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On the Minimal Free Resolution of Multiple Curves

Edoardo Ballico (*)

Summary. - *Here we give a sufficient condition to obtain the property N_p for the minimal free resolution of a projective embedding of certain multiple (i.e. locally Cohen - Macaulay but not reduced) curves (ropes in the sense of Chandler and some generalizations).*

1. Introduction

Let C be a purely one-dimensional locally Cohen - Macaulay projective scheme. We are interested in the cohomological properties of the embeddings of C in a projective space and in particular we want to study the associated minimal free resolution (e.g. property N_p in the sense of [12] or [3]). We study it for certain C: the ropes. Set $D := C_{red}$ and call $\mathcal{I}$ the ideal sheaf of D in C. Notice that if $\mathcal{I}^2 = 0$, then $\mathcal{I}$ is a coherent $\mathcal{O}_D$-module. In [5] C is called a rope if $\mathcal{I}^2 = 0$ and D is smooth and connected. Notice that if C is a rope, then $\mathcal{I}$ is a locally free $\mathcal{O}_D$-module. The rope C is said to have multiplicity $r + 1$ if the locally free $\mathcal{O}_D$-module $\mathcal{I}$ has rank r. In [1], Def. 0.1, C was called a generalized rope if $\mathcal{I}^2 = 0$ and D is irreducible. In this paper we will say that C is a very generalized rope if $\mathcal{I}^2 = 0$ and D is connected. For general definitions and results on minimal free resolutions of curves (e.g. property N_p), see [12] and [3]. In section 2 we will give numerical assumptions which imply property N_p for an embedding of a generalized rope (see Theorem 2.8). We state

(*) Author's address: Edoardo Ballico, Dept. of Mathematics, University of Trento, 38050 Povo (TN), Italy, e-mail: ballico@science.unitn.it
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below as Theorem 1.1 the corresponding result for the simpler case of ropes.

We work over an algebraically closed field $\mathbf{K}$. Let C be a rope with multiplicity $r+1 \geq 2$. Set $E := \mathcal{I}$ seen as a rank r vector bundle on D. Let $\mu^+(E)$ (resp. $\mu^-(E)$) be the maximal (resp. minimal) slope of a graded subquotient of the Harder - Narasimhan filtration of E. Hence $\mu^-(E) \leq \mu(E) \leq \mu^+(E)$ and $\mu^-(E) = \mu^+(E)$ if and only if E is semistable.

THEOREM 1.1. *Assume char*$(\mathbf{K}) = 0$. *Let* C *be a rope and set* $D := C_{red}$ *and* $q := p_a(D)$. *Fix* $L \in Pic(C)$ *and an integer* $p \geq 0$. *Set* $y := deg(L|D)$. *Assume* $y \geq \max\{2q+2+p, 2q+1+(p+2)\mu^-(E)\}$. *Then* L *has property* N_p.

For a curve $Y \subset \mathbf{P}^n$ very weak informations on the general hyperplane section of Y (e.g. the so - called Uniform Position Property) were used to obtain upper bounds for $p_a(Y)$ in terms of $\deg(Y)$ and n (see [5], [3], [6] and [1] for the case of multiple curves). In section 3 we will use a different approach to obtain upper bounds for $p_a(Y)$ when Y is a generalized rope.

2. Property N_p

In this section we will prove Theorem 1.1 and an extension of it to the case of generalized ropes (see Theorem 2.8).

REMARK 2.1. *Let* C *be a very generalized rope. Since* $h^0(D, \mathcal{O}_D) = 1$, *we have* $h^0(C, \mathcal{O}_C) = 1$ *if and only if* $h^0(D, \mathcal{I}) = 0$.

REMARK 2.2. *If* $R \in Pic^z(D)$ *and* $z > 2p_a(D)$, *then* R *is very ample by a very particular case of [4], Th. 1.1.*

To prove Theorem 1.1 we need the following two lemmas. In the statements of lemmas 2.3 and 2.4 we will use the notations introduced in the statement of Theorem 1.1. Thus we have $q = p_a(D)$ and the exact sequence

$$0 \to E \to \mathcal{O}_C \to \mathcal{O}_D \to 0 \qquad (1)$$

Set $g := p_a(C)$ and $x := \deg(E)$. We have $g = 1 - \chi(\mathcal{O}_C) = 1 - \chi(\mathcal{O}_D) - \chi(E) = q - x + r(q-1) = (r+1)q - r - x$. Since $\mathcal{I}^2 = 0$, we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_C^* \to \mathcal{O}_D^* \to 0 \qquad (2)$$

in which the map $\mathcal{O}_C^* \to \mathcal{O}_D^*$ is the exponential map sending t into $1+t$ (see [10], p.179). Since $\dim(C) = 1$ and $H^0(D, \mathcal{O}_D^*) \cong \mathbf{K}^*$, we obtain $\mathrm{Pic}(C) \cong \mathrm{Pic}(D) \times H^1(D, E)$. From the exact sequence (1) we obtain at once the following lemma.

Lemma 2.3. *Fix $L \in Pic(C)$.*

(a) If $deg(L|D) + \mu^-(E) > 2q-2$, then the restriction map $H^0(C,L) \to H^0(D, L|D)$ is surjective.

(b) If $deg(L|D) + \mu^-(E) > 2q-2$ and $deg(L|D) \geq 2q-1$, then $H^1(C,L) = 0$.

Lemma 2.4. *Fix $L \in Pic(C)$ and set $z := deg(L|D)$. Assume $z > \max\{2q-1, 2q-2+\mu^-(E)\}$. Then L is very ample.*

Proof. By [4], Remark 2.1, it is sufficient to prove that for every zero-dimensional subscheme B of C with $\mathrm{length}(B) = 2$ we have $h^1(C, L \otimes \mathcal{I}_B) = 0$. If $B \subset D$ this follows from the very ampleness of $L|D$, the surjectivity of the restriction map $H^0(C,L) \to H^0(D, L|D)$ (Lemma 2.3) and Remark 2.2. Hence we may assume $\mathrm{card}(B_{red}) = 1$, say $B_{red} = \{P\}$. Since $\mu^-(E(-P)) > 2q-2$ and any semistable bundle on D with slope < 0 has no non-zero section, we conclude using (1). □

Proof of Theorem 1.1. By lemma 2.3 we have $h^1(C,L) = 0$ and $h^0(C,L) = (r+1)y + 1 - g = (r+1)y - (r+1)q + r + x$. By lemma 2.4 L is very ample. Let M_L be the kernel of the evaluation map $H^0(C,L) \otimes \mathcal{O}_C \to L$. Hence M_L is a vector bundle on C with $\mathrm{rank}(M_L) = h^0(C,L) - 1$. Set $R := L|D$. Let M_R be the kernel of the evaluation map $H^0(D,R) \otimes \mathcal{O}_D \to R$. By [3], Th. 1.2, or the first few lines of [7], §3, M_R is semistable and stable unless $q = 0$ or $x = 2q$ and D is hyperelliptic. We have $\mu(M_R) = -\deg(R)/(h^0(D,R)-1)0 - y/(y-q)$. $M_L|D$ is the direct sum of M_R and of a trivial bundle with rank $h^0(C,L) - h^0(D,R)$. By [3], Lemma 2.5, for every integer $k \geq 1$ and all vector bundles A, B on D we have $\mu^-(\Lambda^k(A)) \geq k\mu^-(A))$ and $\mu^-(A \otimes B) = \mu^-(A) + \mu^-(B)$. Consider the tensor product of (1) with $\Lambda^{p+1}(M_L) \otimes L^{\otimes k}$, $k \geq 1$. Since a vector bundle A on D with $\mu^-(A) > 2q-2$ satisfies $h^1(D, A) = 0$, we obtain $H^1(C, \Lambda^{p+1}(M_L) \otimes L^{\otimes k}) = 0$ for every $k \geq 1$. Hence L satisfies N_p by [12], Prop.1.3.3; indeed, the proof of [12], Prop.1.3.3, does not use that C is a smooth curve, but just $\dim(C) = 1$. Hence the proof of Theorem 1.1 is over. □

To generalize Theorem 1.1 to the case in which D is not smooth but only irreducible we need to handle the case in which $\mathcal{I}$, seen as a coherent $\mathcal{O}_D$-sheaf, is not locally free. For each $P \in \mathrm{Sing}(D)$ call $\mathbf{m}_P$ the maximal ideal of $\mathcal{O}_{D,P}$ and let δ_P the codimension (as $\mathbf{K}$-vector space) of $\mathcal{O}_{D,P}$ in its normalization. Set $\delta'(D) = \max_{P \in \mathrm{Sing}(D)}\{\delta_P\}$. For any torsion free sheaf A on D set

$$\mu(A, P) := \dim_{\mathbf{K}}(A/\mathbf{m}_P)/\mathrm{rank}(A)$$

. Notice that $\mu(A, P) \geq 1$ and $\mu(A, P) = 1$ if and only if A is locally free at P. Set $\mu(A, \mathrm{Sing}(D)) = \max_{P \in \mathrm{Sing}(D)}\{\mu(A, P)\}$. Notice that for every locally free sheaf B on D we have $\mu(A \otimes B, P) = \mu(A, P)$ and hence $\mu(A \otimes B, \mathrm{Sing}(D)) = \mu(A, \mathrm{Sing}(D))$. By [13], p. 165, we have $\mu(A, P) \leq 1 + \delta_P$. Thus $\mu(A, \mathrm{Sing}(D)) \leq 1 + \delta'$.

Lemma 2.5. *Fix $L \in Pic(C)$.*

(a) If $deg(L|D) + \mu^-(E) > 2q-2$, then the restriction map $H^0(C, L) \to H^0(D, L|D)$ is surjective.

(b) If $deg(L|D) + \mu^-(E) > 2q-2$ and $deg(L|D) \geq 2q-1$, then $H^1(C, L) = 0$.

Proof. It is sufficient to check that a torsion free sheaf A on D with $\mu^-(A) > 2q-2$ has $h^1(D, A) = 0$. Taking the Harder - Narasimhan filtration of A we reduce to the case A semistable. Assume $h^1(D, A) \neq 0$. By duality we have a non-zero map $f : A \to \omega_D$. If $\mathrm{rank}(A) = 1$, f must be injective and hence $\deg(A) \leq 2q-2$, contradiction. Assume $\mathrm{rank}(A) \geq 2$. We have $\mathrm{rank}(\mathrm{Ker}(f)) = \mathrm{rank}(A) - 1 > 0$ and $\deg(\mathrm{Ker}(f)) \geq \deg(A) - 2q + 2$. Thus $\mu(\mathrm{Ker}(f)) > \mu(A)$, contradicting the semistability of A. Alternatively, the result was claimed in [13], first two lines of Lemma 5.2, p.166. □

The proof of Lemma 2.5 gives the following result.

LEMMA 2.6. *Fix $L \in Pic(C)$ and set $x := deg(L|D)$. Assume $x > \max\{2q-1, 2q-2+\mu^-(E)\}$. Then L is spanned and it induces an embedding at each point of D_{reg}.*

LEMMA 2.7. *Fix $L \in Pic(C)$ and set $x := deg(L|D)$. Assume $x > \max\{2q-1, 2q-2+rank(E)\delta' + \mu^-(E)\}$. Then L is very ample. If E is locally free the same is true under the weaker assumption $x > \max\{2q-1, 2q-2+\mu^-(E)\}$.*

Proof. By Remark 2.2 $L|D$ is very ample. By Lemma 2.6 it is sufficient to show that for every $P \in \mathrm{Sing}(D)$ and every zero-dimensional subscheme B of C with $\mathrm{length}(B) = 2$ and $B_{red} = \{P\}$ we have $h^0(C, L \otimes \mathcal{I}_B) = h^0(C, L) - 2$. If $B \subset D$ this equality follows from the very ampleness of $L|D$ and the surjectivity of the restriction map $H^0(C, L) \to H^0(D, L|D)$ (Lemma 2.5, part (a)). If B is not contained in D, it is sufficient to prove that $H^0(D, E \otimes (L|D))$ spans $E \otimes (L|D))$ at P. Since $L|D$ is locally free, we have $\mu(E \otimes (L|D)) = \mu(E) + x$. Hence we may apply [13], Lemma 5.2' (a) at p. 166, to the Harder - Narasimhan filtration of $E \otimes (L|D)$. □

THEOREM 2.8. *Assume $char(\mathbf{K}) = 0$. Let C be a generalized rope. Fix $L \in Pic(C)$ and an integer $p \geq 0$. Set $D := C_{red}$, $q := p_a(D)$, $y := deg(L|D)$ and let E be the ideal sheaf of D in C. Assume $y \geq \max\{2q+2+p, 2q+1+(p+2)\mu^-(E)\}$ if E is locally free and $y \geq \max\{2q+2+p, 2q+1+rank(E)\delta'(D) + (p+2)\mu^-(E)\}$*

Proof. Look at the proof of Theorem 1.1. We quote lemmas 2.5, 2.6 and 2.7 instead of lemmas 2.3 and 2.4. By [8], Appendix with J. Harris, Clifford's theorem for special line bundles is true for an arbitrary integral projective curve. Hence the proof of [7], Lemma 2.4, works verbatim and gives the semistability of $M_{L|D}$ if $p_a(D) \geq 2$; the case $p_a(D) = 1$ is similar. We have $\mu(A \otimes B) = \mu(A) + \mu(B)$ for every torsion free sheaf A on D and every locally free sheaf B on D. Hence we may copy the proof of Theorem 1.1. □

3. Bounds for the genus

Here we try to use the methods of the previous section and of [2] to obtain upper bounds for $p_a(Y)$ refining Castelnuovo's method.

Indeed, we will see a few cases in which assuming $m := \dim(\langle Y_{red}\rangle) < n$ but large, say $m \geq n-2$, we obtain better upper bounds than the ones obtained in [5] and below the range of gaps found in [1] in the case $m = n$. We are mainly interested in the case of generalized ropes or when Y is generically surfilinear, i.e. when the sheaf $\mathcal{I}/\mathcal{I}^2$ on Y_{red} is, outside finitely many points, a line bundle.

DEFINITION 3.1. *Let $Y \subset \mathbf{P}^n$ be a non-degenerate curve with Y_{red} irreducible. Fix a general $P \in Y_{red}$ and a general hyperplane H through P. Let Z be the connected component of the scheme $Y^{(1)} \cap H$ with $Z_{red} = \{P\}$. Set $f := dim(\langle Y_{red} \cup Z\rangle) - dim(\langle Y_{red}\rangle)$. The integer f does not depend from the choices of P and H and will be called the generic fattening dimension of Y.*

EXAMPLE 3.2. *Let $Y \subset \mathbf{P}^n$ be a non-degenerate generalized rope with Y_{red} irreducible. Set $m{:=}dim(\langle Y_{red}\rangle)$ and $t{:=}deg(Y)/deg(Y_{red}) \in \mathbf{N}$. Let f be the generic fattening dimension of Y. Assume $f = t-1$,i.e. assume that general $P \in Y_{red}$ and a general hyperplane H through P, $\langle Z(P)\rangle \cap \langle H \cap Y_{red}\rangle = \{P\}$, where $Z(P)$ is the connected component of $Y \cap H$ with P as support, this condition is always satisfied if $t = 2$. Notice that by definition of generic fattening dimension we have $m + f \leq n$. Let $\mathcal{I}$ be the ideal sheaf of Y_{red} in Y. Hence $\mathcal{I}^2 = 0$, $\mathcal{I}$ is the conormal sheaf of Y_{red} in Y and $rank(\mathcal{I}) = t-1$. The conormal sheaf of Y_{red} in $\mathbf{P}^n$ has a factor $\mathcal{O}_{Y_{red}}^{\oplus(n-m)}(-1)$. By the definition of fattening dimension and the assumption $f = t-1$ the image of $\mathcal{I}$ in this factor by the map associated to the standard exact sequence of conormal sheaves ([9], EGA IV 16.4.21) is generically surjective. Hence $deg(\mathcal{I}) \geq f(deg(Y_{red})) = (t-1)(deg(Y))/t$. By the exact sequence*

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to \mathcal{O}_{Y_{red}} \to 0 \tag{3}$$

and Riemann - Roch we obtain $1 - p_a(Y) = t - t(p_a(Y_{red})) + deg(\mathcal{I})$ and hence

$$p - a(Y) \leq t(p_a(Y_{red})) - t + 1 + (t-1)(deg(Y))/t \tag{4}$$

In several cases (e.g. if $m = n-1$ and $t = 2$) applying Castelnuovo's upper bound for $p_a(Y_{red})$ we obtain in this way an upper bound for $p_a(Y)$ better than the one obtained in [5] and below the range of gaps found in [1] in the case $m = n$.

EXAMPLE 3.3. *Let $Y \subset \mathbf{P}^n$ be a non-degenerate generically surfilinear curve such that Y_{red} is irreducible. Set $m := dim(\langle Y_{red}\rangle)$ and assume $m < n$. Let $\mathcal{I}$ be the ideal sheaf of Y_{red} in Y. Let z be the first integer such that $\mathcal{I}^{z+1} = 0$, i.e. such that $Y^{(z)} = Y$. Since Y_{red} is irreducible and Y is generically surfilinear, we have $deg(Y) = (z+1)deg(Y_{red})$. There is a natural map $S^t(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}^t/\mathcal{I}^{t+1}$ whic is an isomorphism at each smooth point of Y_{red}. As in Example 3.2 the assumption $m < n$ implies $h^0(Y_{red}, \mathcal{I}/\mathcal{I}^2) \neq 0$. Hence $deg(\mathcal{I}^t/\mathcal{I}^{t+1}) \geq -t(deg(Y_{red}))$ for every positive integer t. The sheaf $\mathcal{I}^t/\mathcal{I}^{t+1}$ has generically rank one because Y is generically surfilinear. From the exact sequences*

$$0 \to \mathcal{I}^t/\mathcal{I}^{t+1} \to \mathcal{O}_Y/\mathcal{I}^{t+1} \to \mathcal{O}_Y/\mathcal{I}^t \to 0 \tag{5}$$

and Riemann - Roch we obtain $\chi(\mathcal{O}_Y) = \sum_{t=0}^{z}\chi(\mathcal{I}^t/\mathcal{I}^{t+1}) = (z+1)\chi(\mathcal{O}_{Y_{red}}) + \sum_{t=0}^{z} deg(\mathcal{I}^t/\mathcal{I}^{t+1}) \geq (z+1)\chi(\mathcal{O}_{Y_{red}}) - z(z+1)(deg(Y_{red}))$. In a few cases with low z and high m this is a far better upper bound for $p_a(Y)$ than the one obtained in a straightforward way using Castelnuovo's method.

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