41}$$

which is an analytic function of t, defined in a suitable neighborhood of the origin: $|t| < T_0$. Suppose further that the function $F(x,t)$ is analytically integrable, namely

$$\int_a^b F(x,t)\,dx = \Phi(t), \qquad |t| < T_0, \tag{42}$$

so that the coefficients of the series expansion $\Phi(t) = \sum_{n=0}^{\infty} a_n t^n$ are known.

Then, by using the uniform convergence of the expansion (41), the generating function method allows us to write:

$$\begin{aligned} \int_a^b F(x,t)\,dx &= \sum_{n=0}^{\infty} t^n \left(\int_a^b f_n(x)\,dx \right) = \\ &= \sum_{n=0}^{\infty} t^n a_n, \end{aligned} \tag{43}$$

and hence

$$\int_a^b f_n(x)\,dx = a_n, \qquad (n \in \mathbf{N}). \tag{44}$$

To give an idea of the importance of p-base functions and p-base polynomials we will present in this section a number of integrals, often appearing in applications, which can be written in terms of Hermite-base Bessel, Laguerre-base Bessel functions and also in terms of truncated exponential polynomials.

The examples given below are a consequence of the previously quoted generating function method [10]

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2)} J_n(x+b)\,dx = \sqrt{\frac{\pi}{\alpha}}\, {}_H J_n\left(b, \frac{1}{4\alpha}\right), \tag{45}$$

where ${}_H J_n\left(b, \frac{1}{4\alpha}\right)$ are the Hermite-Bessel functions, furthermore [6]

$$\int_{0}^{\infty} e^{-(\alpha x^2)} L_n(x+d)\,dx = \sqrt{\frac{\pi}{\alpha}}\, {}_H \mathcal{L}_n\left(d, \frac{1}{4\alpha}\right), \tag{46}$$

where ${}_H \mathcal{L}_n\left(d, \frac{1}{4\alpha}\right)$ are the Hermite-Laguerre polynomials. Moreover

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2)} He_n(x+d)\,dx = \frac{n!}{\alpha^{n+1}} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k e_{n-2k}(\alpha d)\alpha^{2k}}{2^k k!}, \tag{47}$$

where $e_{n-2k}(\alpha d)$ are the exponential truncated polynomials.
The integral representation (29) can also be obtained by using the generating function method, which yields

$$\frac{1}{n!}\int_0^\infty e^{-\alpha} H_n(x+\alpha)\, d\alpha = \frac{e^{xt+yt^2}}{1-t} = \sum_{n=0}^{\infty} t^n \, {}_He_n(x,y), \qquad (48)$$

i.e. the generating function of ${}_He_n(x,y)$.
It is easy to derive a similar identity for the ${}_{\mathcal{L}}e_n(x,y)$ polynomials.

5. Concluding Remarks

We have shown that the monomiality principle offer a quite powerful tool to explore new families of functions and new classes of isospectral problems leading to non trivial generalizations of special functions.
Presenting selected examples we have proved the usefulness of the point of view based on the concept of quasi monomiality and of the generating function method. It is worth to note that the methods we have used to treat generalized forms of truncated polynomials provide noticeable advantages with respect to the ordinary techniques. Most of the flexibility of the method is associated with its operational nature and this offer a fairly natural derivation of the associated differential equations, whose derivation would become quite awckward using e.g. the procedure described in ref. [8].
Furthermore the introduction of Hermite and Laguerre base truncated polynomials is naturally framed within the context of the formalism underlying the monomiality principle and therefore their properties emerge quite naturally in such a framework.
A final word of comment is relevant to the applications of this family of polynomials which may become a powerful tool in problems involving overlapping integrals of Hermite-Gauss and Laguerre-Gauss modes in optics, this aspect of the problem has been preliminarily discussed in refs. [1, 10, 6] and will be the topic of a more carefull forthcoming investigation.

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